Riv. Mat. Univ. Parma (5) 5 (1996), 135-142

MAURIZIO CIAMPA (*)

Affine response functions for the stress in incompressible elastic points (**)

1 - Introduction

Many authors (R. L. Fosdick and J. Serrin [4], J. E. Dunn [3], F. Bampi and A. Morro [1], P. Podio-Guidugli [7]) studied and solved the problem to know whether or not in finite elasticity the response function for the stress could be an affine function of the deformation gradient. As the most important finitely deformable elastic materials—the rubberlike materials—are commonly considered *incompressible* (see, L. R. G. Treloar [8]) the above problem needs a particular attention in this case.

The results by J. E. Dunn and P. Podio-Guidugli are applicable to elastic points subject to internal constraint as defined by W. Noll in [9]—in particular incompressible elastic points. Indeed, J. E. Dunn and P. Podio-Guidugli are concerned with response functions that *satisfy the principle of objectivity* and whose domain can be an objective manifold. They prove that the *constrained affine elastic points* are those with a response function for the Cauchy stress which is a *constant* function proportional to the identity tensor, or those with a response function for the Piola-Kirchhoff stress which is a right-multiplication by a constant symmetric tensor.

Our aim is to study the problem for an *incompressible* elastic point under the *weaker* assumptions on the response function given by H. Cohen and C-C. Wang in [2] with the hope to get a *wider* class of incompressible affine elastic points.

We recall that an elastic point is *incompressible* when all its admissible configurations have the same mass density. The deformation gradients at these configurations, relative to one as a reference, are called *admissible*. As emphasized

---

(*) Dip. di Matem. Appl. U. Dini, Univ. Pisa, via Bonanno 25/B, 56126 Pisa, Italia.

(**) Received July 17, 1996. AMS classification 73 B 05.

by H. Cohen and C-C. Wang, a response function for the determinate stress is nothing but an arbitrary function that to each admissible deformation gradient associates one of the possible stress tensors in the corresponding configuration. This function is *not* required to satisfy the principle of objectivity, but this principle restricts the form such a function may assume. The principle of objectivity is indeed verified by the set of all possible stress tensors as a function of the deformation gradient.

The main result of this note is that the weakening of the assumptions on the response function does *not* widen the class of incompressible affine elastic points.

In Section 2 we give a brief summary of the basic notions of the theory of incompressible elastic points as given by H. Cohen and C-C. Wang in [2].

In Sections 3 and 4 we are concerned with incompressible elastic points admitting a response function for the determinate (Cauchy or Piola-Kirchhoff) stress which is an affine function of the deformation gradient.

In Section 3 we prove that the response function for the Cauchy stress must be a multiple of the identity tensor by a scalar *affine* function. Although this result is slightly different from the above-mentioned one by J. E. Dunn and P. Podio-Guidugli under W. Noll's assumptions, both results specify the *same* class of incompressible affine elastic points.

In Section 4 we prove that the response function for the Piola-Kirchhoff stress must be a right-multiplication by a constant symmetric tensor—i.e. the same result of J. E. Dunn and P. Podio-Guidugli under W. Noll's assumptions.

2 - Incompressible elastic points

In this section we recall the basic notions of the theory of incompressible elastic points as given by H. Cohen and C-C. Wang in [2].

Let $\mathcal{V}$ be a three-dimensional inner product vector space. The space of all linear transformations from $\mathcal{V}$ into itself is denoted by Lin and its elements are called tensors. If A and B are two tensors, their inner product $A \cdot B$ is the trace of $A^T B$. The subset of Lin of all proper rotations is denoted by Rot. The subspace of all symmetric tensors is denoted by Sym. The subspace of Sym of all real multiples of the identity tensor I is denoted by $\langle I \rangle$. The set of real numbers is denoted by $\boldsymbol{R}$.

Let X be an incompressible elastic point and let ν_* be an admissible local reference configuration of X. The deformation gradient from ν_* to a local configuration ν will be denoted by F.

The set $\mathcal{C}$ of all admissible deformation gradients is given by

$$\mathcal{C} = \{F \in \mathrm{Lin} \mid \det F = 1\}. \tag{1}$$

For each F in $\mathcal{C}$, let $\mathcal{T}(F)$ be the set of all possible Cauchy stress tensors in the configuration whose deformation gradient from ν_* is F. The principle of objectivity requires the map $\mathcal{T}$ to satisfy

$$\mathcal{T}(QF) = Q\mathcal{T}(F)\,Q^T \tag{2}$$

for every $F \in \mathcal{C}$ and $Q \in \mathrm{Rot}$ (see [2], Sect. 3).

