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**On the connections between kinetic theory
and a statistical model for the distribution of dominance
in populations of social organisms (**)**

1 - Introduction

It is well known that the idea of *dominance* plays an important role in studies of animal behavior. On the other hand the concept of dominance can be introduced into any population of anonymous organisms, as an attribute possessed by each individual which characterizes its social life, and may undergo variation due to *encounters* (interactions) between the individuals themselves. *Individuals* are to be understood in the broadest sense (animals, persons, communities, states, ...) and the attribute might refer to any kind of skill, strenght, or endowment (physical, cultural, economical, technological, ...). Moreover, the establishment of a dominance distribution in a *population* is analogous to the establishment of ratings in competitive events such as chess, tennis, and golf. The interested reader is referred to [5] and to the bibliography therein.

In order to study the dynamics of dominance, the dominance itself is usually associated with a single nonnegative real number x , ranging, without loss of generality, in the closed interval $[0, 1]$.

In the simplest situation, assuming suitable *reasonable* rules governing the development of dominance and its variation due to encounters, Jäger and Segel provided a stochastic model for the time evolution of the *distribution function* $f(x, t)$, where $f(x, t) dx$ is the number of individuals that at time t have dominance values in the interval $(x, x + dx)$. In the frame of the theory of Markov

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processes, by use of the Chapman-Kolmogoroff equation, they derived the *basic equation for the evolution of dominance* [5]

$$(1) \quad \frac{\partial f}{\partial t} = -f(x, t) \int_0^1 \eta(x, y) f(y, t) dy + \int_0^1 \int_0^1 \eta(x', y') \psi(x', y'; x) f(x', t) f(y', t) dx' dy' \equiv I[f, f]$$

where $\eta(x, y) = \eta(y, x) \geq 0$ denotes the microscopic *collision* frequency between individuals with dominance x and y , and $\psi(x', y'; x) \geq 0$ is the probability density that, after an (x', y') encounter, the x' individual ends up with dominance in the interval $(x, x + dx)$, with the obvious normalization

$$(2) \quad \int_0^1 \psi(x', y'; x) dx = 1 \quad \forall x', y' \in [0, 1].$$

The hypotheses underlying equation (1) are of course satisfactory for those societies where correlations between individuals are weak enough, and indeed the model finds a very good verification for primitive communities like insects or similar. In any case, *such hypotheses are much the same as for the derivation of the nonlinear Boltzmann equation of gas kinetic theory.*

Some kinetic like applications in this respect are available in the literature [2], and also immunology problems have been dealt with in such a spirit [1]. However, it is worth considering concepts which are typical of kinetic theory, but are usually not considered in equation (1), like conservation of dominance and microscopic reversibility in each encounter.

The first applies whenever dominance is neither destroyed nor created during interactions, and implies conservation of the total amount of dominance in the population.

The second refers to the situation when the *collision* $(x', y') \rightarrow (x, y)$ and its inverse $(x, y) \rightarrow (x', y')$ are equally likely, as it occurs to the standard particle collisions according to classical mechanics.

Indeed, discussion of these concepts and their consequences is just the main topics of the present paper. It is shown in particular that, as expected, an *H*-theorem can be proved, *Maxwellians* are explicitly found, and trend to equilibrium is established. Possible generalizations are also discussed and commented on.

Under hypothesis of continuity of η on $[0, 1] \times [0, 1]$, and of measurability of ψ on $[0, 1] \times [0, 1] \times [0, 1]$, existence and uniqueness of a global L_1 valued continuous positive mild solution to the Cauchy problem associated to equation (1), with positive L_1 initial datum, has been proved in [3]. Such a solution is also C^1 when ψ is continuous.

We define *total number of individuals* ϱ and *average dominance per individual* u as

$$(3) \quad \varrho = \int_0^1 f(x, t) dx \quad u = \frac{1}{\varrho} \int_0^1 xf(x, t) dx$$

and notice that condition (2) immediately implies $\varrho = \text{constant}$ (the size of the population is conserved). In addition, there results in general that $\psi(x', y'; x) \neq \psi(y', x'; x)$, but the unique solution above is also unique solution of a different version of equation (1), where only the probability distribution ψ is replaced by the symmetrized form

$$(4) \quad \psi^*(x', y'; x) = \frac{1}{2} [\psi(x', y'; x) + \psi(y', x'; x)].$$

If necessary, it is thus not restrictive considering ψ as symmetric with respect to its first two arguments.

