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A new family of pseudo-orthogonal polynomials (**)

1 - Introduction

In an investigation of the exact solution of two new types of Schrödinger equation (see [2]), H. Exton has recently discussed an explicit solution of the second order linear differential equation

$$xy'' + (ax^2 + bx + 3)\,y' + (cx^2 + 2ax + 2b)\,y = 0 \tag{1.1}$$

in which a is non-zero, and where the notation has been slightly modified for convenience.

If we put $y = Y'$, the Laplace transform of (1.1) is found to be of the first order, namely

$$t(at + c)\,v' = [t^3 + bt^2 - 2at - 2c]\,v \tag{1.2}$$

where

$$Y = \int_C \exp(xt)\,u(t)\,dt\,. \qquad v = u' \tag{1.3}$$

Hence

$$v = \exp[\frac{1}{2}t^2 a^{-1} + (ba^{-1} - ca^{-2})\,t]\,t^{-2}(t + ca^{-1})^{c^2a^{-3} - bca^{-2}}\,. \tag{1.4}$$

The contour of integration in (1.3) is taken to be a simple loop beginning and ending at $-\infty$ if $\mathrm{Re}(x) > 0$, or at $+\infty$ if $\mathrm{Re}(x) < 0$, and encircling the origin once. As shown in [2] after some working out, it may be found that (apart from any

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constant factors)

$$Y = \exp(-cxa^{-1}) \sum (2a)^{-m} \left(-\frac{b}{a}\right)^n \left(-\frac{c}{a}\right)^p (2, p) \cdot (-N, 2m + n + p) x^{N-2m-n-p} (m!\, n!\, p!)^{-1} \tag{1.5}$$

where, for convergence, $N = 0, 1, 2, \ldots$. As usual, the Pochhammer symbol (a, n) is given by

$$(a, n) = a(a + 1)(a + 2) \ldots (a + n - 1) = \Gamma(a + n)/\Gamma(a) \qquad (a, 0) = 1\,.$$

As a simple consequence of the binomial theorem (1.5) may be expressed as double series

$$Y = \exp(-cxa^{-1}) \sum (2a)^{-m} (-ca^{-1})^p (2, p) \cdot (-N, 2m + p)(x + ba^{-1})^{N-2m-p} (m!\, p!)^{-1}. \tag{1.6}$$

The indices of summation run over all of the non-negative integers and values of parameters leading to results which do not make sense are tacitly excluded. We have put

$$1 + c^2 a^{-3} - bca^{-2} = -N \tag{1.7}$$

see H. Exton [2], to which the reader is referred for more details in these preliminaries.

2 - Orthogonality of the solutions of (1.1)

We first present (1.1) in its self-adjoint form, namely

$$[x^3 \exp(bx + \frac{1}{2}ax^2) y']'' + \exp(bx + \frac{1}{2}ax^2) x^2 (cx^2 + 2ax + 2b) y = 0 \tag{2.1}$$

as may readily be confirmed by carrying out the differentiations and dividing by $x^2 \exp(bx + \frac{1}{2}ax^2)$.

In accordance with the usual Sturm-Liouville theory, the solutions of (2.1) and, in turn, (1.1) are *orthogonal* with respect to Riemann integration over the limits $\pm\infty$ and with weight function $x^4 \exp(bx + \frac{1}{2}ax^2)$, provided that $\mathrm{Re}(a) < 0$, and where the eingenvalue c is determined by (1.7). See E. L. Ince [3], Chapter 7, for example.

If (1.5) is written in the form

$$Y = \exp(-cxa^{-1}) P(N; a, b; x) \tag{2.2}$$

then the fact that $y = Y'$ implies

$$y = \exp(-cxa^{-1})[-ca^{-1}P(N;\, a,\, b;\, x) + P'(N;\, a,\, b;\, x)].$$

From the series representation (1.5), term-by-term differentation gives $P(N;\, a,\, b;\, x) = NP(N-1;\, a,\, b;\, x)$ and

$$(2.3)\quad y = y(N;\, x) = \exp(-cxa^{-1})[NP(N-1;\, a,\, b;\, x) - ca^{-1}P(N;\, a,\, b;\, x)]$$

recalling that c is determined by (1.7).