Let $\widetilde{T}: \mathcal{C} \to \mathrm{Lin}$ be a *response function for the determinate Cauchy stress relative to* ν_*, i.e. a function such that $\widetilde{T}(F) \in \mathcal{T}(F)$. The function $\widetilde{T}$ is *not* required to satisfy the principle of objectivity.

At each F in $\mathcal{C}$ it is (see [2], Sect. 3)

$$\mathcal{T}(F) = \widetilde{T}(F) + \langle I \rangle. \tag{3}$$

Since each element of $\mathcal{T}(F)$ is symmetric, then by (3) we have

$$\widetilde{T}(F) \in \mathrm{Sym} \qquad \text{for every } F \in \mathcal{C}. \tag{4}$$

For each F in $\mathcal{C}$, let $\mathcal{S}(F) = \mathcal{T}(F)\,F^{-T}$ be the set of all possible Piola-Kirchhoff stress tensors in the configuration whose deformation gradient from ν_* is F. By (2), the principle of objectivity requires the map $\mathcal{S}$ to satisfy

$$\mathcal{S}(QF) = Q\mathcal{S}(F) \tag{5}$$

for every $F \in \mathcal{C}$ and $Q \in \mathrm{Rot}$.

Let $\widetilde{S}: \mathcal{C} \to \mathrm{Lin}$ be a *response function for the determinate Piola-Kirchhoff stress relative to* ν_*, i.e. a function such that $\widetilde{S}(F) \in \mathcal{S}(F)$. The function $\widetilde{S}$ is *not* required to satisfy the principle of objectivity.

At each F in $\mathcal{C}$, since $\mathcal{S}(F) = \widetilde{T}(F)\,F^{-T} + \langle I \rangle F^{-T}$ (see (3)), and $\widetilde{S}(F) \in \mathcal{S}(F)$, we have

$$\mathcal{S}(F) = \widetilde{S}(F) + \langle I \rangle F^{-T}. \tag{6}$$

Since $\widetilde{S}(F) \in \mathcal{T}(F)\,F^{-T}$, then

$$\widetilde{S}(F)\,F^T \in \mathrm{Sym} \qquad \text{for every } F \in \mathcal{C}. \tag{7}$$

3 - Affine response functions for the Cauchy stress

In this section we find the most general form of the affine response functions for the determinate Cauchy stress of incompressible elastic points.

Statement 1. *Let X be an incompressible elastic point and let ν_* be an admissible local reference configuration of X. Let $\widetilde{T}$ be a response function of X for the determinate Cauchy stress relative to ν_*. If $\widetilde{T}$ is the restriction to the set $\mathcal{C}$ of an affine function whose domain is* Lin, *then there exist a real number γ and a tensor Γ such that $\widetilde{T}(F) = (\gamma + \Gamma \cdot F)I$ for all $F \in \mathcal{C}$.*

Proof. For all $F \in \mathcal{C}$ and $Q \in \mathrm{Rot}$, by (2) and (3) it follows that

$$\widetilde{T}(QF) + \langle I \rangle = Q\widetilde{T}(F)Q^T + Q\langle I \rangle Q^T = Q\widetilde{T}(F)Q^T + \langle I \rangle$$

hence

$$\widetilde{T}(QF) - Q\widetilde{T}(F)Q^T \in \langle I \rangle . \tag{8}$$

For each $F \in \mathcal{C}$, let

$$\widetilde{T}(F) = \widetilde{T}_0 + \mathsf{T}(F - I) \tag{9}$$

where $\widetilde{T}_0 \in \mathrm{Lin}$ and $\mathsf{T} : \mathrm{Lin} \to \mathrm{Lin}$ is a *linear* function.

Since ν_* is an admissible local configuration, then $I \in \mathcal{C}$ and, by (9) and (4) we have $\widetilde{T}_0 = \widetilde{T}(I) \in \mathrm{Sym}$.

