Riv. Mat. Univ. Parma (5) 5 (1996), 159-168

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A variational formulation of the oblique dipole layer model for cardiac potentials (**)

1 - Introduction

The spreading of the excitation in an anisotropic bidomain model for the cardiac tissue can be macroscopically described by the evolution of the trasmembrane and extracellular potentials that are solutions of a reaction-diffusion system (RD-system), characterized by fast reaction and slow diffusion [3], [4].

During the excitation process, also called depolarization phase, the trasmembrane potential exhibits a traveling wavefront behaviour, characterized by a propagating internal layer. The so-called *excitation wavefront* is the macroscopic surface related to this layer and may be considered the median surface of the layer. Starting from the knowledge of the excitation wavefront S_t at time t, a *far-field* approximation of the extracellular potential u can be obtained, by means of the so-called oblique dipole layer model [4], [5], [6]. This macroscopic model is suitable for simulations of potentials at a distance from the cardiac electric sources, mainly localized on the excitation wavefront. The model can be described by an elliptic equation for the extracellular potential u coupled with jump conditions on S_t.

In this work we consider the Neumann elliptic problem with jump relationships representing the oblique dipole layer model. We propose a variational formulation which is suitable for a finite element approximation of the oblique dipole layer model, and we investigate existence and uniqueness of a weak solution.

Numerical simulations of extracellular potential distributions obtained from

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(**) Received September 9, 1996. AMS classification 92 C 50.

this model of cardiac sources appear in [7]. In a simplified three-dimensional model of anisotropic myocardium we computed the approximate solution by means of the Galerkin finite element method applied to the weak formulation presented in this paper. Using isoparametric elements of degree one we built an adaptive mesh with respect to the shape of the wavefront surface.

The author gratefully thanks P. Colli Franzone and G. Di Cola for pointing to this problem and for providing many suggestions during the research.

2 - The far-field model for the extracellular potential

Let $\Omega \in \mathbf{R}^3$ be an insulated three dimensional block of ventricular myocardium and let $W(\mathbf{x}, t) = 0$ be the equation of the surface of the excitation wavefront S_t propagating in the cardiac tissue. We suppose that S_t is a regular and orientable surface. Let Ω_r, $\Omega_a \subset \Omega$ be, respectively, the resting and the activated sides of S_t at time t, defined by

$$\Omega_r = \{\mathbf{x} \in \Omega \,|\, W(\mathbf{x}, t) < 0\} \qquad \Omega_a = \{\mathbf{x} \in \Omega \,|\, W(\mathbf{x}, t) > 0\}. \tag{1}$$

Let $\mathbf{n} = -\dfrac{\nabla W}{|\nabla W|}$ be the unit normal to S_t directed toward the resting region Ω_r.

For a function f regular outside S_t, we consider the jump through the surface $S_t : [f]_{S_t} = f^r - f^a$, where f^r and f^a are the traces of f on S_t taken from Ω_r and Ω_a respectively. Then it follows that the extracellular potential u satisfies the *elliptic boundary problem* and *jump conditions* [4], [5]:

$$\operatorname{div}(\mathbf{M}\nabla u) = 0 \qquad \text{in } \Omega - S_t \tag{2}$$

$$[u]_{S_t} = v_p \frac{\mathbf{n}^T \mathbf{M}_i \mathbf{n}}{\mathbf{n}^T \mathbf{M} \mathbf{n}} \tag{3}$$

$$[\mathbf{n}^T \mathbf{M} \nabla u]_{S_t} = v_p \operatorname{div}_{S_t} \boldsymbol{\omega} \tag{4}$$

$$\mathbf{n}_\Omega^T \mathbf{M} \nabla u = 0 \qquad \text{on } \partial\Omega \tag{5}$$

where: $\mathbf{M}_i(\mathbf{x})$, $\mathbf{M}_e(\mathbf{x})$ are the conductivity tensors in the intra- and extracellular media (i.e the two interpenetrating continuous conducting media composing the *bidomain* macroscopic model for the cardiac tissue). They are defined as:

$$\mathbf{M}_{i,e} = \mathbf{M}_{i,e}(\mathbf{x}) = \sigma_t^{i,e}(\mathbf{x})\,\mathbf{I} + (\sigma_l^{i,e}(\mathbf{x}) - \sigma_t^{i,e}(\mathbf{x}))\,\mathbf{a}_l(\mathbf{x})\,\mathbf{a}_l^T(\mathbf{x}) \tag{6}$$

with $\sigma_l^{i,e}(\mathbf{x})$, $\sigma_t^{i,e}(\mathbf{x})$ conductivity coefficients parallel and transverse, respectively, to the cardiac fiber direction (assuming axial symmetry around fiber), and $\mathbf{a}_l(\mathbf{x})$ unit vector tangent to the fiber passing through point $\mathbf{x}$

$\mathbf{M} = \mathbf{M}(\mathbf{x}) = \mathbf{M}_i + \mathbf{M}_e$ is the bulk conductivity tensor

v_p is the plateau value of the transmembrane potential v (and we set the resting value $v_r = 0$)

$\mathbf{n}_\Omega$ denotes the outward unit vector normal to the boundary $\partial\Omega$

$\boldsymbol{\omega}$ is the tangential vector field given by

$$\boldsymbol{\omega} = (\beta_e \mathbf{M}_i - \beta_i \mathbf{M}_e)\,\mathbf{n} = \mathbf{M}_i \mathbf{n} - \beta_i \mathbf{M}\mathbf{n} \qquad \text{with } \beta_{i,e} = \frac{\mathbf{n}^T \mathbf{M}_{i,e}\mathbf{n}}{\mathbf{n}^T \mathbf{M}\mathbf{n}} \tag{7}$$

and div_{S_t} is the surface divergence operator.

Equations (2), (3), (4) define the oblique dipole layer model suitable for far-field potential simultations, given the excitation wavefront S_t and the tensor $\mathbf{M}_i$, $\mathbf{M}_e$. By means of a perturbation analysis it has been shown in [3] that relationships (2), (3), (4) represent an approximation of zero order of an appropriately scaled form of the RD-system, in which a dimensionless small parameter appears.

Under the assumption $\mathbf{M} = \sigma_0 \mathbf{I}$ this reduced potential model was introduced by P. Colli Franzone et al. in [5], [6] and an existence and uniqueness result for the solution, up to an additive constant, was established in [2]. It was proved that the solution belongs to the Sobolev space $W^{\frac{1}{2}-\varepsilon, 2}(\Omega)$, for any $\varepsilon > 0$.

3 - Weak formulations for the oblique dipole layer problem

We now investigate existence conditions for solutions of the problem (2)-(5), which are acceptable in a physiological point of view, and we propose two equivalent weak formulations.

In order to study the problem (2)-(5) let us introduce the *regularity hypotheses*:

$\mathbf{H}_1$. Ω is an open connected bounded domain of $\mathbf{R}^3$, with Lipschitzian boundary

$\mathbf{H}_2$. For each $\mathbf{x}$, the unit vector $\mathbf{a}_l(\mathbf{x})$ tangent to the fiber passing through that point is a function $\mathbf{a}_l : \overline{\Omega} \to \mathbf{R}^3$ of class $C^1(\overline{\Omega})$

$\mathbf{H}_3$. The conductivity coefficients $\sigma_l^{i,e}(\mathbf{x})$, $\sigma_t^{i,e}(\mathbf{x}) : \overline{\Omega} \to \mathbf{R}^3$ are functions of class $C^1(\overline{\Omega})$ and the following property holds

$$0 < \lambda_{i,e} \leq \sigma_t^{i,e}(\mathbf{x}) < \sigma_l^{i,e}(\mathbf{x}) \leq \Lambda_{i,e} \qquad \forall \mathbf{x} \in \overline{\Omega} \tag{8}$$

where: $\lambda_{i,e} = \min\limits_{\overline{\Omega}} \sigma_t^{i,e}(\mathbf{x}) \qquad \Lambda_{i,e} = \max\limits_{\overline{\Omega}} \sigma_l^{i,e}(\mathbf{x})$

