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Existence and uniqueness theorems in the linear magnetohydrodynamics with dissipative boundary conditions (**)

1 - Introduction

This paper presents some *existence and uniqueness theorems* for a linear differential problem which characterizes the evolution of a magnetohydrodynamics system, without neglecting the current displacement and the separation of charges. Such a system is usually called a *plasma* [1], [3], [4], [7]. In particular, a condition on the boundary characterizing a large class of *dissipative boundary conditions* is also considered.

Furthermore, being the domain Ω unbounded [2], the existence and uniqueness are studied both in the case of a *finite energy* all over Ω and in the case of a *locally finite energy*. For this purpose, we will use a theorem on the domain dependence, which guarantees a finite speed of propagation and a theorem of continuous dependence on data.

The method, used in the following, refers to [5], [6], [8], [9], which are related to linear symmetric hyperbolic systems considered in this paper.

2 - Problem statement

Let Ω be a domain of the ordinary three dimensional space $\boldsymbol{R}^3$, and $x = (x_1, x_2, x_3)$ an arbitrary point of $\boldsymbol{R}^3$; I is a finite interval of a timelike variable t. Vectors $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{v}$ represent the *electric field*, the *magnetic field* and the

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speed of electrons respectively. The *pressure of the electrons* is characterized by a scalar function p.

This set of variables defines the electromagnetic state of a *magnetohydrodynamics system* (*plasma*) by means of the function $\mathcal{H} = [\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{v}, p]$, which is related both to a supply of *magnetic current* $\boldsymbol{K}$, *electric current* $\boldsymbol{J}$, *external body force* $\boldsymbol{F}$, and to a *supply of flow* Φ.

Furthermore, on the plasma is applied a *magnetic induction field* denoted by $\boldsymbol{B}_0$, while n_0 and v_0 will characterize the *mean density* and *mean speed* of electrons respectively.

For a given plasma, we will consider a differential problem with initial-boundary conditions, which satisfies the following system:

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E} - \boldsymbol{K} \tag{1}$$

$$\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} + e n_0 \boldsymbol{v} - \boldsymbol{J} \qquad Q = \Omega \times I \tag{2}$$

$$m n_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p - e n_0 (\boldsymbol{E} + \boldsymbol{B}_0 \times \boldsymbol{v}) + \boldsymbol{F} \qquad Q = \Omega \times I \tag{3}$$

$$\frac{1}{n_0 m v_0^2} \frac{\partial p}{\partial t} = -\nabla \cdot \boldsymbol{v} + \Phi \tag{4}$$

e being the *charge of the electron* whereas ε_0 and μ_0 represent the *dielectric constant* and the *magnetic permeability*, in vacuum, which are ruled by the condition $\frac{1}{\varepsilon_0 \mu_0} = c^2 > v_0^2$. The initial conditions are

$$\begin{aligned} &\boldsymbol{H}(x, 0) = \boldsymbol{h}(x) \qquad \boldsymbol{E}(x, 0) = \boldsymbol{e}(x) \qquad \text{in } \Omega \\ &\boldsymbol{v}(x, 0) = \boldsymbol{\nu}(x) \qquad p(x, 0) = P(x) \qquad \text{in } \Omega \end{aligned} \tag{5}$$

where the vectors $\boldsymbol{h}(x)$, $\boldsymbol{e}(x)$, $\boldsymbol{\nu}(x)$ and the scalar function $P(x)$ are assumed to be known in Ω, so that the initial state $\mathcal{H}^0 = [\boldsymbol{h}(x), \boldsymbol{e}(x), \boldsymbol{\nu}(x), P(x)]$ is given.

Furthermore the function $\mathcal{H}$ satisfies a linear and homogeneous boundary conditions Γ such that the inequalities:

$$\boldsymbol{E} \times \boldsymbol{H} \cdot \boldsymbol{n} \leqslant 0 \qquad \text{in } \partial\Omega \times I \tag{6}$$

$$p \boldsymbol{v} \cdot \boldsymbol{n} \leqslant 0 \qquad \text{in } \partial\Omega \times I \tag{7}$$

are satisfied on the boundary $\partial\Omega$, being $\boldsymbol{n}$ the outward normal vector. The field source is represented by the set of functions $\mathcal{F}(x, t) = [-\boldsymbol{K}, -\boldsymbol{J}, \boldsymbol{F}, \Phi]$.

Together with the differential system (1)-(4) we must consider, for the plasma state $\mathcal{H}$, the *rate of the energy* in the domain $\Omega \cap K$ at time t, i.e.

$$\frac{1}{2} \int_{\Omega \cap K} \frac{\partial}{\partial t} (\mu_0 \boldsymbol{H}^2 + \varepsilon_0 \boldsymbol{E}^2 + n_0 m v^2 + \frac{p^2}{n_0 m v_0^2}) \, dx \tag{8}$$

for every cube $K \subset \boldsymbol{R}^3$.

If the integral (8) converges to a finite value for any measurable set K, then $\mathcal{H}$ is called a *plasma state with finite energy*. If the integral (8) converges only for a bounded measurable set K then $\mathcal{H}$ is called a *plasma state with locally finite energy*.