Both (8) for $F = I$ and (9) imply that for any $Q \in \mathrm{Rot}$ we have

$$\widetilde{T}_0 - Q\widetilde{T}_0 Q^T + \mathsf{T}(Q - I) \in \langle I \rangle .$$

This relation is equivalent to require that there exists a function $\alpha : \mathrm{Rot} \to \boldsymbol{R}$ such that for all $Q \in \mathrm{Rot}$

$$\widetilde{T}_0 - Q\widetilde{T}_0 Q^T + \mathsf{T}(Q - I) = \alpha(Q) I . \tag{10}$$

By (10) it follows

$$\widetilde{T}_0 \cdot I - Q\widetilde{T}_0 Q^T \cdot I + \mathsf{T}(Q - I) \cdot I = \mathsf{T}(Q - I) \cdot I = \alpha(Q) I \cdot I = 3\alpha(Q)$$

and so

$$\alpha(Q) = \frac{1}{3}\,[\mathsf{T}(Q - I) \cdot I] . \tag{11}$$

The map $X \to \mathsf{T}(X) \cdot I$ is defined on Lin and it is linear. Then, by the Representation Theorem for linear forms ([5], Sect. 67), there exists a tensor $\overline{\Gamma}$ such that $\mathsf{T}(X) \cdot I = \overline{\Gamma} \cdot X$ for all $X \in \mathrm{Lin}$. Defining $\Gamma = \frac{1}{3}\overline{\Gamma}$, by (11) it follows that $\alpha(Q) = \Gamma \cdot (Q - I)$ for all $Q \in \mathrm{Rot}$, and (10) may be rewritten as

$$\widetilde{T}_0 - Q\widetilde{T}_0 Q^T + \mathsf{T}(Q - I) - [\Gamma \cdot (Q - I)] I = 0 . \tag{12}$$

For any $X \in \mathrm{Lin}$, let $\mathsf{F}(X) = \mathsf{T}(X) - (\Gamma \cdot X) I$. This map is linear. By (12) we have $\widetilde{T}_0 - Q\widetilde{T}_0 Q^T + \mathsf{F}(Q - I) = 0$ for all $Q \in \mathrm{Rot}$. Then we have (see [7], Sect. 3):

a. there exists a real number $\overline{\gamma}$ such that $\widetilde{T}_0 = \overline{\gamma} I$

b. for any $Q \in \mathrm{Rot}$, $\mathsf{F}(Q - I) = 0$.

From **b** it follows that F is the null map (see [7], Sect. 2), hence $\mathsf{T}(X) = (\Gamma \cdot X) I$ for all $X \in \mathrm{Lin}$. Hence by **a** and (9), defining $\gamma = \overline{\gamma} - \Gamma \cdot I$, it follows $\widetilde{T}(F) = (\gamma + \Gamma \cdot F) I$ for any $F \in \mathcal{C}$.

4 - Affine response functions for the Piola-Kirchhoff stress

In this section we find the most general form of the affine response functions for the determinate Piola-Kirchhoff stress of incompressible elastic points.

Statement 2. *Let X be an incompressible elastic point and let ν_* be an admissible local reference configuration of X. Let $\widetilde{S}$ be a response function of X for the determinate Piola-Kirchhoff stress relative to ν_*. If $\widetilde{S}$ is the restriction to the set $\mathcal{C}$ of an affine function whose domain is* Lin, *then there exists a symmetric tensor Σ such that $\widetilde{S}(F) = F\Sigma$ for all $F \in \mathcal{C}$.*

Proof. For all $F \in \mathcal{C}$ and $Q \in \mathrm{Rot}$ by (5) and (6) it follows

$$\widetilde{S}(QF) + \langle I \rangle QF^{-T} = Q\widetilde{S}(F) + Q\langle I \rangle F^{-T}$$

hence

$$\widetilde{S}(QF) - Q\widetilde{S}(F) \in Q\langle I \rangle F^{-T} . \tag{13}$$

For any $F \in \mathcal{C}$, let

$$\widetilde{S}(F) = \widetilde{S}_0 + \mathsf{S}(F - I) \tag{14}$$

where $\widetilde{S}_0 \in \mathrm{Lin}$ and $\mathsf{S} : \mathrm{Lin} \to \mathrm{Lin}$ is a *linear* function.