2 - Dominance conservation and microreversibility in an encounter

If the dominance of the interacting partners has to be the same before and after the encounter, bearing in mind the meaning of the fundamental parameters η and ψ , it is spontaneous to define *conservation of dominance* by

$$(5) \quad \eta(x', y') \psi(x', y'; x) = \eta(y', x') \psi(y', x'; x' + y' - x)$$

namely the final dominance attained by the y' -individual is $x' + y' - x$, if x is the dominance of x' after the encounter. Since both final levels must belong to $[0, 1]$, the support of ψ must be restricted, since, according to whether $x' + y' \leq 1$ or $x' + y' \geq 1$, there are dominance intervals not admissible starting from x' . More precisely $\psi(x', y'; x) = 0$ for $x < \max(0, x' + y' - 1)$ or $x > \min(x' + y', 1)$. Since η is symmetric, dominance conservation corresponds to

$$(6) \quad \psi(x', y'; x) = \psi(y', x'; x' + y' - x) \quad \forall (x', y'; x) \in \mathcal{D}$$

where the domain \mathcal{D} is defined by

$$0 \leq x' \quad y' \leq 1 \quad \max(0, x' + y' - 1) \leq x \leq \min(x' + y', 1).$$

Upon defining the *average dominance of x' after interaction with y'*

$$(7) \quad g(x', y') = \int_0^1 x \psi(x', y'; x) dx$$

subject thus to the constraint $\max(0, x' + y' - 1) \leq g(x', y') \leq \min(x' + y', 1)$, and resorting to equation (2), it is easy to prove that (6) implies

$$(8) \quad g(x', y') + g(y', x') = x' + y'$$

of clear physical meaning (if the symmetrized probability distribution (4) were

used, one would get correspondingly $g^*(x', y') = \frac{x' + y'}{2}$. Equation (8) is sufficient to guarantee conservation of the global dominance in the population.

Multiplying equation (1) by x and integrating over x yields in fact, after simple manipulations

$$(9) \quad \frac{du}{dt} = \frac{1}{2\varrho} \int_0^1 \int_0^1 [g(x', y') + g(y', x') - x' - y'] \cdot \eta(x', y') f(x', t) f(y', t) dx' dy' = 0$$

so that u is, like ϱ , a conserved quantity.

In the Boltzmann equation, reversibility of the equations of mechanics entails microreversibility in the particle collision. In the present context the equivalent formulation is that, if an encounter $(x', y') \rightarrow (x, x' + y' - x)$ occurs with a given probability, the *inverse* encounter $(x, x' + y' - x) \rightarrow (x', y')$ must be endowed with exactly the same probability. We are thus led to the following definition of *microscopic reversibility*

$$(10) \quad \eta(x', y') \psi(x', y'; x) = \eta(x, x' + y' - x) \psi(x, x' + y' - x; x')$$

for any $\forall(x', y', x) \in \mathcal{O}$.

If this condition is used into (1), it is not difficult to show that the collision term I can be recast in a kind of kinetic form

$$(11) \quad \begin{aligned} I[f, f] &= - \int_0^1 \int_0^1 \eta(x, y) \psi(x, y; x') f(x) f(y) dx' dy \\ &+ \int_0^1 \int_0^1 \eta(x', y') \psi(x', y'; x) f(x') f(y') dx' dy' \\ &= \int_0^1 \int_0^1 \eta(x', y') \psi(x', y'; x) [f(x') f(y') - f(x) f(x' + y' - x)] dx' dy'. \end{aligned}$$

Appropriate additional algebra leads also to

$$(12) \quad \begin{aligned} &\int_0^1 \varphi(x) I[f, f] dx \\ &= \frac{1}{2} \int_{\mathcal{O}} [\varphi(x) + \varphi(x' + y' - x) - \varphi(x') - \varphi(y')] \\ &\quad \cdot \eta(x', y') \psi(x', y'; x) f(x') f(y') dx' dy' dx \\ &= - \frac{1}{4} \int_{\mathcal{O}} [\varphi(x) + \varphi(x' + y' - x) - \varphi(x') - \varphi(y')] \\ &\quad \cdot [f(x) f(x' + y' - x) - f(x') f(y')] \eta(x', y') \psi(x', y'; x) dx' dy' dx \end{aligned}$$

for the production of any property φ , function of x , due to encounters.