$\mathbf{H}_4$. For each fixed $\mathbf{x}$, $\mathbf{M}_{i,e}(\mathbf{x})$ are symmetric positive definite matrices; they have at most two eigenvalues: $\sigma_l^{i,e}(\mathbf{x}) > \sigma_t^{i,e}(\mathbf{x}) > 0$, and they are diagonalized by the same unit matrix $\mathbf{A}(\mathbf{x}) = [\mathbf{a}_1(\mathbf{x}), \mathbf{a}_2(\mathbf{x}), \mathbf{a}_3(\mathbf{x})]$, where $\mathbf{a}_1(\mathbf{x}), \mathbf{a}_2(\mathbf{x}), \mathbf{a}_3(\mathbf{x})$ are three mutually orthogonal unit vectors with $\mathbf{a}_3(\mathbf{x})$ parallel to $\mathbf{a}_l(\mathbf{x})$.

For semplicity of notations, in the sequel we will omit the dependence on $\mathbf{x}$ of all quantities, when it is not necessary. For a fixed $\hat{t}$ let $W(\mathbf{x}, \hat{t}) = 0$ be the equation of the surface of the excitation wavefront $S_{\hat{t}}$ and let us denote by Ω_a, Ω_r respectively the region behind and ahead $S_{\hat{t}}$ (see (1)).

First, we remark that a realistic model of the extracellular potential must give rise to bounded potentials. Using results of potential theory and of elliptic boundary problems (see [9], [8]) and proceeding as in [2], the boundedness of u is garanteed if $S_{\hat{t}}$ is a closed surface; otherwise, if $S_{\hat{t}}$ is an open surface, the imposition of some geometrical constraints on the wavefront boundary $\partial S_{\hat{t}}$ with respect to the fiber orientation on $\partial\Omega$ is required.

More precisely, if $S_{\hat{t}}$ is an open surface, having the rim on $\partial\Omega$, in [2] it has been noted that u is bounded if and only if

$$\boldsymbol{\omega}^T \mathbf{n}_b = 0 \qquad \text{on } \partial S_{\hat{t}} \tag{9}$$

where $\mathbf{n}_b$ is the unit vector tangent to $S_{\hat{t}}$ and perpendicular to the rim $\partial S_{\hat{t}}$ and $\boldsymbol{\omega}$ is the tangential vector on $S_{\hat{t}}$ (i.e. $\boldsymbol{\omega}^T \mathbf{n} = 0$) defined in (7).

We now discuss under what kind of geometrical hypotheses the boundedness condition (9) is verified. In a physiological point of view, we can assume that the boundary $\partial\Omega$ represents the epi-endocardium surfaces, and then it is correct to suppose that the fibers are tangent to $\partial\Omega$, that is

$$\mathbf{n}_\Omega^T \mathbf{a}_l(\mathbf{x}) = 0 \qquad \forall \mathbf{x} \in \partial\Omega. \tag{10}$$

The following theorems hold:

Theorem 1. *If $\mathbf{n}_\Omega^T \mathbf{a}_l(\mathbf{x}) = 0$ $\forall \mathbf{x} \in \partial\Omega$, then the wavefront surfaces S_t are perpendicular to $\partial\Omega$, that is $\mathbf{n}_\Omega^T \mathbf{n} = 0$.*

Proof. We notice that the wavefront surfaces S_t defined by $W(\mathbf{x}, t) = 0$ are isopotential sets with respect to the transmembrane potential v, i.e. at time t, if point $\mathbf{x}$ satisfies $W(\mathbf{x}, t) = 0$, then $v(\mathbf{x}, t) = \text{const.} = \dfrac{v_p}{2}$.

Supposing W of class $C^1(\Omega)$, we can locally define S_t as $t = t(\mathbf{x})$. Supposing moreover W strictly increasing with respect to t and differentiating $v(\mathbf{x}, t)$, for $(\mathbf{x}, t) \in S_t$ we obtain

$$\nabla_{\mathbf{x}} v + \partial_t v \cdot \left(-\frac{\nabla W}{\partial_t W}\right) = 0 .$$

Neumann boundary conditions for the potentials v, u solutions of the RD-system imply that (see [3], [4] for motivation and derivation)

$$0 = \mathbf{n}_\Omega^T \mathbf{M}_i \nabla v = -\partial_t v \, \mathbf{n}_\Omega^T \mathbf{M}_i \left(\frac{-\nabla W}{\partial_t W} \right) \qquad \text{on } \partial\Omega \cap \partial S_t$$

hence $\mathbf{n}_\Omega^T \mathbf{M}_i \nabla W = 0$ on $\partial\Omega \cap \partial S_t$.