Since ν_* is an admissible local configuration, then $I \in \mathcal{C}$ and, by (14) and (7) we have $\widetilde{S}_0 = \widetilde{S}(I) \in \mathrm{Sym}$.

Both (13) for $F = I$ and (14) imply that for any $Q \in \mathrm{Rot}$

$$\widetilde{S}_0 + \mathsf{S}(Q - I) - Q\widetilde{S}_0 \in Q\langle I \rangle$$

i.e.

$$\mathsf{S}(Q - I) - (Q - I)\widetilde{S}_0 \in Q\langle I \rangle . \tag{15}$$

For any $X \in \mathrm{Lin}$, let $\mathsf{L}(X) = \mathsf{S}(X) - X\widetilde{S}_0$. The map L is linear and we may rewrite (15) as $\mathsf{L}(Q - I) \in Q\langle I\rangle$. This relation is equivalent to require that there exists a function $\lambda\colon \mathrm{Rot} \to \boldsymbol{R}$ such that for all $Q \in \mathrm{Rot}$

$$\mathsf{L}(Q - I) = \lambda(Q)\,Q\,. \tag{16}$$

By the lemma we will prove in Section 5, L is the null map. Then, for any $X \in \mathrm{Lin}$ we have $\mathsf{S}(X) = X\widetilde{S}_0$. Hence by (14), defining $\Sigma = \widetilde{S}_0 \in \mathrm{Sym}$, it follows

$$\widetilde{S}(F) = \Sigma + (F - I)\Sigma = F\Sigma$$

for any $F \in \mathcal{C}$.

5 - Appendix

Now we prove the lemma used at the end of Section 4.

Lemma. *Let* $\mathsf{L} : \mathrm{Lin} \to \mathrm{Lin}$ *be a linear map, and* $\lambda\colon \mathrm{Rot} \to \boldsymbol{R}$ *be a function such that for any* $Q \in \mathrm{Rot}$ *we have*

$$\mathsf{L}(Q - I) = \lambda(Q)\,Q\,. \tag{17}$$

Then L *is the null map.*

Proof. As a first claim we prove that $\mathsf{L}(I) = 0$.

Let $\{v_1, v_2, v_3\}$ be an orthonormal basis for $\mathcal{V}$. If L is a tensor, let $[L]$ denote its matrix relative to the given basis of $\mathcal{V}$ and L_{ij} the element ij of $[L]$.

For every $i, j = 1, 2, 3$, by Representation Theorem for linear forms [5] Sect. 67, there exist $L^{ij} \in \mathrm{Lin}$ such that for all $X \in \mathrm{Lin}$ the element ij of $[\mathsf{L}(X)]$ is given by $L^{ij} \cdot X$.

Let Q_3 be the rotation through the angle θ about the axis spanned by v_3. Its matrix is

$$[Q_3] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows from (17) that

$$[\mathsf{L}(Q_3 - I)] = \lambda(Q_3)[Q_3]\,. \tag{18}$$

In particular

$$\lambda(Q_3) = L^{33}\cdot(Q_3 - I) = (L^{33}_{11} + L^{33}_{22})\cos\theta + (L^{33}_{21} - L^{33}_{12})\sin\theta - (L^{33}_{11} + L^{33}_{22}) \tag{19}$$

and $$L^{11}\cdot(Q_3 - I) = \lambda(Q_3)\cos\theta\,.$$

Then for all $\theta \in \boldsymbol{R}$ we have

$$(L^{11}_{11} + L^{11}_{22})\cos\theta + (L^{11}_{21} - L^{11}_{12})\sin\theta - (L^{11}_{11} + L^{11}_{22})$$

$$= (L^{33}_{11} + L^{33}_{22})\cos^2\theta + (L^{33}_{21} - L^{33}_{12})\sin\theta\cos\theta - (L^{33}_{11} + L^{33}_{22})\cos\theta$$

hence $$L^{33}_{11} + L^{33}_{22} = 0 \qquad L^{11}_{11} + L^{11}_{22} = 0 \qquad L^{33}_{21} - L^{33}_{12} = 0 \qquad L^{11}_{21} + L^{11}_{12} = 0\,.$$

By (19) if follows $\lambda(Q_3) = 0$ and then by (18) $L^{ij}\cdot(Q_3 - I) = 0$ i.e.