Equation (12) implies that a function φ satisfying

$$(13) \quad \varphi(x) + \varphi(x' + y' - x) = \varphi(x') + \varphi(y') \quad \forall(x', y'; x) \in \mathcal{D}$$

is necessarily a *collision invariant*, i.e. there is no net production of φ in an encounter, a property which has been established already for $\varphi = 1$ and $\varphi = x$, which evidently fulfil equation (13).

It is possible now to show by standard methods that, if φ is restricted to the class of the continuous functions on $[0, 1]$, all solutions of (13) are just linear combinations of 1 and x . In fact, equation (13) must be in order also for $x' + y' \leq 1$ and $x = 0$. Since φ is also bounded, for $\phi(x) = \exp[\varphi(x)]$ ($\exp[\varphi(0)]$)⁻¹ we get

$$(14) \quad \phi(x' + y') = \phi(x') \phi(y')$$

with $\phi(x) > 0$ for any x . Setting $\phi(1) = e^\lambda$ and $\Phi(x) = e^{-\lambda x} \phi(x)$ yields

$$(15) \quad \Phi(x' + y') = \Phi(x') \Phi(y') \quad \Phi(1) = 1$$

from which $\Phi(x) = 1$ for any rational x in the interval $[0, 1]$, and then $\Phi(x) = 1$ identically. The only continuous solutions to (14) are $\phi(x) = e^{\lambda x}$, with λ arbitrary real number, which all satisfy (13) with arbitrary $\varphi(0)$. Since relaxing the conditions $x' + y' \leq 1$, $x = 0$ can only restrict the class of solutions, the continuous solutions to (13) constitute just the two parameter family $\mu + \lambda x$, with $\mu, \lambda \in \mathbf{R}$.

Moreover, if f is smooth enough to have the involved integrals make sense (in particular, if f is positive), equation (12) gives

$$(16) \quad \int_0^1 (\log f) I[f, f] dx = -\frac{1}{4} \int_{\omega} \log \frac{f(x)f(x' + y' - x)}{f(x')f(y')} \\ \cdot [f(x)f(x' + y' - x) - f(x')f(y')] \cdot \eta(x', y') \psi(x', y'; x) dx' dy' dx \leq 0$$

i.e. the Boltzmann inequality relevant to the present context.

If additionally the functions η and ψ are assumed, as usual, to be almost everywhere positive, then the equal sign is in order, if and only if $\log f$ satisfies condition (13). The same requirement is also sufficient for $I[f, f]$ to vanish, in force of equation (11).

3 - Equilibrium states and stability

If, in the class of admissible solutions to equation (1), we introduce the *functional*

$$(17) \quad H[f] = \int_0^1 f \log f dx$$

then, following a solution of (1) itself, standard algebra leads to the *H-theorem*

$$(18) \quad \dot{H}[f] = \int_0^1 \log f I[f, f] dx \leq 0$$

by which H is a good candidate to prove dissipativity and to provide a Liapunov functional.

In particular, there follows that equilibrium distribution functions, defined by $I[f, f] = 0$, are only those functions for which $\log f$ satisfies equation (13), and this constitutes the present version of the *detailed balance principle*. In the class of the continuous functions, the only equilibrium distributions are thus the ∞^2 family of *Maxwellians* [4], exponential functions of dominance

$$(19) \quad f_M(x) = a \exp(bx),$$

with $a > 0$ and b real, but otherwise arbitrary. Then \dot{H} vanishes at the equilibria, and is negative at any other f .

The constants a and b are in a one to one relationship with the two conserved quantities q and u . Using in fact (19) for f in (3) yields easily

$$(20) \quad a = \frac{b}{e^b - 1} q \quad u = \frac{e^{-b} - 1 + b}{b(1 - e^{-b})} = F(b)$$

which are valid also in the limit $b \rightarrow 0$. The second is a transcendental equation for the exponent b , where

$$(21) \quad F'(b) = \frac{\cosh b - 1 - \frac{b^2}{2}}{2b^2 \sinh^2 \frac{b}{2}} > 0 \quad \text{for any } b$$

so that F is monotonically increasing versus b , ranging between its asymptotic limits 0 at $-\infty$ and 1 at $+\infty$, with $F(0) = \frac{1}{2}$. Therefore, the unique root b of equation (20) is positive or negative according to whether the average dominance per individual is greater or less than $\frac{1}{2}$, producing an increasing or decreasing function of x as *Maxwellian*. For $u = \frac{1}{2}$, individuals are equally distributed over all values of dominance.