From a differential point of view, system (1)-(4) is an hyperbolic system of differential equations of the type:

$$E(x) \frac{\partial \boldsymbol{u}}{\partial t} = \mathcal{A}^i(x) \frac{\partial \boldsymbol{u}}{\partial x_i} + B(x) \boldsymbol{u} + \boldsymbol{f}(x, t) \qquad i = 1, \ldots, n \tag{9}$$

where the $m \times m$ matrices

$$E = (E_{\alpha\beta}) \qquad \mathcal{A}^i = (A^i_{\alpha\beta}) \qquad B = (B_{\alpha\beta}) \qquad \alpha, \beta = 1, \ldots, m$$

and the m vectors $\boldsymbol{u} = (u_\alpha), \boldsymbol{f} = (f_\alpha)$ $(\alpha = 1, \ldots, m)$, are functions of $x = (x_\alpha)$ and t.

When E and $\mathcal{A}^i$, $(i = 1, \ldots, n)$ are symmetric matrices and E is definitely positive then system (9) is an hyperbolic symmetric system. Thus system (1)-(4) is an *hyperbolic symmetric system* like (9), assuming $n = 3$, $m = 10$ and with $\boldsymbol{u} = \mathcal{H}$, $f = \mathcal{F}$.

The differential operator of the first order A acting on the class $\mathcal{C}^1$ of the continuous functions with continuous first derivative is defined as

$$A\mathcal{H} = \mathcal{A}^i \frac{\partial \mathcal{H}}{\partial x_i} \qquad i = 1, 2, 3$$

and verifies the condition on the formal adjoint $A^* = A$.

3 - Preliminary definitions

In this section we give preliminary definitions and recall standard notations [8].

Let us consider only functions $u: D \to H$, defined in an arbitrary domain $D \subset \boldsymbol{R}^n$ and values into an Hilbert separable space H, where the scalar product is $(f, g)_H$ and the norm of a function is $\|f\|_H = (f, f)_H^{\frac{1}{2}}$. The space

$$L_2(D, H) = \{u \mid u \text{ is measurable in } H, \int_D \|u\|_H^2 \, dx < \infty\}$$

is an *Hilbert space*; moreover

$$L_2^{\text{loc}}(D, H) = \{u \mid u \in L_2(K \cap D, H) \quad \forall \text{ bounded measurable set } K \subset \boldsymbol{R}^n\}$$

is the *space of locally square-integrable functions*. Let $\Lambda : H \to H'$ be a linear differential operator with bounded and measurable coefficients and $\Lambda^+ : H' \to H$ its formal adjoint. Let us now consider the linear subspaces:

$$L_2(\Lambda, D, H) = \{u \in L_2(D, H) \mid \Lambda u \in L_2(D, H')\}$$

$$L_2^{\text{loc}}(\Lambda, D, H) = \{u \in L_2^{\text{loc}}(D, H) \mid \Lambda u \in L_2^{\text{loc}}(D, H')\}$$

$$L_2^{\text{vox}}(\Lambda, D, H) = L_2(\Lambda, D, H) \cap \{u \mid u = 0 \text{ outside a bounded set}\}.$$

In particular, if $\boldsymbol{A}, \boldsymbol{B} \in L_2(\nabla \times, \Omega, \boldsymbol{R}^3)$ (where $\nabla \times$ is the *curl operator*) then

$$\int_\Omega (\boldsymbol{A} \cdot \nabla \times \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \times \boldsymbol{A})\, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{n}\, \mathrm{d}s \tag{10}$$

there follows that $\nabla \times$ is formally selfadjoint, i.e.

$$\int_\Omega \boldsymbol{A} \cdot \nabla \times \boldsymbol{\Psi} \mathrm{d}x = \int_\Omega \nabla \times \boldsymbol{A} \cdot \boldsymbol{\Psi} \mathrm{d}x \tag{11}$$

for any $\boldsymbol{A} \in L_2(\nabla \times, \Omega, \boldsymbol{R}^3)$ and $\boldsymbol{\Psi} \in C_1^0(\Omega, \boldsymbol{R}^3)$.

In fact, when $\boldsymbol{A}$ fulfills the boundary condition $\boldsymbol{A} \times \boldsymbol{n} = 0$ in $\partial\Omega$ (inequality (6) is satisfied when $\boldsymbol{A} = \boldsymbol{E}$ or $\boldsymbol{A} = \boldsymbol{H}$), then equation (10) implies (11) for all $\boldsymbol{B}$. As consequence one is led to consider the space

$$L_2^0(\nabla \times, \Omega, \boldsymbol{R}^3) = \{\boldsymbol{A} \mid \boldsymbol{A} \in L_2(\nabla \times, \Omega, \boldsymbol{R}^3), \int_\Omega \boldsymbol{A} \cdot \nabla \times \boldsymbol{B}\, \mathrm{d}x = \int_\Omega \boldsymbol{B} \cdot \nabla \times \boldsymbol{A}\, \mathrm{d}x, \forall \boldsymbol{B} \in L_2(\nabla \times, \Omega, \boldsymbol{R}^3)\}$$

as a generalization of the class of fields satisfying the boundary condition

$$\boldsymbol{A} \times \boldsymbol{n} = 0 \qquad \text{in } \partial\Omega.$$

The space $L_2^0(\nabla \times, \Omega, \boldsymbol{R}^3)$ is obviously a linear closed subspace of $L_2(\nabla \times, \Omega, \boldsymbol{R}^3)$. Analogously for the spaces $L_2(\nabla \cdot, \Omega, \boldsymbol{R}^3)$, $L_2(\nabla, \Omega, \boldsymbol{R}^3)$, according to the formula