$$(L^{ij}_{11} + L^{ij}_{22})\cos\theta + (L^{ij}_{21} - L^{ij}_{12})\sin\theta - (L^{ij}_{11} + L^{ij}_{22}) = 0$$

for all $\theta \in \boldsymbol{R}$. Hence

$$L^{ij}_{11} + L^{ij}_{22} = 0 \qquad L^{ij}_{21} - L^{ij}_{12} = 0 \tag{20}$$

for $i, j = 1, 2, 3$. Considering the rotations Q_1 and Q_2 about the axis spanned by v_1 and v_2 respectively, we get:

$$L^{ij}_{11} + L^{ij}_{33} = 0 \qquad L^{ij}_{31} - L^{ij}_{13} = 0 \tag{21}$$

and

$$L^{ij}_{22} + L^{ij}_{33} = 0 \qquad L^{ij}_{32} - L^{ij}_{23} = 0\,. \tag{22}$$

Using $(20)_1$, $(21)_1$ and $(22)_1$, the element ij of $[\mathsf{L}(I)]$ is given by

$$L^{ij}\cdot I = L^{ij}_{11} + L^{ij}_{22} + L^{ij}_{33} = \frac{1}{2}\,[(L^{ij}_{11} + L^{ij}_{22}) + (L^{ij}_{22} + L^{ij}_{33}) + (L^{ij}_{11} + L^{ij}_{33})] = 0\,.$$

This proves the claim.

Since $\mathsf{L}(I) = 0$, then by (17)

$$\mathsf{L}(Q) = \lambda(Q)\,Q \qquad \text{for every } Q \in \mathrm{Rot}\,. \tag{23}$$

By (23) the function $\lambda\colon \mathrm{Rot} \to \boldsymbol{R}$ satisfies the properties:

a. λ assumes only a finite number of values, indeed $\lambda(Q)$ is an eigenvalue of L and Lin is a finite-dimensional vector space

b. λ is a continuous function, indeed $\lambda(Q) = \dfrac{1}{3}\,\mathsf{L}(Q)\cdot Q$.

Since Rot is a connected set (see [6], p. 172) then λ is a constant function. Since $\lambda(I) = \dfrac{1}{3}\,\mathsf{L}(I)\cdot I = 0$, then λ is the null function and hence L has constant value over Rot. By [7] Sect. 2, L is then the null map.

References

[1] F. BAMPI and A. MORRO, *Objective constitutive relations in elasticity and visco-elasticity*, Meccanica **17** (1982), 138-142.

[2] H. COHEN and C-C. WANG, *On the response and symmetry of elastic materials with internal constraints*, Arch. Rational Mech. Anal. **99** (1987), 1-36.

[3] J. E. DUNN, *Possibilities and impossibilities in affine constitutive theories for stress in solids* (1981), unpublished.

[4] R. L. FOSDICK and J. SERRIN, *On the impossibility of linear Cauchy and Piola-Kirchhoff constitutive theories for stress in solids*, J. Elasticity **9** (1979), 83-89.

[5] P. R. HALMOS, *Finite-dimensional vector spaces*, Van Nostrand-Reinhold, New York 1958.

[6] A. I. KOSTRIKIN and Y. I. MANIN, *Linear algebra and geometry*, Gordon and Breach, New York 1989.

[7] P. PODIO-GUIDUGLI, *The Piola-Kirchhoff stress may depend linearly on the deformation gradient*, J. Elasticity **17** (1987), 183-187.

[8] L. R. G. TRELOAR, *The physics of rubber elasticity*, Clarendon Press, Oxford 1975.

[9] C. TRUESDELL and W. NOLL, *Non-linear field theories of mechanics*, in Handbuch der Physik **3**, Springer, Berlin 1965.

Sommario

In questa nota si determina la forma più generale che la funzione risposta per lo sforzo determinato (di Cauchy o di Piola-Kirchhoff) può avere, in un materiale elastico incomprimibile, assumendo per essa una dipendenza affine dal gradiente di deformazione. Alcuni autori hanno risolto questo problema per i materiali elastici soggetti a vincolo interno come definito da Noll. Qui si risolve il problema sotto le ipotesi più deboli sulla funzione risposta, proposte da Cohen e Wang. I risultati provano che l'indebolimento delle ipotesi non porta ad una più anpia classe di materiali elastici incomprimibili affini.

* * *