Riv. Mat. Univ. Parma (5) 5 (1996), 103-107

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Simpson points in normed spaces (**)

1 - Introduction

The main subject of this paper is the study of the so called *minimax set*.

A way to describe this set is to consider the situation where there is a given set of customers at fixed positions and where each customer will purchase one unit of commodity from the firm closest to his location. The problem for a firm is to choose a location that will garantee to it at least as many customers as to its competitor, regardless of where its competitor locates (see [3], [6]).

These problems lead to the following model: for any position x in the set of alternative locations X define $W(x) = \sup_{y \in X} W(y, x)$ where $W(y, x)$ is the number of customers who prefer y to x. A *Simpson point* is a point which minimize $W(x)$ on X and the minimax set is the set of Simpson points (see [3], [8]).

By identifying customers as voters and firms as candidates and by assuming that each voter will vote for the candidate closest to his position the above question leads to the problem in voting theory of the existence of a candidate's position that is unbatible in an election (see [4], [5], [9]).

The minimax set problem will be studied from a theoretical point of view. The set of alternatives will be a real normed space X, the distance will be measured by the norm and the set of customers will be a compact non singleton set $A \subset X$ with a probability measure m on the Borel σ-algebra of A supported by the whole set (that is: for every open set $V \subset X$ satisfying $V \cap A \neq \emptyset$, we have: $m(V \cap A) > 0$).

For $x, y \in X$, we consider the set $A(y, x) = \{a \in A \colon \|y - a\| < \|x - a\|\}$ and define $W_A(y, x) = m(A(y,x))$, $W_A(x) = \sup_{y \in X} W_A(y, x)$ and for $k \in (0, 1]$ $C_k = \{x \in X \colon W_A(x) \leq k\}$.

We simply write $W(y, x)$ and $W(x)$ when no confusion can arise.

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(**) Received June 14, 1996. AMS classification 90 A 28.

Note that a point x belongs to C_k if for any other location y the *percentage* of users who prefer x to y is not smaller then $1-k$. Obviously $C_1 = X$.

The following *properties* of C_k are easy consequences of the above definitions except the last result which is taken from [3]:

1. There exists $k_0 > 0$ such that $C_k = \emptyset$ for $k \leqslant k_0$.
2. If $m(\{a\}) > 0$ $(a \in A)$ then $a \in C_k$ if $k > 1 - m(\{a\})$.
3. $C_{k_1} \subset C_{k_2}$ if $k_1 \leqslant k_2$.
4. C_k is a bounded and closed subsed of X for any $k \in (0, 1)$.

If X is a finite dimensional space and there exists $k \in (0, 1)$ such that $C_k \neq \emptyset$, then the last two properties imply that C_k forms a nested family of compact sets. Therefore the set of Simpson points is a non empty proper subset of X.

Now the following question seems to be natural: are, in general, the sets C_k non empty for some $k \in (0, 1)$? Partial results are known. In particular in [**1**] it is proved that if $x \in \operatorname{int}(\operatorname{convex}(A))$ then $W(x) < 1$ if X is a finite dimensional Hilbert space and in [**3**] some constructions of C_k are given for finite sets A when X is a two-dimensional polyhedral space.

Our paper is divided in two parts. In the first one we give definitions and preliminary results, in the second one we prove a general result.

2 - Notations and preliminary results

Let X be a real Banach space and X^* its topological dual. Put

$$B'(x, r) = \{y \in X\colon\ 0 < \|x - y\| < r\} \qquad \text{and} \qquad S = \{x \in X\colon\ \|x\| = 1\}.$$

If A is a subset of X, then we indicate with $\operatorname{int}(A)$ its interior. If $x, y \in X$, we set

$$\tau(x, y) = \inf_{t>0} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t \searrow 0} \frac{\|x + ty\| - \|x\|}{t}.$$

We list some *properties of the function* τ (for other properties of the function τ see [2], [7]):

1. $\tau(x, y) = \sup\{f(y) \mid f \in X^*,\ \|f\| = 1,\ f(x) = \|x\|\}$
2. $\tau(\lambda x, \mu y) = \mu \tau(x, y)$ for $\lambda \in \boldsymbol{R}$, $\mu \geqslant 0$
3. $\tau\colon X \times X \to \boldsymbol{R}$ is upper semicontinuous.

Now we give a formula to describe the function $W(x)$ in a different way.

Theorem 1. $W_A(x) = \sup_{y \in S} m(\{a \in A\colon \tau(a - x, y) < 0\})$.

Proof. If $x, y \in X$ and $\lambda \in [0, 1]$, set $y_\lambda = \lambda y + (1 - \lambda)x$ and let $\lambda_1 < \lambda_2$ with $\lambda_1, \lambda_2 \in (0, 1]$.

The convexity of $f(t) = \|a - x + t(x - y)\|$ implies that $A(y_{\lambda_1}, x) \supset A(y_{\lambda_2}, x)$. In fact if $a \in A(y_{\lambda_2}, x)$ then

$$f(\lambda_1) = f(s\lambda_2) \leqslant sf(\lambda_2) + (1 - s)f(0) \leqslant sf(0) + (1 - s)f(0) = f(0)$$

that is $\|a - x + \lambda_1(x - y)\| < \|a - x\|$ and so $a \in A(y_{\lambda_1}, x)$.

In particular we have $A(y, x) \subseteq A(y_\lambda, x)$ for any $\lambda \in (0, 1]$. This implies the monotonicity of $W(y_\lambda, x)$ and so $\lim_{\lambda \searrow 0} W(y_\lambda, x)$ exists. Moreover

$$W(y, x) \leqslant \lim_{\lambda \searrow 0} W(y_\lambda, x) \leqslant \sup_{x \neq y} \lim_{\lambda \searrow 0} W(y_\lambda, x)$$

and so $W(x) \leqslant \sup_{x \neq y} \lim_{\lambda \searrow 0} W(y_\lambda, x)$.

