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ROGER YUE CHI MING (*)

On p-injectivity and generalizations (**)

To Renée Yue Chi Ming in memoriam

The concept of *injective modules*, introduced by B. Eckmann - A. Schopf in 1953, is among the most fundamental tools in module and ring theory (cf. for example [7]). In [**11**], a generalization of injective modules, noted p-injective modules, is introduced to study von Neumann regular rings and associated rings, leading naturally to the area of V-rings and p-V-rings. The concept of p-injectivity is weakened to YJ-injectivity in [**15**]. Rings whose simple left modules are YJ-injective are studied in [**19**].

The aim of this paper is to improve several known results on p-injectivity and YJ-injectivity and to answer two questions raised in [**19**]:

1. *Is A strongly regular if A is a left quasi-duo ring whose simple right modules are* YJ-*injective* ([**19**], Question 2)?

2. *Is A left self-injective if A contains an injective maximal left ideal and every simple left A-modules is* YJ-*injective* ([**19**], Question 4)?

Throughout, A represents an associative ring with identity and A-modules are unital. J, Z will stand respectively for the Jacobson radical and the left singular ideal of A. As usual, an ideal of A will always mean a two-sided ideal. A is called *left quasi-duo* (after S. H. Brown) if every maximal left ideal of A is an ideal. A left(right) ideal of A is called *reduced* if it contains no non-zero nilpotent element.

Recall that a left A-module M is *p-injective* if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to A. ${}_AM$ is YJ-*injective* if, for any $0 \neq a \in A$, there exists a positive integer n (depending on a) such that $a^n \neq 0$ and any left A-homomorphism of Aa^n into M extends to A [**15**].

(*) Univ. Paris VII D. Diderot, UFR de Math., UMR 9994, CNRS, 2 Place Jussieu, 75251 Paris, France.

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A is called a *left* V-*ring* (following C. Faith) if every simple left A-module is injective. A well-known theorem of I. Kaplansky asserts that a commutative ring A is von Neumann regular iff A is a V-*ring*. A is called a *left* p-V-*ring* if every simple left A-module is p-injective. P-V-rings are studied in [6], [9], [10], [12]. In the context of von Neumann regular rings, V-rings and associated rings, p-injectivity has drawn the attention of many authors (cf. for example [2]-[6], [7] p. 340, [8]-[10], [20], [21]).

Let us note that the GP-*injective modules* considered in [4] coincide with our definition of YJ-injective modules in [15].

Next lemma extends Lemma 1 (iii) of [12] and strengthens Lemma 4 of [4].

Lemma 1. *Let A be a ring whose simple right modules are* YJ-*injective. Then $J = 0$.*

Proof. First suppose that J is non-zero reduced.

If $0 \neq c \in J$, then we must have $cA + r(c) \neq A$. Let R be a maximal right ideal containing $cA + r(c)$. Since A/R_A is YJ-injective, given c, there exists a positive integer n such that $c^n \neq 0$ and any right A-homomorphism of $c^n A$ into A/R extends to A. Since J is reduced, we may define a right A-homomorphism $f\colon c^n A \to A/R$ by $f(c^n a) = a + R$ for all $a \in A$. This implies that $f(c^n) = yc^n + R$ for some $y \in A$. Then $1 + R = yc^n + R$, which implies that $1 \in R$ (because $yc^n \in J \subseteq R$), contradicting $R \neq A$.

This proves that if J is reduced, then J must be zero. Now suppose that $J \neq 0$. Then there exist $0 \neq d \in J$ such that $d^2 = 0$. Let M be a maximal right ideal containing $r(d)$. Since A/M_A is YJ-injective, if $g\colon dA \to A/M$ is the right A-homomorphism defined by $g(da) = a + M$ for all $a \in A$, there exists $w \in A$ such that $g(d) = wd + M$. Then $1 + M = wd + M$ implies that $1 \in M$ (in as much as $wd \in J \subseteq M$), which contradicts $M \neq A$. This proves that $J = 0$.

Consider the following example where $J \neq 0$.

Example. Let $A = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$, where K is a field. $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ are the only maximal left ideals of A. ${}_A A/M$ is injective but ${}_A A/N$ is not even YJ-injective (according to Lemma 1), because the Jacobson radical is $J = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$. A is not left self-injective although ${}_A N$ is injective.

In [16], we defined GP-injectivity as follows:

A left A-module M is GP-*injective* if, for any $a \in A$, there exists a positive integer m such that any left A-homomorphism of Aa^m into M extends to A. (Note that a^m may be zero; consequently, this does *not* coincide with the GP-injective modules defined in [4]).

Obviously, YJ-injectivity implies GP-injectivity. It is clear that in case of reduced rings, GP-injectivity coincides with YJ-injectivity.

Contrary to the assertion made in [4], p. 5442, Proposition 4 of [**16**], remains valid.

In the above example, A is π-regular and every left A-module is GP-injective in the sense of [**16**]. Consequently, GP-injectivity (in the sense of [**16**]) does not imply YJ-injectivity. (Otherwise, $J = 0$ in the above example, according to Lemma 1.)

Strongly regular rings, which were defined some fifty years ago by R. F. Arens and I. Kaplansky (cf. [**1**]), have been extensively studied in recent years. It is well-known that a left quasi-duo, left or right V-ring is strongly regular. In [**19**], Question 2, it is asked whether A is strongly regular if A is a left quasi-duo ring whose simple right modules are YJ-injective. An affirmative answer is given in the next theorem which may be considered as a sequel to Theorem 10 of [4].