Since ∇W is parallel to $\mathbf{n}$, where $\mathbf{n}$ is the unit vector normal to S_t, we obtain $\mathbf{n}_\Omega^T \mathbf{M}_i \mathbf{n} = 0$ on $\partial\Omega \cap \partial S_t$.

Now, if $\mathbf{n}_\Omega^T \mathbf{a}_l(\mathbf{x}) = 0 \ \forall \mathbf{x} \in \partial\Omega$, then using (6) we obtain

$$0 = \mathbf{n}_\Omega^T \mathbf{M}_i \mathbf{n} = \sigma_t^i \mathbf{n}_\Omega^T \mathbf{n} + (\sigma_l^i - \sigma_t^i)(\mathbf{n}_\Omega^T \mathbf{a}_l)(\mathbf{a}_l^T \mathbf{n}) = \sigma_t^i \mathbf{n}_\Omega^T \mathbf{n}$$

which is the thesis.

Remark 1. In a physiological point of view, the orthogonality of S_t and $\partial\Omega$ is the generic collision situation with the epi-endocardial surface.

Theorem 2. *If* $\mathbf{n}_\Omega^T \mathbf{a}_l(\mathbf{x}) = 0 \ \forall \mathbf{x} \in \partial\Omega$, *then the boundedness condition* $\boldsymbol{\omega}^T \mathbf{n}_b = 0$ *on* ∂S_t *holds.*

Proof. First, we rewrite the quantity $\boldsymbol{\omega}^T \mathbf{n}_b$ in equivalent way. We define:

$$\sigma_l = \sigma_l^i + \sigma_l^e \qquad \sigma_t = \sigma_t^i + \sigma_t^e$$

and then we have $\mathbf{M} = \sigma_t \mathbf{I} + (\sigma_l - \sigma_t)\mathbf{a}_l \mathbf{a}_l^T$.

Then, using (4) an easy calculation show that

$$M_i = \frac{\sigma_t^i}{\sigma_t} M + (\sigma_l^i - \frac{\sigma_l}{\sigma_t} \sigma_t^i)\mathbf{a}_l \mathbf{a}_l^T \tag{11}$$

hence

$$\beta_i = \frac{\mathbf{n}^T \mathbf{M}_i \mathbf{n}}{\mathbf{n}^T \mathbf{M} \mathbf{n}} = \frac{\sigma_t^i}{\sigma_t} + (\sigma_l^i - \frac{\sigma_l}{\sigma_t} \sigma_t^i) + \frac{(\mathbf{a}_l^T \mathbf{n})^2}{\mathbf{n}^T \mathbf{M} \mathbf{n}} . \tag{12}$$

Then we obtain

$$\boldsymbol{\omega} = \mathbf{M}_i \mathbf{n} - \frac{\mathbf{n}^T \mathbf{M}_i \mathbf{n}}{\mathbf{n}^T \mathbf{M} \mathbf{n}} \mathbf{M} \mathbf{n} = (\sigma_l^i - \frac{\sigma_l}{\sigma_t} \sigma_t^i)\mathbf{a}_l^T \mathbf{n} [\mathbf{a}_l - \frac{\mathbf{a}_l^T \mathbf{n}}{\mathbf{n}^T \mathbf{M} \mathbf{n}} \mathbf{M} \mathbf{n}]$$

from which:

$$\boldsymbol{\omega}^T \mathbf{n}_b = (\sigma_l^i - \frac{\sigma_l}{\sigma_t} \sigma_t^i)(\mathbf{a}_l^T \mathbf{n})[\mathbf{a}_l^T \mathbf{n}_b - \frac{\mathbf{a}_l^T \mathbf{n}}{\mathbf{n}^T \mathbf{M} \mathbf{n}} \mathbf{n}^T \mathbf{M} \mathbf{n}_b] \tag{13}$$

$$\mathbf{n}^T \mathbf{M} \mathbf{n}_b = \sigma_t \mathbf{n}^T \mathbf{n}_b + (\sigma_l - \sigma_t)(\mathbf{a}_l^T \mathbf{n})(\mathbf{a}_l^T \mathbf{n}_b) . \tag{14}$$