$$\int_\Omega [(\nabla \cdot \boldsymbol{v}) p + \nabla p \cdot \boldsymbol{v}]\, \mathrm{d}x = \int_{\partial\Omega} p \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}s \tag{12}$$

the boundary condition $p\boldsymbol{v} \cdot \boldsymbol{n} = 0$ is satified in a generalized sense by the class of spaces:

$$L_2^0(\nabla \cdot, \Omega, \boldsymbol{R}^3) = \{\boldsymbol{v} \in L_2(\nabla \cdot, \Omega, \boldsymbol{R}^3) \mid \int_\Omega (\phi \nabla \cdot \boldsymbol{v} + \nabla \phi \cdot \boldsymbol{v})\, \mathrm{d}x = 0, \ \forall \phi \in L_2(\nabla, \Omega, \boldsymbol{R}^3)\}$$

$$L_2^0(\nabla, \Omega, \boldsymbol{R}^3) = \{p \in L_2(\nabla, \Omega, \boldsymbol{R}^3) \mid \int_\Omega (p \nabla \cdot \boldsymbol{\Phi} + \nabla p \cdot \boldsymbol{\Phi})\, \mathrm{d}x = 0, \ \forall \boldsymbol{\Phi} \in L_2(\nabla \cdot, \Omega, \boldsymbol{R}^3)\}.$$

In the following we shall assume $I=\{t\,|\,0<t<T\}$ and $H^1(I,H)=L_2(\frac{\partial}{\partial t},I,H)$. Let $\Gamma=\Gamma_1\times\Gamma_2\times\Gamma_3\times\Gamma_4$ be a closed linear subspace of $L_2(A,\,\Omega,\,\boldsymbol{R}^{10})$, where $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4$ are closed linear subspaces of $L_2(\nabla\times,\,\Omega,\,\boldsymbol{R}^3)$, $L_2(\nabla\times,\,\Omega,\,\boldsymbol{R}^3)$, $L_2(\nabla\cdot,\,\Omega,\,\boldsymbol{R}^3)$, $L_2(\nabla,\,\Omega,\,\boldsymbol{R})$. According to these definitions, we have

$$\boldsymbol{H}\in\Gamma_1 \qquad \boldsymbol{E}\in\Gamma_2 \qquad v\in\Gamma_3 \qquad p\in\Gamma_4 .$$

The boundary conditions (6), (7) have to be generalized in order to include the dissipative condition derived from the identities (10), (11), i.e.:

$$\int_\Omega(\boldsymbol{E}\cdot\nabla\times\boldsymbol{H}-\boldsymbol{H}\cdot\nabla\times\boldsymbol{E})\,dx\leqslant 0 \qquad \forall\boldsymbol{H}\in\Gamma_1,\quad \boldsymbol{E}\in\Gamma_2,$$

$$\int_\Omega(p\nabla\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\nabla p)\,dx\leqslant 0 \qquad \forall\boldsymbol{v}\in\Gamma_3,\quad p\in\Gamma_4 .$$

Together with the condition that Φ be locally dissipative we must have $\Phi\mathcal{H}\in\Gamma$ whenever $\Phi\in C_0^1(\boldsymbol{R}^3)$ and $\mathcal{H}\in\Gamma$. We assume also

$$L_2^0(\nabla\times,\,\Omega,\,\boldsymbol{R}^3)\subset\Gamma_1;\; L_2^0(\nabla\times,\,\Omega,\,\boldsymbol{R}^3)\subset\Gamma_2;\; L_2^0(\nabla\cdot,\,\Omega,\,\boldsymbol{R}^3)\subset\Gamma_3;\; L_2^0(\nabla,\,\Omega,\,\boldsymbol{R})\subset\Gamma_4$$

so that any boundary condition Γ implies a further adjoint boundary condition Γ^* defined as

$$\Gamma^*=L_2(A,\Omega,\boldsymbol{R}^{10})\cap\{\mathcal{K}:\mathcal{K}\in L_2(A,\Omega,\boldsymbol{R}^{10}),\ \int_\Omega[\langle\mathcal{H},A\mathcal{K}\rangle+\langle A\mathcal{H},\mathcal{K}\rangle]\,dx=0\ \forall\mathcal{H}\in\Gamma\}.$$

4 - Solutions with finite energy

In this section we define the solution with finite energy (FE-*solution*) and the solution with locally finite energy (LFE-*solution*) that satisfy given initial-boundary conditions.