In order to guarantee relaxation to the equilibrium characterized by the same values of q and u of any initial distribution, one has also to prove that H is actually a *Liapunov functional*, namely that it attains its minimum at that equilibrium for all admissible distribution functions which share the same values of the total amounts of population and dominance.

To this end we evaluate $H[f]$ for f in such a class, and set $h = (f - f_M)f_M^{-1}$, with $h > -1$. Owing to the fact that

$$(22) \quad \int_0^1 h f_M dx = 0 \quad \int_0^1 x h f_M dx = 0$$

it is not difficult to obtain

$$(23) \quad H[f] - H[f_M] = \int_0^1 \{ [1 + h(x)] \log [1 + h(x)] - h(x) \} f_M(x) dx.$$

Analogously to equation (16), the content of the curly brackets is a positive definite function of h for $-1 \leq h < +\infty$, vanishing at $h = 0$ and positive elsewhere. There follows that the right hand side of (23) is equal to zero for $f = f_M$, and is positive for any other f in the considered class.

The required minimum property is so established.

4 - Possible generalizations

The previous theory can be extended in several directions. The most immediate one concerns the possibility of a multi-species society, in which individuals of each species may interact, according to the previous scheme, with individuals of any other, equal or not, species.

The evolution equation for the distribution vector $f = \{f_i, i = 1, 2, \dots, N\}$ would read as

$$(24) \quad \frac{\partial f_i}{\partial t} = -f_i(x, t) \sum_{j=1}^N \int_0^1 \eta_{ij}(x, y) f_j(y, t) dy \\ + \sum_{j=1}^N \int_0^1 \int_0^1 \eta_{ij}(x', y') \psi_{ij}(x', y'; x) f_i(x', t) f_j(y', t) dx' dy' \equiv I_i[f, f]$$

with

$$(25) \quad \eta_{ij}(x', y') = \eta_{ji}(y', x') \quad \int_0^1 \psi_{ij}(x', y'; x) dx = 1.$$

One may introduce now

$$(26) \quad \varrho_i = \int_0^1 f_i(x, t) dx \quad \varrho = \sum_{i=1}^N \varrho_i \quad u_i = \frac{1}{\varrho_i} \int_0^1 x f_i(x, t) dx \quad u = \frac{1}{\varrho} \sum_{i=1}^N \varrho_i u_i$$

where all ϱ_i are constant because of (25). The symmetrization (3) is not permissible here.

Conservation of dominance can be introduced by imposing that, if x' undergoes a transition to x , the transition $y' \rightarrow x' + y' - x$ must take place for its partner. There follows that each ψ_{ij} must have the same support \mathcal{O} as for single species, and that

$$(27) \quad \psi_{ij}(x', y'; x) = \psi_{ji}(y', x'; x' + y' - x) \quad \forall (x', y'; x) \in \mathcal{O}, \quad \forall i, j.$$

This implies

$$(28) \quad g_{ij}(x', y') + g_{ji}(y', x') = x' + y' \quad g_{ij}(x', y') = \int_0^1 x \psi_{ij}(x', y'; x) dx$$

for any pair i, j and any dominance x', y' . By integration of (24) one gets

$$(29) \quad \frac{du_i}{dt} = \frac{1}{\varrho_i} \sum_{j=1}^N \int_0^1 \int_0^1 [g_{ij}(x', y') - x'] \eta_{ij}(x', y') f_i(x', t) f_j(y', t) dx' dy'$$

which is nonzero in general. The average dominance over all species is instead conserved under the only assumption that equation (28) holds, since

$$(30) \quad \frac{du}{dt} = \frac{1}{2Q} \sum_{i=1}^N \sum_{j=1}^N \int_0^1 \int_0^1 [g_{ij}(x', y') + g_{ji}(y', x') - x' - y'] \cdot \eta_{ij}(x', y') f_i(x', t) f_j(y', t) dx' dy' = 0.$$