We remark that if $\mathbf{n}_\Omega^T \mathbf{a}_l(\mathbf{x}) = 0$, then from Theorem 1 $\mathbf{n}_\Omega^T \mathbf{n} = 0$, and since we have $\partial S_t \subset \partial\Omega$, then $\mathbf{n}_\Omega^T \boldsymbol{\tau}_b = 0$ on ∂S_t where $\boldsymbol{\tau}_b$ is a unit vector tangent to ∂S_t and oriented in such a way that the triplet $(\mathbf{n}, \mathbf{n}_b, \boldsymbol{\tau}_b)$ is orthogonal and right handed. Then we obtain

$$\mathbf{n}_\Omega^T \boldsymbol{\tau}_b = \mathbf{n}_\Omega^T \mathbf{n} = \mathbf{n}_\Omega^T \mathbf{a}_l = 0\,.$$

Hence $\mathbf{n}_\Omega$ is parallel to $\mathbf{n}_b$, and thus $\mathbf{a}_l^T \mathbf{n}_b = 0$.

Since $\mathbf{n}^T \mathbf{n}_b = 0$, from (13), (14) it is easy to see that $\mathbf{a}_l^T \mathbf{n}_b = 0$ on ∂S_t is equivalent to

$$\boldsymbol{\omega}^T \mathbf{n}_b = 0 \qquad \text{on } \partial S_t \tag{15}$$

which is the thesis.

Now we first consider the variational formulation of (2)-(5). Let $S_{\hat t}$ be the wavefront surface at fixed time $\hat t$. Multiplying each side of (2) for a test function $v \in W^{1,2}(\Omega)$ and integrating over $\Omega_a \cup \Omega_r$ we obtain

$$0 = \int_{\Omega_r} v \operatorname{div}(\mathbf{M}\nabla u)\,d\mathbf{x} + \int_{\Omega_a} v \operatorname{div}(\mathbf{M}\nabla u)\,d\mathbf{x}$$

$$= \int_{\partial\Omega_r} v(\mathbf{n}^T \mathbf{M}\nabla u)^r\,d\sigma - \int_{\partial\Omega_a} v(\mathbf{n}^T \mathbf{M}\nabla \mathrm{u})^a\,d\sigma - \int_{\Omega_r} (\nabla v)^T \mathbf{M}\nabla u\,d\mathbf{x} - \int_{\Omega_a} (\nabla v)^T \mathbf{M}\nabla u\,d\mathbf{x}\,.$$

Imposing Neumann boundary condition (5) on $\partial\Omega$ we obtain

$$\int_{S_{\hat t}} [\mathbf{n}^T \mathbf{M}\nabla u]_{S_{\hat t}} v\,d\sigma - \int_{\Omega_r} (\nabla v)^T \mathbf{M}\nabla u\,d\mathbf{x} - \int_{\Omega_a} (\nabla v)^T \mathbf{M}\nabla u\,d\mathbf{x} = 0\,.$$

Then we consider the *variational formulation* of the problem (2)-(5):

$$\begin{aligned} &\textit{Find } u \in W^{1,2}(\Omega_a) \times W^{1,2}(\Omega_r), \quad [u]_{S_{\hat t}} = v_p\,\frac{\mathbf{n}^T \mathbf{M}_i \mathbf{n}}{\mathbf{n}^T \mathbf{M}\mathbf{n}}, \ \textit{such that} \\ &\int_{\Omega_r} (\nabla u)^T \mathbf{M}\nabla v\,d\mathbf{x} + \int_{\Omega_a} (\nabla u)^T \mathbf{M}\nabla v\,d\mathbf{x} = \int_{S_{\hat t}} (v_p \operatorname{div}_{S_{\hat t}} \boldsymbol{\omega})\, v\,d\sigma \end{aligned} \tag{16}$$

for any test function $v \in W^{1,2}(\Omega)$.

For the solvability of (16) a compatibility of data is required [**10**]. More precisely the *compatibility condition*

$$\int_{S_{\hat t}} v_p \operatorname{div}_{S_{\hat t}} \boldsymbol{\omega}\,d\sigma = 0 \tag{17}$$

must be verified.