Since we have that any linear homogeneous locally dissipative boundary condition Γ is made of a closed linear subspace of $L_2(A,\,\Omega,\,\boldsymbol{R}^{10})$, Γ is a *separable Hilbert space* with respect to the scalar product of $L_2(A,\,\Omega,\,\boldsymbol{R}^{10})$. The space

$$F=L_2(I,\,\Gamma)\cap H^1(I,\,L_2(A,\,\Omega,\,\boldsymbol{R}^{10}))$$

denotes the class of functions $\mathcal{H}(x,t)$ for which $E\,\dfrac{\partial\mathcal{H}}{\partial t}$, $A\mathcal{H}$, $B\mathcal{H}$ exist in $L_2(Q;\,\boldsymbol{R}^{10})$, and satisfies the boundary condition Γ, in the sense that $\mathcal{H}(t)\in\Gamma$ for almost all $t\in I$. Let us define the following sets:

$$\Gamma^{\text{vox}}=\Gamma\cap\{\mathcal{H}\,|\,\mathcal{H}=0 \text{ outside a bounded set } K\subset\Omega\}$$

$$\Gamma^{\text{loc}}=\Gamma_2^{\text{loc}}(A,\,\Omega)\cap\{\mathcal{H}\,|\,\int_\Omega(\langle\mathcal{H},\,A\mathcal{K}\rangle+\langle A\mathcal{H},\,\mathcal{K}\rangle)\,dx=0\} \qquad \forall\mathcal{K}\in(\Gamma^*)^{\text{vox}}$$

$$F^{\text{loc}}=\{\mathcal{H}\,|\,\mathcal{H}\in L_2(I,\,L_2(A,\,K\cap\Omega))\cap H^1(I,\,L_2(A,\,K\cap\Omega))$$

for any bounded measurable set $K \subset \boldsymbol{R}^3$ and $\mathcal{H}(t) \in \Gamma^{\mathrm{loc}}\ \forall t \in I\}$

$$F^* = L_2(I, \Gamma^*) \cap H^1((I, L_2 \Omega))$$

$$(F^*)^{\mathrm{vox}} = F^* \cap \{\mathcal{K} : \mathcal{K} = 0 \text{ outside } K \times I \subset Q\}.$$

According to the above definitions we can distinguish the following kind of solutions:

Definition 1. $\mathcal{H}$ is a FE-*solution* of system (1)-(4) with boundary condition Γ and $\mathcal{F}(x, t) \in L_2(Q; \boldsymbol{R}^{10})$, $\mathcal{H}^0(x) \in L_2(\Omega)$ given, iff $\mathcal{H}(t) \in F$ satisfies

$$\text{(13)} \qquad \begin{aligned} E\,\frac{\partial \mathcal{H}}{\partial t} &= A\mathcal{H} + B\mathcal{H} + \mathcal{F} && \text{almost everywere in } Q \\ \mathcal{H}(0) &= \mathcal{H}^0 && \text{almost everywere in } \Omega\,. \end{aligned}$$

Definition 2. $\mathcal{H}$ is a *weak* FE-*solution* of system (1)-(4) with boundary condition Γ and $\mathcal{F}(x, t) \in L_2(Q; \boldsymbol{R}^{10})$, $\mathcal{H}^0(x) \in L_2(\Omega)$ given, iff $\mathcal{H}(t) \in L_2(Q; \boldsymbol{R}^{10})$ satisfies

$$\text{(14)} \qquad \begin{aligned} &\int_Q \langle E\,\frac{\partial \mathcal{K}}{\partial t} - A\mathcal{K} - B\mathcal{K}, \mathcal{H}\rangle + \langle \mathcal{F}, \mathcal{K}\rangle \,\mathrm{d}x\,\mathrm{d}t \\ &- \int_\Omega \langle E(x)\,\mathcal{K}(x, T), \mathcal{H}(x, T)\rangle \,\mathrm{d}x + \int_\Omega \langle E(x)\,\mathcal{K}(x, 0), \mathcal{H}^0(x)\rangle \,\mathrm{d}x = 0 \end{aligned}$$

for any $\mathcal{K} \in F^k$.

Definition 3. $\mathcal{H}$ is a LFE-*solution* of system (1)-(4) with boundary condition Γ and $\mathcal{F}(x, t) \in L_2^{\mathrm{loc}}(Q, \boldsymbol{R}^{10})$, $\mathcal{H}^0(x) \in L_2^{\mathrm{loc}}(\Omega)$ given, iff $\mathcal{H}(t) \in F^{\mathrm{loc}}$ satisfies system (13).

Definition 4. $\mathcal{H}$ is a *weak* LFE-*solution* of system (1)-(4) with boundary condition Γ and $\mathcal{F}(x, t) \in L_2^{\mathrm{loc}}(Q, \boldsymbol{R}^{10})$, $\mathcal{H}^0(x) \in L_2^{\mathrm{loc}}(\Omega)$ given, iff $\mathcal{H}(t) \in L_2^{\mathrm{loc}}(Q, \boldsymbol{R}^{10})$ satisfies system (14) $\forall \mathcal{K} \in (F^*)^{\mathrm{vox}}$.

Following Wilcox methods [8], we give now some theorems on the existence, uniqueness and domain of dependence.

Theorem 1 (Existence and uniqueness). *Let Γ be a locally dissipative boundary condition for the system* (1)-(4) *and Γ^* its adjoint boundary condition, then in the initial-boundary problem of a plasma propagation there exists a unique:*

a. *weak* LFE-*solution* $\forall \mathcal{F} \in L_2^{\mathrm{loc}}(Q, \boldsymbol{R}^{10})$ *and* $\mathcal{H}^0 \in L_2^{\mathrm{loc}}(\Omega, \boldsymbol{R}^{10})$

b. *weak* FE-*solution* $\forall \mathcal{F} \in L_2(Q, \boldsymbol{R}^{10})$ *and* $\mathcal{H}^0 \in L_2(\Omega, \boldsymbol{R}^{10})$

c. FE-*solution* $\forall \mathcal{F} \in H^1(I, L_2(\Omega))$ *and* $\mathcal{H}^0 \in \Gamma$

d. LFE-*solution* $\forall \mathcal{F} \in H^1(I, L_2(\Omega))$ *where* K *is any arbitrary bounded set* $K \subset \boldsymbol{R}^3$ *and* $\mathcal{H}^0 \in \Gamma^{\mathrm{loc}}$.