Theorem 1. *The following conditions are equivalent*:

1. *A is strongly regular*
2. *A is a left quasi-duo ring whose simple right modules are* YJ-*injective*
3. *A is a left quasi-duo ring whose simple left modules are* YJ-*injective*
4. *A is a left quasi-duo ring whose principal left ideals are* YJ-*injective*
5. *A is a left quasi-duo ring whose principal right ideals are* YJ-*injective.*

Proof. Obviously, **1** implies **2** through **5**.

Assume 2. By Lemma 1, $J = 0$. Since A is left quasi-duo, the proof of **2** *implies* **3** in Theorem 2.1 of [**13**] shows that A must be reduced. Suppose there exists $d \in A$ such that $Ad + l(d) \neq A$. If M is a maximal left ideal containing $Ad + l(d)$, then M is a maximal right ideal of A and A/M_A is YJ-injective. There exists a positive integer n such that $d^n \neq 0$ and if g: $d^nA \to A/M$ is the map $d^n a \to a + M$ for all $a \in A$, then g is a well-defined right A-homomorphism (because A is reduced). Therefore $1 + M = g(d^n) = cd^n + M$ for some $c \in A$ which implies that $1 - cd^n \in M$, whence $1 \in M$, contradicting $M \neq A$. This proves that $Aa + l(a) = A$ for every $a \in A$ and hence $a = ua^2$ for some $u \in A$, proving that A is strongly regular. Thus **2** implies **1**. Similarly, **3** implies **1**.

Assume 4. Since A/J is left quasi-duo with zero Jacobson radical, then A/J is a reduced ring (cf. the proof of 2 *implies* 3 in Theorem 2.1 of [13]). Now suppose there exists $0 \neq b \in A$ such that $b^2 = 0$. Then $b \in J$.

Since ${}_AAb$ is YJ-injective, if j: $Ab \to Ab$ is the identity map, there exists $d \in A$ such that $b = j(b) = bdb$. Then db is an idempotent in J which implies that $db = 0$. Therefore $b = 0$. This contradiction proves that A is reduced.

Since ${}_AA$ is YJ-injective, for any $0 \neq c \in A$, we have $c^2 \neq 0$ and there exists a positive integer n such that $c^{2n}A$ is a non-zero right annihilator ([15], Lemma 3). Since A is reduced, $l(c) = l(c^{2n})$ and therefore

$$cA \subseteq r(l(cA)) = r(l(c)) = r(l(c^{2n}))r(l(c^{2n}A)) = c^{2n}A$$

which implies that $c = c^2u$ for some $u \in A$. This proves that A is strongly regular and 4 implies 1. Similarly, 5 implies 1.

Self-injective rings play an important role in ring theory. Well-known examples are quasi-Frobeniusean rings, pseudo-Frobeniusean rings and the maximal quotient rings of non-singular rings. Those rings have been extensively studied in recent years. Call A a left (resp. right) MI-ring if A contains an injective maximal left (resp. right) ideal. In the above example, A is left MI but not left self-injective (cf. [17] and [21]).

In [19] Question 4, it is asked whether A is left self-injective regular if A is a left MI-ring whose simple left modules are YJ-injective. We here give a positive answer.

Theorem 2. *The following conditions are equivalent:*

1. *A is left self-injective regular with non-zero socle*
2. *A is a left* MI-*ring whose simple left modules are* YJ-*injective*
3. *A is a left* MI-*ring whose simple right modules are* YJ-*injective*
4. *A is a left* MI-*ring whose principal right ideals are* YJ-*injective.*

Proof. 1 implies 2, 3 and *4* evidently.

Assume 2. By Lemma 1, $J = 0$. If M is an injective maximal left ideal, $A = M \oplus U$, where $M = Ae$, $e = e^2 \in A$, $U = Au$, $u = 1 - e$. Since A is semi-prime and Au is a minimal left ideal, then uA is a minimal right ideal of A. Now $uA = r(M)$ and by Theorem 1.2 of [18] A is left self-injective. Since $J = 0$, A is regular and 2 implies 1. Similarly, 3 implies 1.

Assume 4. Suppose there exists a non-zero right ideal R such that $R^2 = 0$. If $0 \neq b \in R$, since bA_A is YJ-injective, given the identity map j: $bA \to bA$, there exist $c \in A$ such that $b = j(b) = bcb \in R^2 = 0$. This contradiction proves that A must be semi-prime. By Theorem 1.2 of [18] A is a left self-injective ring. Suppose that $Z \neq 0$. By Lemma 7 of [**14**] there exists $0 \neq z \in Z$ such that $z^2 = 0$. Since zA_A is YJ-injective, if i: $zA \to zA$ is the identity map, there exists $w \in A$ such that $z = i(z) = zwz$. Since $Az \cap l(wz) = 0$, then $Az = 0$ which contradicts $z \neq 0$. This proves that $Z = 0$. Thus 4 implies **1**.

Remark. Since primitive left self-injective regular rings with non-zero socle need not be simple Artinian (K. R. Goodearl), according to Theorem 2, left MI-rings need not be right MI.

If A is π-regular, it is easily seen that every left (right) A-module is GP-injective in the sense of [**16**].

If A is von Neumann regular, it is obvious that every left (right) A-module is YJ-injective.

Question.

a. Is A π-regular if every left (and right) A-module is GP-injective?

b. Is A von Neumann regular if every left (and right) A-module is YJ-injective?

Remark that **a** holds if A is a weakly left duo-ring and that **b** holds in case A is reduced or left quasi-duo.

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Sommario

In questa nota viene studiata la YJ-*iniettività (che è una effettiva generalizzazione della iniettività) nell'ambito degli anelli regolari autoiniettivi o fortemente regolari. Il lavoro risponde anche a due domande relative alla* YJ-*iniettività.*

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