The associated orthogonality relation may be written as

$$(2.4)\qquad \int_{-\infty}^{\infty} x^4 \exp(bx + \frac{1}{2}ax^2)\, y(N;\, x)\, y(M;\, x)\, dx = K_N\, \delta_{M,N}$$

where K_N is given by

$$(2.5)\qquad K_N = \int_{-\infty}^{\infty} x^4 \exp(bx + \frac{1}{2}ax^2)[y(N;\, x)]^2\, dx\,.$$

The expression (2.5) can be written as a rather cumbersome linear combination of convergent series by means of term-by-term integration. So far, no compact representation for K_N has been deduced.

A rather unusual feature of this family of orthogonal functions is that the eigenvalue c, as related to N by (1.7), is associated to the exponential part $\exp(-cxa^{-1})$, as well as with the degree of the polynomial

$$NP(N-1;\, a,\, b;\, x) - ca^{-1}P(N;\, a,\, b;\, x),$$

so that the exponential factor of (2.3) cannot be subsumed into the weight function of (2.4). The function $y(N;\, x)$ is not completely polynomial in form. The polynomial

$$(2.6)\qquad R(N;\, a,\, b;\, x) = NP(N-1;\, a,\, b;\, x) - ca^{-1}P(N;\, a,\, b;\, x)$$

is therefore described as being *pseudo-orthogonal* in character.

3 - A pure recurrence relation for $P(N;\, -2a^2,\, b;\, X)$

For convenience, replace a by $-2a^2$ and put $x - \frac{1}{2}ba^{-2} = X$, when from the double series expression (1.6) and bearing in mind (2.2), we have

$$(3.1)\qquad \begin{aligned} &P(N;\, -2a^2,\, b;\, x) = P(N;\, X) \\ &= \sum(-4a^2)^{-m}(\frac{1}{2}ca^{-2})^p(2,\, p)(-N,\, 2m+p)X^{N-2m-p}(m!\,p!)^{-1}. \end{aligned}$$

This double series may easily be written as a finite series of Hermite polynomials, so that, after a little algebra

$$P(N;X)=(2a)^{-N}\sum(2,p)(-N,p)(ca^{-1})^{p}H_{N-p}(aX)(p!)^{-1}$$

and if p is replaced by $N-k$. it follows that

(3.2) $$P(N;X)=(-\tfrac{1}{2}ca^{-2})^{N}N!\sum(2,N-k)(ac^{-1})^{k}(-1)^{k}[k!(N-k)!]^{-1}H_{k}(ax).$$

This series has certain features in common with a case discussed by E. D. Rainville in [4], Section 128 (1) p. 240. Following Rainville's approach, put

(3.3) $$\frac{1}{N!}P(N;X)(-2a^{2}c^{-1})^{N}=s(N;X)$$

and $$\frac{(-1)^{k}}{k!}(ac^{-1})^{k}(2,N-k)H_{k}(aX)=\nu(k;X)$$

bearing in mind that c is a function of N.

From the well-known recurrence relation for the Hermite polynomials, it may easily be shown that

(3.4) $$\begin{gathered}(ca^{-1})^{2}k(2+N-k)(3+N-k)\nu(k;X)\\+2cX(3+N-k)\nu(k-1;X)+2\nu(k-2;X)=0\,.\end{gathered}$$

Proceeding further with Rainville's technique, an expression is deduced for the series $\sum\frac{1}{(N-k)!}k(2+N-k)(3+N-k)v(k;X)$ in the form $As(N;X)+Bs(N-1;X)+Cs(N-2;X)+Ds(N-3;X)$ noting that

$$s(N;X)=\sum\frac{1}{(N-k)!}\nu(k;X)$$

and that $$s(N-1;X)=\sum\frac{1}{(N-k)!}(N-k)\nu(k;X)/(N-k)!\,,\quad\text{etc.}$$