To prove this theorem, we have to show first the

Theorem 2 (Domain of dependence). *The* LFE-*solutions* (*as well as the* FE-*solutions*) *with locally dissipative boundary conditions fulfill the following inequality*

$$\begin{aligned}&\int_{\Omega \cap S(x^0,\, a)} \langle E\mathcal{H}(T), \mathcal{H}(T)\rangle \,\mathrm{d}x \\ &\leqslant e^{kT}\Big(\int_{\Omega \cap S(x^0,\, a+cT)} \langle E\mathcal{H}(0), \mathcal{H}(0)\rangle \,\mathrm{d}x + 2\int_{Q \cap C(x^0,\, a)} e^{-kt}\langle \mathcal{F}, \mathcal{H}\rangle \,\mathrm{d}x\,\mathrm{d}t\Big)\end{aligned} \tag{15}$$

where: $S(x^0, a) = \{x : |x - x^0| \leqslant a\}$

$C(x^0, a) = \{(x, t) : |x - x^0| \leqslant a + c(T - t),\ 0 \leqslant t \leqslant T\}$.

Proof. The proof makes use of an auxiliary function $\mu(x, t)$ defined as

$$\mu(x, t) = \mu_\delta(\tau) \qquad \text{where} \qquad \tau(x, t) = a - |x - x^0| + (T - t)$$

where the real-valued function μ_δ has the following properties:

$$\begin{aligned}&\mu_\delta \in C^1(-\infty, \infty) \qquad \mu_\delta'(\tau) = \frac{\mathrm{d}\mu}{\mathrm{d}\tau} \geqslant 0 \\ &\mu_\delta(\tau) = 0 \quad \text{for } |\tau| \geqslant \delta \qquad 0 \leqslant \mu_\delta(\tau) \leqslant 1 \quad \text{for } |\tau| \leqslant \delta .\end{aligned} \tag{16}$$

According to this definition $\mu(x, t) \in C^1(Q \cap C(x^0, a))$ and $\mu(x, t) = 0$ outside $Q \cap C(x^0, a + \delta)$. We might also consider, without restrictions, that $k = 0$, so that from equation (14), assuming $\mathcal{K} = \mu\mathcal{H}$, we have

$$\int_{Q \cap C(x^0,\, a)} [\langle E \frac{\partial \mathcal{H}}{\partial t} - A\mathcal{H} - B\mathcal{H}, \mathcal{H}\rangle \mu + \langle \mathcal{F}, \mathcal{H}\rangle \mu + \langle (E \frac{\partial \mu}{\partial t} - A\mu)\mathcal{H}, \mathcal{H}\rangle]\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_{Q \cap S(x^0,\, a)} [\langle E(x)\mathcal{H}(x, T), \mathcal{H}(x, T)\rangle - \langle E(x)\mathcal{H}(x, 0), \mathcal{H}^0(x)\rangle]\mu \,\mathrm{d}x = 0 .$$

and, according to (13) we have

$$2\int_{Q\cap C(x^0,\,a)} \langle \mathcal{F}, \mathcal{H}\rangle \mu \,dx\,dt + \int_{Q\cap C(x^0,\,a)} \langle (E\frac{\partial \mu}{\partial t} - A\mu)\,\mathcal{H}, \mathcal{H}\rangle \,dx\,dt$$

$$= \int_{Q\cap S(x^0,\,a)} \langle E(x)\,\mathcal{H}(x, T), \mathcal{H}(x, T)\rangle \mu \,dx - \int_{Q\cap S(x^0,\,a)} \langle E(x)\,\mathcal{H}(x, 0), \mathcal{H}^0(x)\rangle \mu \,dx = 0\,.$$

Taking into account the definition of μ and the value of its derivatives $(\frac{\partial \mu}{\partial t} = -\mu'_\delta(\tau),\ |A\mu| = \mu'_\delta(\tau)|A\tau| = \mu'_\delta(\tau))$, it results

$$2\int_{Q\cap C(x^0,\,a)} \langle \mathcal{F}, \mathcal{H}\rangle \mu \,dx\,dt = \int_{\Omega\cap S(x^0,\,a)} \langle (E(x)\,\mathcal{H}(x, T), \mathcal{H}(x, T)\rangle \mu \,dx$$

$$- \int_{Q\cap S(x^0,\,a)} \langle E(x)\,\mathcal{H}(x, 0), \mathcal{H}^0(x)\rangle \mu \,dx - \int_{Q\cap C(x^0,\,a)} (\langle E\mathcal{H}, \mathcal{H}\rangle + \langle \mathcal{H}, \mathcal{H}\rangle)\,\mu'_\delta \,dx\,dt$$

from where, as a consequence of the definition of μ and of the properties of the last integral (≥ 0), the inequality (15) easily follows.

We now give a result for the FE-solutions.