Applying the divergence theorem to the manifold $S_{\hat{t}}$ embedded in Ω, if $S_{\hat{t}}$ is a closed surface, then (17) is satisfied; otherwise, if $S_{\hat{t}}$ is open, then

$$\int_{S_{\hat{t}}} v_p \,\mathrm{div}_{S_{\hat{t}}} \boldsymbol{\omega} \,\mathrm{d}\sigma = \int_{\partial S_{\hat{t}}} \mathbf{n}_b^T \boldsymbol{\omega} \,\mathrm{d}s \tag{18}$$

and so the compatibility condition is assured under the hypoteses of Theorems 1, 2. From (16) uniqueness follows up to an additive constant.

Moreover we can consider the *equivalent formulation*:

Find $\tilde{u} \in W^{1,2}(\Omega)$ *such that*

$$\int_{\Omega} (\nabla \tilde{u})^T \mathbf{M} \nabla v \,\mathrm{d}\mathbf{x} = \int_{S_{\hat{t}}} (v_p \,\mathrm{div}_{S_{\hat{t}}} \boldsymbol{\omega}) \, v \,\mathrm{d}\sigma - \int_{\Omega_r} (\nabla \tilde{\alpha})^T \mathbf{M} \nabla v \,\mathrm{d}\mathbf{x} \tag{19}$$

for any test function $v \in W^{1,2}(\Omega)$, *where* $\tilde{\alpha} \in W^{1,2}(\Omega_r)$ *and* $\tilde{\alpha} = v_p \dfrac{\mathbf{n}^T \mathbf{M}_i \mathbf{n}}{\mathbf{n}^T \mathbf{M} \mathbf{n}}$ *on* $\partial \Omega_r$.

An easy calculation shows that, up to an additive constant, we have $u = \tilde{u}$ in Ω_a and $u = \tilde{u} + \tilde{\alpha}$ in Ω_r and we show that u is independent of the kind of prolongation of α in Ω_r in the following theorem

Theorem 3. *Let* $\tilde{\alpha}_1, \tilde{\alpha}_2 \in W^{1,2}(\Omega_r)$ *be two prolongations of* $\alpha = v_p \dfrac{\mathbf{n}^T \mathbf{M}_i \mathbf{n}}{\mathbf{n}^T \mathbf{M} \mathbf{n}}$ *in* Ω_r; *let* $\tilde{u}_1, \tilde{u}_2$ *be the solutions of* (19) *related to* $\tilde{\alpha}_1, \tilde{\alpha}_2$ *respectively and*

$$u_h = \tilde{u}_h \;\; in \; \Omega_a \qquad u_h = \tilde{u}_h + \tilde{\alpha}_h \;\; in \; \Omega_r \qquad h = 1, 2 .$$

Then, up to an additive constant, we have $u_1 = u_2$ *in* Ω.

Proof. We consider the function $z = u_1 - u_2$ defined as

$$z = z_a = \tilde{u}_1 - \tilde{u}_2 \quad \text{in } \Omega_a , \qquad z = z_r = (\tilde{u}_1 - \tilde{u}_2) + (\tilde{\alpha}_1 - \tilde{\alpha}_2) \quad \text{in } \Omega_r$$

and define the following bilinear forms:

$$a(u, v) = \int_{\Omega} (\nabla v)^T \mathbf{M} \nabla u \,\mathrm{d}\mathbf{x} \tag{20}$$

$$a_r(u, v) = \int_{\Omega_r} (\nabla v)^T \mathbf{M} \nabla u \,\mathrm{d}\mathbf{x} \qquad a_a(u, v) = \int_{\Omega_a} (\nabla v)^T \mathbf{M} \nabla u \,\mathrm{d}\mathbf{x} . \tag{21}$$