Under the additional hypothesis of microreversibility

$$(31) \quad \psi_{ij}(x', y') \eta_{ij}(x', y'; x) = \eta_{ij}(x, x' + y' - x) \psi_{ij}(x, x' + y' - x; x')$$

for any $(x', y', x) \in \mathcal{O}$ and any i, j one has

$$(32) \quad \begin{aligned} & \sum_{i=1}^N \int_0^1 w_i(x) I_i[\mathbf{f}, \mathbf{f}] dx \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \int_{\mathcal{O}} [w_i(x) + w_j(x' + y' - x) - w_i(x') - w_j(y')] \\ & \quad \cdot \eta_{ij}(x', y') \psi_{ij}(x', y'; x) f_i(x') f_j(y') dx' dy' dx \\ &= -\frac{1}{4} \sum_{i=1}^N \sum_{j=1}^N \int_{\mathcal{O}} [w_i(x) + w_j(x' + y' - x) - w_i(x') - w_j(y')] \\ & \quad \cdot [f_i(x) f_j(x' + y' - x) - f_i(x') f_j(y')] \eta_{ij}(x', y') \psi_{ij}(x', y'; x) dx' dy' dx \end{aligned}$$

from which all vectors $\mathbf{w} = \{w_i(x), i = 1, \dots, N\}$ such that

$$(33) \quad w_i(x) + w_j(x' + y' - x) = w_i(x') + w_j(y') \quad \forall (x', y'; x) \in \mathcal{O} \quad \forall i, j$$

are collisional invariants.

Futhermore, upon defining

$$(34) \quad H[\mathbf{f}] = \sum_{i=1}^N \int_0^1 f_i \log f_i dx$$

one gets, after some algebra

$$(35) \quad \begin{aligned} \dot{H}[\mathbf{f}] &= \sum_{i=1}^N \int_0^1 \log f_i I_i[\mathbf{f}, \mathbf{f}] dx = -\frac{1}{4} \sum_{i=1}^N \sum_{j=1}^N \int_{\omega} \log \frac{f_i(x)f_j(x'+y'-x)}{f_i(x')f_j(y')} \\ &\cdot [f_i(x)f_j(x'+y'-x) - f_i(x')f_j(y')] \eta_{ij}(x', y') \psi_{ij}(x', y'; x) dx' dy' dx \leq 0. \end{aligned}$$

Skipping all details, it is possible to prove that the equilibrium distribution vectors \mathbf{f} , defined by $I[\mathbf{f}, \mathbf{f}] = 0$, coincide with the solutions of the equation $\dot{H}[\mathbf{f}] = 0$, and are those for which the vector $\{\log f_i\}$ satisfies (33).

Again, in the class of the continuous functions, one ends up with the *Maxwellians*

$$(36) \quad f_{iM}(x) = a_i \exp(bx)$$

where the exponent b must be the same for all species, as imposed by (33) itself. The $N + 1$ free constants a_i and b are related one-to-one to the $N + 1$ first integrals Q_i and u

$$(37) \quad a_i = \frac{b}{e^b - 1} Q_i \quad u = \frac{e^{-b} - 1 + b}{b(1 - e^{-b})} = F(b)$$

where the transcendental equation for b is the same as for (20). In particular, the average dominance u_i in the general species is, at equilibrium, the same for all species and equal to the overall average value u .

In this multispecies frame it is possible to model also the presence of a *background species* ($i = 0$), taking part in the interactions but not being appreciably affected by any variation due to encounters. One could even consider the linear problem of a single population embedded in a much larger host population and deal with the equivalent of the linear (neutron) Boltzmann equation

$$(38) \quad \frac{\partial f}{\partial t} = -\nu(x)f(x, t) + \int_0^1 \nu(x')k(x', x)f(x', t) dx' \equiv I[f]$$

with

$$\nu(x) = \int_0^1 \eta_{10}(x, y)f_0(y) dy \quad \nu(x')k(x', x) = \int_0^1 \eta_{10}(x', y')\psi_{10}(x', y'; x)f_0(y') dy'$$

$$(39) \quad \int_0^1 k(x', x) dx = 1.$$

Linearity makes the mathematical problem much easier, but several of the previous results fail. For instance ϱ is conserved, but u is not, in general, even if there is global dominance conservation in each encounter, since there could be transfer of dominance to or from the host population. Assuming $f_0(x) = a_0 \exp(b_0 x)$ and taking the functional (17), it is not difficult to show that