Theorem 3. *Let $\mathcal{H}$ be a* FE-*solution with arbitrary given values* $(\mathcal{F}, \mathcal{H}^0)$, *then for any $t \in I$, we have*

$$\int_\Omega \langle E\mathcal{H}(T), \mathcal{H}(T)\rangle \,dx \leq C(\int_\Omega \langle E\mathcal{H}^0, \mathcal{H}^0\rangle dx + \int_Q \langle E^{-1}\mathcal{F}, \mathcal{F}\rangle \,dx\,dt) \tag{17}$$

where E^{-1} denotes the inverse of matrix E.

Proof. Since Theorem 2 is still valid also for the FE-solutions we can use the inequality (15) and let $a \to \infty$, so we obtain

$$\int_\Omega \langle E\mathcal{H}(T), \mathcal{H}(T)\rangle \,dx \leq e^{kT}(\int_\Omega \langle E\mathcal{H}^0, \mathcal{H}^0\rangle dx + 2\int_Q e^{-kt}\langle \mathcal{F}, \mathcal{H}\rangle \,dx\,dt)\,. \tag{18}$$

If we write $\mathsf{E}(t) = \int_0^t \int_\Omega \langle E\mathcal{H}(T), \mathcal{H}(\tau)\rangle \,dx\,d\tau$, by using the Schwarz inequality and the expression of E

$$2\int_Q e^{-kt}\langle \mathcal{F}, \mathcal{H}\rangle \,dx\,dt \leq 2(\int_Q \langle E^{-1}\mathcal{F}, \mathcal{F}\rangle \,dx\,dt)^{\frac{1}{2}}(\int_Q \langle E\mathcal{H}, \mathcal{H}\rangle \,dx\,dt)^{\frac{1}{2}}$$

$$\leq \int_Q \langle E^{-1}\mathcal{F}, \mathcal{F}\rangle \,dx\,dt + \int_Q \langle E\mathcal{H}, \mathcal{H}\rangle \,dx\,dt = \int_Q \langle E^{-1}\mathcal{F}, \mathcal{F}\rangle \,dx\,dt + \mathsf{E}(t)\,.$$

From (18), exchanging T with any arbitrary $t \in I$, there follows

$$\mathsf{E}'(t) \leq e^{kT}(\mathsf{E}'(0) + \mathsf{E}(t) + \int_0^T S(\tau)\,d\tau) \quad \text{where} \quad S(t) = \int_\Omega \langle E^{-1}\mathcal{F}, \mathcal{F}\rangle \,dx\,.$$

The above inequality can be written as

$$\mathsf{E}'(t) - k_1 \mathsf{E}(t) \leqslant k_2 = e^{kT}(\mathsf{E}'(0) + \int_0^T S(\tau)\,\mathrm{d}\tau) \qquad k_1 = e^{kT}$$

then
$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-k_1 t}\mathsf{E}(t)) \leqslant e^{-k_1 t}(\mathsf{E}'(t) - k_1 \mathsf{E}(t)) \leqslant e^{-k_1 t} k_2 .$$

After integration, with respect to t, and since $\mathsf{E}(0) = 0$, we have

$$e^{-k_1 t}\mathsf{E}(t) \leqslant \frac{1 - e^{-k_1 t}}{k_1} k_2 , \qquad k_1 \mathsf{E}(t) \leqslant (e^{k_1 t} - 1) k_2 , \qquad \mathsf{E}'(t) \leqslant k_2 e^{k_1 t}$$

which is equivalent to the inequality (17).

5 - Proof of the uniqueness

In this section the proof of the uniqueness of Theorem 1 is given. Since the initial and boundary value problem is a linear differential system, we have only to show that if $\mathcal{F} = 0$ almost everywhere in Q and $\mathcal{H}^0 = 0$ almost everywhere in Ω, then $\mathcal{H} = 0$ almost everywhere in Q.

Theorem 4. *Given the data* $\mathcal{F} = 0$ *and* $\mathcal{H}^0 = 0$, *we derive*:

a. *if* $\mathcal{H}$ *is a* LFE-*solution, then* $\mathcal{H} = 0$ *almost everywhere in* Q.

b. *if* $\mathcal{H}$ *a weak* LFE-*solution, then* $\mathcal{H} = 0$ *almost everywhere in* Q.

Proof. The proof of **a** follows from Theorem 2, since the inequality (15) holds for any LFE-solution and arbitrary positive numbers a, T. In order to obtain the proof of the second part of Theorem 4 we give first

Theorem 5. *If* $\mathcal{H}$ *is a weak* LFE-*solution of problem* (1)-(4) *with boundary condition* Γ *and given* $(\mathcal{F}, \mathcal{H}^0)$ *and if*

$$\mathcal{H}_1 = \int_0^t \mathcal{H}(\tau)\,\mathrm{d}\tau \qquad \mathcal{F}_1 = \int_0^t \mathcal{F}(\tau)\,\mathrm{d}\tau \tag{19}$$

then $\mathcal{H}_1$ *defines a* LFE-*solution with the same boundary conditions and values* $(\mathcal{F}_1(t) + E\mathcal{H}^0, 0)$ *given.*

Proof. Assuming $\tau \in I$ and $\Psi = (\Psi_a) \in (\Gamma^*)^{\mathrm{vox}}(\Omega, \boldsymbol{R}^{10})$ we define

$$\Phi(x, t) = \begin{matrix} (\tau - t)\,\Psi(x) & \quad 0 \leqslant t \leqslant \tau \\ 0 & \quad \tau \leqslant t \leqslant T . \end{matrix} \tag{20}$$