We then obtain the *identity*

$$A+B(N-k)+C(N-k)(N-k-1)+D(N-k)(N-k-1)(N-k-2)=E$$

where $E=k(k-2-n)(k+3-n)$, from which expressions for the quantities A, B, C and D may be worked out. Hence

$$-3N(N^{2}+2N+2)\,s(N;X)+3(N^{2}+2N+2)\,s(N-1;X)+(N-2)\,s(N-2;X)-s(N-3;X)$$

$$=\sum\frac{1}{(N-x)!}k(k-2-N)(k-3-N)\,\nu(k;X)\,.$$

Similarly, it may be established that

$$3s(N-1;X)+s(N-2;X)=\sum\frac{1}{(N-x)!}(3+N-k)\,\nu(k-1;X)$$

and that
$$s(N-2;X)=\sum\frac{1}{(N-x)!}\nu(k-2;X)\,.$$

If the corrisponding terms of the above series are now added, the recurrence relation (3.4) becomes

$$3(ca^{-1})^2N(N^2+2N+2)\,s(N;X)-3[(ca^{-1})^2(N^2+2N+2)+6cX]\,s(N-1;X)$$
$$-[(ca^{-1})^2(N-2)+2cX+2]\,s(N-2;X)+(ca^{-1})^2s(N-3;X)=0\,.$$

Since, by (3.2), $s(N;X)=(N!)^{-1}P(N;x)(-2a^2c^{-1})^N$, we may deduce the required recurrence relation for $P(N;X)$. Recall that $c=c(N)$ is itself a function of N determined by $1-[c(N)]^2(8a^6)^{-1}-b[c(N)](4a^4)^{-1}=-N$. See (1.7), in which a has been replaced by $-2a^2$.

Hence we have finally the *recurrence relation* with irrational coefficients

$$\begin{aligned}&24a^6\{c(N)\}^{2-N}(N^2+2N+2)P(N;\,-2a^2,\,b;\,X)\\ &+12a^4[\{c(N)\}^2(N^2+2N+2)+6a^2c(N)X]\{c(N-1)\}^{1-N}P(N-1;-2a^2,b;X)\\ &-2a^2[\{c(N)\}^2(N-2)+2c(N)X+2](N-1)\{c(N-2)\}^{2-N}P(N-2;-2a^2,b;X)\\ &-\{c(N)\}^2(N-1)(N-2)\{c(N-3)\}^{3-N}P(N-3;\,-2a^2,\,b;\,X)=0\end{aligned}\tag{3.5}$$

where $X=\frac{1}{2}(x-ba^{-2})$.

For more details of the above method, the reader should consult E. D. Rainville [4], Chapter 14.

4 - Relationship with the biconfluent Heun equation

In conclusion, the close relation between (1.1) and the biconfluent Heun equation is briefly indicated. If the process of confluence is carried out twice to the Heun equation, such that two of the finite regular singularities are made to coalesce with that at infinity, the so-called *biconfluent Heun equation* is obtained. By general consensus, its canonical form is taken to be

$$xy''+(1+\alpha-\beta x-2x^2)\,y'+[(\gamma-\alpha-2)\,x+\frac{1}{2}[\delta+(1+\alpha)\beta]\}\,y=0\,.\tag{4.1}$$

For a general discussion of Heun's equation and its various confluent forms, see the comprehensive treatise [5] of A. Ronveaux. In general, explicit solutions of (4.1) can be obtained, but they are extremely complicated (see H. Exton [1]). It will now be briefly indicated that (1.1) can be reduced to a non-trivial special case of the biconfluent Heun equation

In (1.1), put $y = \exp(\lambda x)\,Y$, and obtain the equation

$$xY'' + [ax^2 + (b+2\lambda)\,x + 3]\,Y' + [(c+a\lambda)\,x^2 + (2a + b\lambda + \lambda^2)\,x + 2b + 3\lambda]\,Y = 0\,. \tag{4.2}$$

Let $\lambda = -ca^{-1}$ and replace x by $(2a^{-1})^{\frac{1}{2}}iX$, when (4.2) is clearly of biconfluent Heun form, and $R(N;\,a,\,b;\,x)$ as given by (2.8) is a biconfluent Heun function, which is not, in itself orthogonal, unless multiplied by $\exp(-cxa^{-1})$.

References

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[3] E. L. INCE, *Ordinary differential equations*, Longmans Green, London 1926.

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Sommario

Viene indicata una soluzione compatta di un'equazione differenziale ordinaria lineare del secondo ordine, strettamente legata ad un caso particolare non banale dell'equazione biconfluente di Heun. Questa soluzione è ortogonale ed ha la forma di una funzione esponenziale che fa intervenire un autovalore associato, moltiplicato per un polinomio, il cui grado è anche esso legato all'autovalore. Il polinomio di per sè non è ortogonale, tuttavia viene chiamato «pseudo-ortogonale». Si deduce anche una relazione di ricorrenza a quattro termini a coefficienti irrazionali.

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