$$(40) \quad \begin{aligned} \dot{H}[f] &= \int_0^1 \log f I[f] dx \\ &= -\frac{1}{2} \int_0^1 \int_0^1 \log \frac{f(x)}{f(x')} [f(x) e^{-b_0 x} - f(x') e^{-b_0 x'}] v(x') k(x', x) e^{b_0 x'} dx' dx \end{aligned}$$

without any definiteness in sign. Even more, if f has the form $a \exp(bx)$, $a > 0$, $b \in \mathbf{R}$, one can easily show that

$$(41) \quad \begin{aligned} &< 0 && \text{for } b > \max(0, b_0) \text{ or } b < \min(0, b_0) \\ \dot{H}[ae^{bx}] &= 0 && \text{for } b = 0 \text{ or } b = b_0 \\ &> 0 && \text{for } \min(0, b_0) < b < \max(0, b_0). \end{aligned}$$

Other possible generalizations might concern the treatment of other phenomena which are typically modeled in mathematical biology, like *birth and death processes* (with an age variable in addition to the time variable), *male and female behaviors*, *predator-prey systems*, *migration effects*, and so on.

Another important generalization is related to *space inhomogeneity* and *border effects*, with additional space variable(s).

Space dependent problems have been actually considered already [2], even though for a rather particular interaction model. This matter will hopefully be subject of future investigation.

5 - Further remarks and comments

It could deserve some attention verifying if the conservation and microreversibility conditions, (8) and (10), are fulfilled in the most popular interaction models quoted in the literature. In most of these models η is taken to be constant (a kind of *Maxwell molecule* assumption of kinetic theory [4]), whereas the probabilities ψ are limiting cases, in distributional sense, of continuous statistical distributions when a variance tends to zero, and are given by *Dirac delta functions*. For instance the model [3]

$$(42) \quad \psi(x', y'; x) = \delta[x - \beta(x', y')]$$

prescribes deterministically the final dominance of x' after an encounter with the y' partner. Since in this case $g = \beta$, condition (8) is satisfied whenever, for any

$x' \geq 0, y' \leq 1$, there results

$$\max(0, x' + y' - 1) \leq \beta(x', y') \leq \min(x' + y', 1) \quad \text{and} \quad \beta(x', y') + \beta(y', x') = x' + y'.$$

As regards condition (10) for microreversibility, it is seen to imply the necessary condition

$$(43) \quad \beta[\beta(x', y'), x' + y' - \beta(x', y')] = x' \quad \forall x', y', \in [0, 1].$$

It is verified at once that such a complicated functional equation is satisfied by $\beta(x', y') = x'$ and by $\beta(x', y') = y'$, in which cases equation (10) is also actually fulfilled. But both cases are trivial, since they describe either conservation of pre-collisional dominances, or their exchange, thus without any net variation of dominance distribution (equation (1) would collapse in fact to $\frac{\partial f}{\partial t} = 0$).

On the other hand, it is only matter of careful and patient analysis to show that equation (43) is always violated whenever $\beta(x', y')$ is either internal or external to the interval $(\min(x', y'), \max(x', y'))$. Therefore, there are no nontrivial microreversible models of the kind (42).

Of course, this is not surprising at all. Take for instance $x' > y'$. Any conservative model in which $g(x', y') > x'$ (and then $g(y', x') < y'$) describes a dictatorial society in which the stronger partner takes advantage of its power in order to improve its rank. If instead $g(x', y') < x'$ the individuals behave democratically, and help weaker partners. In both cases microscopic reversibility has not to be expected, but the irreversible microscopic behavior should enhance the global irreversible trend that was guaranteed by the H -theorem.

Another very popular model is the one which allows for the possibility of winning or losing an encounter, and getting a prize or a penalty for it, respectively [5]. More precisely

$$(44) \quad \psi(x', y'; x) = p(x', y') \delta[x - x' - W(x', y')] + p(y', x') \delta[x - x' + L(x', y')]$$

where $W, L > 0$ are the win/loss functions, subject to the constraints $x' + W(x', y') \leq 1$ and $x' - L(x', y') \geq 0$, and $p(x', y')$, $0 \leq p \leq 1$, is the probability for x' to win, with $p(x', y') + p(y', x') = 1$. There results

$$(45) \quad g(x', y') = x' + p(x', y')W(x', y') - p(y', x')L(x', y')$$

so that condition (8) for conservation of dominance is satisfied for any p by the very understandable option $L(x', y') = W(y', x')$.