It easily follows that $\Phi \in (F^*)^{\mathrm{vox}}$ and $\Phi(T) = 0$, so that Φ can be used in equation (14), which defines the weak LFE-solutions (see Definition 2), instead of $\mathcal{H}$. Thus we obtain

$$\int_0^\tau \int_\Omega (\langle -E\Psi - (\tau - t)(A\Psi - B\Psi), \mathcal{H}\rangle + (\tau - t)\langle \mathcal{F}, \Psi\rangle)\, \mathrm{d}x\, \mathrm{d}\tau + \tau \int_\Omega \langle E\Psi, \mathcal{H}^0\rangle\, \mathrm{d}x = 0 .$$

Deriving the above with respect to τ we obtain

$$- \int_\Omega \langle E\Psi, \mathcal{H}\rangle\, \mathrm{d}x - \int_0^\tau \int_\Omega (\langle A\Psi - B\Psi, \mathcal{H}\rangle + \langle \mathcal{F}, \Psi\rangle)\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega \langle E\Psi, \mathcal{H}^0\rangle\, \mathrm{d}x = 0 \tag{21}$$

for almost any $\tau \in I$. Taking into account Fubini's Theorem and equation

$$\frac{\partial \mathcal{H}_1(\tau)}{\partial t} = \mathcal{H}(\tau) \tag{22}$$

equation (21) can be rewritten as

$$\int_\Omega (\langle A\Psi, \mathcal{H}_1\rangle - \langle E\Psi, \frac{\partial \mathcal{H}_1}{\partial t}\rangle - \langle B\psi, \mathcal{H}_1\rangle + \langle \mathcal{F}_1 \Psi\rangle - \langle E\Psi, \mathcal{H}^0\rangle)\, \mathrm{d}x = 0 \tag{23}$$

for almost any $\tau \in I$ and $\forall \Psi \in (\Gamma^*)^{\mathrm{vox}}$. Since $C_0^\infty(\Omega, \boldsymbol{R}^{10}) \subset (\Gamma^*)^{\mathrm{vox}}$, equation (23) implies the existence of $\mathcal{H}_1$ for almost any $\tau \in I$ and also

$$E\, \frac{\partial \mathcal{H}_1}{\partial t} = A\mathcal{H}_1 + B\mathcal{H}_1 + \mathcal{F}_1 + E\mathcal{H}^0 . \tag{24}$$

Thus $\mathcal{H}_1$ is a LFE-solution because it satisfies equation (24) and from

$$\mathcal{H} \in L_2^{\mathrm{loc}}(Q, \boldsymbol{R}^{10}) \qquad \mathcal{F} \in L_2^{\mathrm{loc}}(Q, \boldsymbol{R}^{10})$$

according to equation (5) and Theorem 2, it results:

$$\mathcal{H}_1 \in F^{\mathrm{loc}} \qquad \mathcal{F}_1 \in F^{\mathrm{loc}} \qquad \mathcal{H}_1(x, 0) = 0 .$$

Now we are able to show that Theorem 4 **b** is a direct consequence of Theorem 5. In fact, $(\mathcal{F}, \mathcal{H}^0) = (0, 0)$ implies $(\mathcal{F}_1 + E\mathcal{H}^0, 0) = (0, 0)$ and thus $\mathcal{H} = 0$ holds according to Theorem 4 **a**. This implies also $\mathcal{H} = \mathcal{H}_1 = 0$.

Let us now consider the linear operator $M : \mathcal{D}(A) \to \mathsf{F}$, defined as

$$M\mathcal{H} = (E\, \frac{\partial \mathcal{H}}{\partial t} - A\mathcal{H} - B\mathcal{H}, E\mathcal{H}(0))$$

where $\qquad \mathcal{D}(M) = \mathsf{F} \qquad \mathsf{F} \in L_2(Q; \boldsymbol{R}^{10})$.

Let $\mathcal{R}(M) \subset \mathsf{F}$ be the range of M; as a consequence of the uniqueness we can prove

Theorem 6. *The set* $\mathcal{R}(M)$ *is dense in* F.

Proof. We have to prove that $\overline{\mathcal{R}}(M) = \mathsf{F}$. Let us assume that $\overline{\mathcal{R}}(M) \neq \mathsf{F}$, then there exists in F a non-zero element $\mathcal{H}$, $\mathcal{H}^0$ orthogonal to $\mathcal{R}(M)$, i.e.

$$\langle E \frac{\partial \mathcal{K}}{\partial t} - A\mathcal{K} - B\mathcal{K}, \mathcal{H}\rangle + \langle E\mathcal{K}(0), \mathcal{H}^0\rangle = 0 \qquad \forall \mathcal{K} \in \mathcal{D}(M).$$

Therefore $$\int_0^T (\langle E \frac{\partial \mathcal{K}}{\partial t} - A\mathcal{K} - B\mathcal{K}, \mathcal{H}\rangle + \langle E\mathcal{K}(0), \mathcal{H}^0\rangle)\, dt = 0 \qquad \forall \mathcal{K} \in \mathcal{D}(M).$$

According to (14), it results that $\mathcal{H}$ is a weak FE-solution with $\mathcal{F} = 0$ and $\mathcal{K}(T) = 0$. In particular, by assuming $\mathcal{K} = 0$ we have $\mathcal{H}^0 = 0$ and, according to Theorem 4, it follows $\mathcal{H} = 0$ almost everywhere in $\mathcal{D}(M)$, thus $(\mathcal{H}, \mathcal{H}^0) = (0, 0)$, which is contrary to the hypothesis. Therefore $\overline{\mathcal{R}}(M) = \mathsf{F}$.