From the equivalent weak formulation (19), written for u_1 and u_2, by subtracting we obtain:

$$a_r(z_r, v) + a_a(z_a, v) = 0 \qquad \forall v \in W^{1,2}(\Omega) .$$

We notice that $z_r \in W^{1,2}(\Omega_r)$, $z_a \in W^{1,2}(\Omega_a)$ and, since on $S_{\hat{t}}$ we have $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \alpha$, then:

$$z_r \,|_{S_{\hat{t}}} = (\tilde{u}_1 - \tilde{u}_2) \,|_{S_{\hat{t}}} = z_a \,|_{S_{\hat{t}}}$$

hence $z \in W^{1,2}(\Omega)$, and moreover $a(z, v) = 0$ for any $v \in W^{1,2}(\Omega)$. Setting $v = z$, it follows that: $(\nabla z)^T \mathbf{M} \nabla z = 0$ almost everywhere on Ω and since $\mathbf{M}$ is not singular, then $z =$ constant on Ω.

4 - Existence of weak solutions

Now we prove the existence of a solution of the variational problem (19), up to an additive constant; this will follow from the Lax-Milgram theorem [**10**].

Setting $V = W^{1,2}(\Omega)$, let $F: V \to \mathbf{R}$ be defined as

$$F(v) = \int_{S_{\tilde{l}}} (v_p \operatorname{div}_{S_{\tilde{l}}} \boldsymbol{\omega}) v \, d\sigma - a_r(\tilde{\alpha}, v) \tag{22}$$

with a_r given by (21), $\tilde{\alpha}$ prolongation of α in $W^{1,2}(\Omega_r)$. Then (19) can be rewritten as

$$\textit{Find } u \in V \textit{ such that } a(u, v) = F(v) \qquad \textit{for any } v \textit{ of } V. \tag{23}$$

The form $a: V \times V \to \mathbf{R}$ is bilinear and is moreover continuous. In fact we have:

$$|a(u, v)| \leq \|\nabla v\|_{L^2(\Omega)} \, \|\mathbf{M}(\mathbf{x}) \nabla u\|_{L^2(\Omega)}$$

$$\Big(\int_\Omega |\mathbf{M}(\mathbf{x}) \nabla u|^2 \, d\mathbf{x}\Big)^{\frac{1}{2}} \leq \Big(\int_\Omega \|\mathbf{M}(\mathbf{x})\|^2 \, |\nabla u|^2 \, d\mathbf{x}\Big)^{\frac{1}{2}} \leq \max_{\mathbf{x} \in \bar{\Omega}} \|\mathbf{M}(\mathbf{x})\| \, \|\nabla u\|_{L^2(\Omega)}$$

where $\|\mathbf{M}(\mathbf{x})\|$ is the euclidean norm of the matrix $\mathbf{M}(\mathbf{x})$. From hypotesis $\mathbf{H}_1$ we have $\|\mathbf{M}(\mathbf{x})\| = \sigma_l^i(\mathbf{x}) + \sigma_l^e(\mathbf{x})$ and setting $\lambda_{\max} = \max_{\mathbf{x} \in \bar{\Omega}} (\sigma_l^i(\mathbf{x}) + \sigma_l^e(\mathbf{x}))$, we obtain

$$|a(u, v)| \leq \lambda_{\max} \|\nabla v\|_{L^2(\Omega)} \, \|\nabla u\|_{L^2(\Omega)} \leq \lambda_{\max} \|\nabla v\|_V \, \|\nabla u\|_V .$$

The functional $F(v)$ given by (22) is linear and also bounded. In fact, setting $\beta = v_p \operatorname{div}_{S_{\tilde{l}}} \boldsymbol{\omega}$ we have

$$|F(v)| = \Big|\int_{S_{\tilde{l}}} \beta v \, d\sigma - \int_{\Omega_r} (\nabla v)^T \mathbf{M} \nabla \tilde{\alpha} \, d\mathbf{x}\Big| \leq \|\beta\|_{L^2(S_{\tilde{l}})} \|v\|_{L^2(S_{\tilde{l}})} + \lambda_{\max} \|\tilde{\alpha}\|_{W^{1,2}(\Omega_r)} \|v\|_{W^{1,2}(\Omega_r)} .$$

From the trace theorem (see, for example, [**10**]) it follows $\|v\|_{L^2(S_{\tilde{l}})} \leq C \|v\|_{W^{1,2}(\Omega)}$ hence

$$|F(v)| \leq C(\|\beta\|_{L^2(S_{\tilde{l}})} + \lambda_{\max} \|\tilde{\alpha}\|_{W^{1,2}(\Omega)}) \, \|v\|_V .$$

We notice that the bilinear form $a(u, v)$ is not V-elliptic. Indeed for any constant $c \in V$, $c \neq 0$ we have $a(c, c) = 0$ while $\|c\|_V > 0$.