In the same way as for (42), the microreversibility condition can be seen to yield the necessary condition on W

$$(46) \quad W(y' - W(x', y'), x' + W(x', y')) = W(x', y') \quad \forall x', y' \in [0, 1].$$

Such an information is sufficient, for instance, to rule out microreversibility for the simple case $W(x', y') = \varepsilon y' (1 - x')$, for any $\varepsilon > 0$ [5].

We would like to conclude by reporting without details some results relevant to a model, which can be dealt with analytically to some extent, and provides a different example of equilibrium distribution and of trend to equilibrium.

With $\eta = \text{constant}$, suitable scalings of variable allow to set $\eta = 1$ and $\varrho = 1$. Take (42) for ψ with

$$(47) \quad \beta(x', y') = \alpha x' + (1 - \alpha) y' \quad 0 < \alpha < 1$$

namely an example of conservative democratic microscopically irreversible behavior.

Another typical tool of kinetic theory, the moment method, can be conveniently applied. Introduce *power moments*

$$(48) \quad M_n(t) = \frac{1}{n!} \int_0^1 x^n f(x, t) dx \quad n = 0, 1, \dots$$

with $M_0 = \text{const.} = 1$ and $M_1 = \text{const.} = u$. Taking moments of equations (1) yields the *moment equations*

$$(49) \quad \dot{M}_n + M_n = \sum_{k=0}^n \alpha^k (1 - \alpha)^{n-k} M_k M_{n-k} \quad n = 2, 3, \dots$$

solvable in cascade, each linear in its unknown. Introduce the *generating function* of the moments

$$(50) \quad G(z, t) = \sum_{n=0}^{\infty} z^n M_n(t) = \int_0^1 e^{xz} f(x, t) dx$$

where z is a complex variable. Then summation of (49), multiplied by z^n , over n leads to the partial-functional equation for G

$$(51) \quad \frac{\partial G}{\partial t} + G(z, t) = G(\alpha z, t) G[(1 - \alpha)z, t]$$

with $G(0, t) = 1$ and $(\frac{\partial G}{\partial z})(0, t) = u$. Equilibrium states are defined by $G(z) = G(\alpha z) G[(1 - \alpha)z]$, where $G(z)$ is analytic at $z = 0$. Therefore its logarithmic derivative $\gamma(z)$ is analytic in turn, and satisfies the linear equation $\gamma(z) = \alpha \gamma(\alpha z) + (1 - \alpha) \gamma[(1 - \alpha)z]$, that is easily seen, by series expansion, to be solved uniquely by $\gamma(z) = \text{constant} = u$, from which $G(z) = e^{uz}$.

We have correspondingly $M_n = u^n (n!)^{-1}$, which allows to prove, on the basis of (48) and of the Weierstrass theorem, that $f(x) = \delta(x - u)$ is the *associated equilibrium distribution*. It is a limiting case in which all individuals possess exactly the average dominance u .

Local stability analysis for (50) is easily performed by Liapunov's first method. Setting $G(z, t) = e^{uz} [1 + H(z, t)]$, with $H(z, t) = \sum_{n=2}^{\infty} z^n h_n(t)$, the linearized equation for H is

$$(52) \quad \frac{\partial H}{\partial t} + H(z, t) = H(\alpha z, t) + H[(1 - \alpha)z, t]$$

and is solved by

$$(53) \quad H(z, t) = \sum_{n=2}^{\infty} z^n h_n(0) e^{-[1 - \alpha^n - (1 - \alpha)^n]t} = O(e^{-2\alpha(1 - \alpha)t})$$

which shows *exponential asymptotic stability*, with relaxation time $\frac{1}{2\alpha(1 - \alpha)}$.

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Sommario

L'equazione di Boltzmann della teoria cinetica e l'equazione per l'evoluzione della dominanza in biomatematica sono basati su concetti analoghi, di natura statistica. In questa ottica vengono esaminati i tipici concetti cinetici di invarianza e di microreversibilità, che generalmente sono ignorati nell'ambito biologico. In particolare, viene derivato e analizzato un «teorema H». Si indicano poi alcune possibili generalizzazioni a situazioni più complesse e si discutono, infine, alcuni esempi.