6 - Proof of the existence

In this section we begin proving the existence of the weak FE-solutions, claimed by Theorem 1. From Theorems 4, 6 it follows that the space F^* is dense in the Hilbert space Γ. A sequence $\mathcal{H}_n$ exists such that if $\mathcal{F}_n \in F^*$ and $\mathcal{F}_n \to \mathcal{F}$ in $L_2(Q; \boldsymbol{R}^{10})$, $\mathcal{F}_n$ being the sequence

$$\mathcal{F}_n = E \frac{\partial \mathcal{H}_n}{\partial t} - A\mathcal{H}_n - B\mathcal{H}_n$$

then $\mathcal{H}_n \to \mathcal{H}$.

Applying Theorem 3 to the differences $\mathcal{F}_n - \mathcal{F}_m$, $\mathcal{H}_n - \mathcal{H}_m$ we have

$$\begin{aligned} &\int_\Omega \langle E(\mathcal{H}_n(T) - \mathcal{H}_m(T)), (\mathcal{H}_n(T) - \mathcal{H}_m(T))\rangle\, dx \\ &\leq C(\int_\Omega \langle E(\mathcal{H}_n^0 - \mathcal{H}_m^0), (\mathcal{H}_n^0 - \mathcal{H}_m^0)\rangle\, dx + \int_Q \langle E^{-1}(\mathcal{F}_n - \mathcal{F}_m), (\mathcal{F}_n - \mathcal{F}_m)\rangle\, dx\, dt). \end{aligned} \tag{25}$$

Therefore, according to the Riesz-Fischer Theorem, since $\mathcal{H}_n$ is a Cauchy sequence there exists a function $\mathcal{H} \in L_2$ such that $\mathcal{H}(t) = \lim_{n \to \infty} \mathcal{H}_n(t)$ in $L_2(\Omega)$ exists, $\forall t \in I$.

Making $m \to \infty$ in equation (25), there follows the uniform convergence and then the existence of the limit in $L_2(Q; \boldsymbol{R}^{10})$ too, i.e.

$$\mathcal{H}(t) = \lim_{n \to \infty} \mathcal{H}_n(t) \quad \text{in} \quad L_2(Q; \boldsymbol{R}^{10}).$$

Thus from definition (14), written for $\mathcal{H}_n$ and $\mathcal{F}_n$, we just let $n \to \infty$, and using the convergence of the above limits we get the existence of the weak FE-solutions.

Now we prove the existence of FE-solutions. These solutions, when exist, are also weak FE-solutions, therefore a sufficient condition for the existence is

Theorem 7. *The weak* FE-*solutions* $\tilde{\mathcal{H}}$ *with data* $(\mathcal{F}, \mathcal{H}^0)$ *with* $\mathcal{F} \in L_2^{\mathrm{loc}}(Q)$ *and* $\mathcal{H}^0 \in L_2^{\mathrm{loc}}(Q)$ *are* FE-*solutions.*

Proof. If $\tilde{\mathcal{H}}$ is a weak FE-solution, then according to Theorem 5 and to the linearity of the operators E and B, the function $\mathcal{H}_1$ of equation (19) is a FE-solution with the same boundary conditions and data $[(A+B)(E^{-1}\mathcal{F}_1 + \mathcal{H}^0), 0]$. From equation (24), and taking into account equation (22) we have

$$E\tilde{\mathcal{H}} = A\mathcal{H}_1 + B\mathcal{H}_1 + (A+B)(E^{-1}\mathcal{F}_1 + \mathcal{H}^0) \tag{26}$$

and

$$E\tilde{\mathcal{H}} = A\mathcal{H}^* + B\mathcal{H}_1 + B(E^{-1}\mathcal{F}_1 + \mathcal{H}^0) \tag{27}$$

where

$$\mathcal{H}^* = \mathcal{H}_1 + E^{-1}\mathcal{F}_1 + \mathcal{H}^0 \tag{28}$$

so that equation (27) becomes

$$E\tilde{\mathcal{H}} = A\mathcal{H}^* + B\mathcal{H}^* \,. \tag{29}$$

On the other hand, deriving $\mathcal{H}^*$ with respect to t, we get

$$E\,\frac{\partial \mathcal{H}^*}{\partial t} = E\tilde{\mathcal{H}} + \mathcal{F} \tag{30}$$

and comparing with (29)

$$E\,\frac{\partial \mathcal{H}^*}{\partial t} = A\mathcal{H}^* + B\mathcal{H}^* + \mathcal{F} \qquad \mathcal{H}^*(0) = \mathcal{H}^0$$

i.e. $\mathcal{H}^*$ is a FE-solution, but for the uniqueness Theorem this solution must coincide with $\mathcal{H}$.

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Sommario

Vengono stabiliti teoremi di esistenza, unicità e dipendenza continua dai dati per un sistema lineare iperbolico, che descrive l'evoluzione di un plasma in un dominio non limitato con condizioni al contorno dissipative.

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