Then we define

$$Q = \{q \in V | \int_\Omega q \, d\mathbf{x} = 0\} . \tag{24}$$

This set is a closed subspace of the Hilbert space V and is the orthogonal complement (in L^2-norm) of the real constant function space P. From hypothesis $\mathbf{H}_1$, since Ω is an open connected bounded domain with Lipschitzian boundary, then the Poincaré-Wirtinger inequality holds (see [1])

$$\|u-\bar{u}\|_{L^2} \leqslant C\|\nabla u\|_{L^2} \qquad \forall u \in W^{1,2}(\Omega) \tag{25}$$

where $\bar{u} = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega u \,\mathrm{d}\mathbf{x}$. Hence it follows

$$\|u\|_{L^2(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in Q\,.$$

Now we can show that $a(u, v)$ is Q-elliptic. From hypothesis $\mathbf{H}_1$ it follows that the minimum eigenvalue of matrix $\mathbf{M}(\mathbf{x})$ is $\sigma_t^i(\mathbf{x}) + \sigma_t^e(\mathbf{x})$.

Setting $\lambda_{\min} = \min\limits_{\mathbf{x}\in\bar{\Omega}} \sigma_t^i(\mathbf{x}) + \sigma_t^e(\mathbf{x})$, since $\mathbf{M}(\mathbf{x})$ is symmetric then

$$a(q, q) = \int_\Omega (\nabla q)^T \mathbf{M}(\mathbf{x}) \nabla q \,\mathrm{d}\mathbf{x} \geqslant \lambda_{\min} \int_\Omega |\nabla q|^2 \,\mathrm{d}\mathbf{x}\,.$$

By (25) we obtain

$$a(q, q) \geqslant \frac{\lambda_{\min}}{2} \|\nabla q\|_{L^2}^2 + \frac{\lambda_{\min}}{2C} \|q\|_{L^2}^2 \geqslant \gamma \|q\|_Q^2$$

with $\gamma = \min\{\lambda_{\min}/2, \lambda_{\min}/(2C)\}$, $\|\cdot\|_Q$ being the restriction of the V-norm to the subspace Q. Hence we can conclude that the bilinear bounded form $a(u, v)$ is Q-elliptic.

From the Lax-Milgram theorem the existence and the uniqueness in Q of the solution of (23) can be deduced if only if we have $F(c) = 0$ for any $c \in P$ [**11**]. More explicitly:

$$\int_{S_{\tilde{t}}} (v_p \operatorname{div}_{S_{\tilde{t}}} \boldsymbol{\omega})\, c \,\mathrm{d}\sigma - a_r(\tilde{\alpha}, c) = 0 \qquad \forall \text{ constant } c \in P.$$

The second term is clearly null if c is a constant, so the above condition is verified if and only if the compatibility condition (17) holds. Then the geometrical assumptions stated in Theorems 1, 2 ensure the existence of a unique solution in Q. We are now in position to state the result proved above.

Theorem 4. *Under hypotheses* $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4$, *if* $S_{\tilde{t}}$ *is a closed surface or if the hypotheses of Theorems* 1, 2 *hold, then there exists a unique solution* $\tilde{u} \in Q$, *with* Q *given by* (24), *of the weak problem* $a(\tilde{u}, v) = F(v)$ *for any* $v \in W^{1,2}(\Omega)$, *with* $a(\cdot,\cdot)$ *and* $F(\cdot)$ *given by* (20), (22).

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Sommario

In questo lavoro si considera il modello dello strato di dipoli obliquo per i potenziali cardiaci extracellulari. Le soluzioni risolvono un problema di Neumann ellittico e relazioni di salto sulla superficie del fronte di eccitazione. Di questo problema si propone una formulazione variazionale idonea all'approssimazione di Galerkin mediante elementi finiti e si studiano l'esistenza e l'unicità delle soluzioni deboli.

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