But obviously $W(x) \geqslant W(y_\lambda, x)$ for any $\lambda \in (0, 1]$ and $y \neq x$ and so $W(x) \geqslant \lim_{\lambda \searrow 0} W(y_\lambda, x)$. This implies $W(x) \geqslant \sup_{x \neq y} \lim_{\lambda \searrow 0} W(y_\lambda, x)$ and therefore

$$W(x) = \sup_{x \neq y} \lim_{\lambda \searrow 0} W(y_\lambda, x).$$

Let now $a \in \bigcup_{\lambda > 0} A(y_\lambda, x)$, hence there exists $\lambda \in (0, 1]$ such that $\|a - x + \lambda(x - y)\| - \|a - x\| < 0$ and so $\inf_{\lambda > 0} \dfrac{\|a - x + \lambda(x - y)\| - \|a - x\|}{\lambda} < 0$ that is $\tau(a - x, x - y) < 0$.

Conversely if $\tau(a - x, x - y) < 0$, recalling the definition of τ, we have immediately that there exists $\lambda \in (0, 1]$ such that: $\|a - x + \lambda(x - y)\| - \|a - x\| < 0$ and so $a \in A(y_\lambda, x)$.

Finally we obtain $\{a \in A \mid \tau(a - x, x - y) < 0\} = \bigcup_{\lambda > 0} A(y_\lambda, x)$ and so

$$m(\{a \in A \mid \tau(a - x, x - y) < 0\}) = m(\bigcup_{\lambda > 0} A(y_\lambda, x)) = \lim_{\lambda \searrow 0} W(y_\lambda, x).$$

Now $\sup_{y \neq x} m(\{a \in A \mid \tau(a - x, x - y) < 0\}) = \sup_{y \neq x} \lim_{\lambda \searrow 0} W(y_\lambda, x) = W(x)$. Replacing y by $x - y$ and using the second property of the function τ listed above we obtain our result.

Since in inner product spaces we have $\tau(x, y) = (x, y)\|x\|^{-1}$ as a consequence we have

Corollary 1. *Let H be an inner product space; then for $x \in H$ we have*

$$W_A(x) = \sup_{y \in H} m(\{a \in A \mid (a, y) < (x, y)\}).$$

3 - A general result

This is our main result.

Theorem 2. *Let X be a finite dimensional normed space and A be a compact subset of X. Iff $x \in \operatorname{int}(A)$ then $W_A(x) < 1$.*

Proof. By translation it is not restrictive to suppose that $x = 0$ and by Theorem 1 $W_A(0) < 1$ if there exists $\alpha > 0$ such that

$$m(\{a \in A \mid \tau(a, y) \geqslant 0\}) \geqslant \alpha \qquad \text{for every } y \in S.$$

Suppose, by contradiction, that there exists a sequence $\{y_n\}$ of unit vectors such that $m(\{a \in A;\ \tau(a, y_n) \geqslant 0\}) < \frac{1}{n}$.

Since X is finite dimensional, we can suppose that $\{y_n\}$ converges to some $y_0 \in S$. Let now $\varepsilon > 0$ such that $B'(0, \varepsilon) \subset A$; then we have again $m(\{a \in B'(0, \varepsilon);\ \tau(a, y_n) \geqslant 0\}) < \frac{1}{n}$. Let now $x, y \in X$ with $x \neq 0$ and $f \in X^*$ be a support functional at x, i.e. a functional such that $\|f\| = 1$ and $f(x) = \|x\|$. Then we obtain

$$\frac{\|x + ty\| - \|x\|}{t} \geqslant \frac{|f(x + ty)| - \|x\|}{t} \geqslant \frac{f(x) + f(ty) - f(x)}{t} = f(y)$$

$$= f(\frac{x}{\|x\|}) + f(y - \frac{x}{\|x\|}) \geqslant 1 - |f(y - \frac{x}{\|x\|})| \geqslant 1 - \|y - \frac{x}{\|x\|}\|.$$

Thus $\tau(x, y) \geqslant 1 - \|y - \frac{x}{\|x\|}\|$. This implies

$$\{a \in B'(0, \varepsilon):\ \|\frac{a}{\|a\|} - y_n\| < 1\} \subset \{a \in B'(0, \varepsilon):\ \tau(a, y_n) \geqslant 0\}.$$

Let now $E = \{a \in B'(0, \varepsilon):\ \|\frac{a}{\|a\|} - y_0\| < \frac{1}{2}\}$. Then if $a \in E$ and if $n > n_0$ we have

$$\|\frac{a}{\|a\|} - y_n\| \leqslant \|\frac{a}{\|a\|} - y_0\| + \|y_0 - y_n\| < \frac{1}{2} + \|y_0 - y_n\| < 1$$

so $a \in \{a \in B'(0, \varepsilon): \|\frac{a}{\|a\|} - y_n\| < 1\}$ for $n > n_0$. So we have:

$$m(E) \leq \lim_{n \to \infty} m(\{a \in B'(0, \varepsilon): \|\frac{a}{\|a\|} - y_n\| < 1\}) = 0 .$$

This is a contradiction since E is an open subset of A and the measure is supported by A.

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Sommario

I punti di Simpson si presentano in modo naturale nella teoria delle competizioni locali. Vengono date condizioni per l'esistenza di tali punti in un quadro astratto di spazi normati di dimensione finita.

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