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A wave splitting approach for a layer (**)

1 - Introduction

A noticeable part of theoretical and experimental research in geophysics and electromagnetism is devoted to wave propagation into inhomogeneous layers. This topic covers a wide class of phenomena such as the propagation of seismic displacements through continuous or discontinuous strata of the earth's surface, the reflection and refraction of acoustic or electromagnetic pulses and the transmission of signals in electromechanical devices.

Concerning to a linear theory, the mathematical description of the problem reduces to a system of second order differential equations for the field variables whose coefficients depend on the depth within the stratified layer.

To this respect, we deal with a one-dimensional problem with assigned boundary conditions at the edges of the layer. If we also account for harmonic waves, where the time dependence of the fields is given by $\exp(-i\omega t)$, $(\omega \in \boldsymbol{R}^{++})$, then we get a system of ordinary linear differential equations.

In the simplest cases, such as waves in continuously layered isotropic media with normal incidence through the layer, we have scalar problems of the type

$$(Gv')' + \omega^2 Sv = 0$$

where v represents the field variable, $G(z)$ and $S(z)$ are suitable continuous functions of the depth z and the prime denotes derivative with respect to z (see for ex. [2]).

If we are dealing with anisotropic media and/or we account for bias external

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fields, we obtain vector problems in which coupling effects among the different field's components take a crucial role. Such is the case of electromagnetic waves in an anisotropic dielectric layer [3] and of electromechanic waves in a biased elastic dielectric layer [5], where two-component fields are considered.

Here we are interested in generalizing the previous models to a n-components time-harmonic field for a continuous layered medium. In particular we extend the approach of [5] to the vector equation

$$(1.1)\qquad (\mathbf{G}\mathbf{v}')' + \omega^2\,\mathbf{S}\mathbf{v} - \omega\mathbf{C}'\mathbf{v} = 0$$

where $\mathbf{G}(z)$ and $\mathbf{S}(z)$ are n-dimensional diagonal matrices and where we have pointed out the contribution of the coupling effect in terms of the spatial derivative of a suitable matrix $\mathbf{C}(z)$, which accounts for bias and/or anisotropy of the medium and whose diagonal entries are constant (see [5]). It is also understood that $\mathbf{G}$, $\mathbf{S}$ and $\mathbf{C}$ are piecewise smooth functions in $(0, d)$, where d is the thickness of the layer, and that $\mathbf{G}$ and $\mathbf{S}$ are positive definite.

A physical motivation of equation (1.1) can be found for example in the problem which is complementary to that solved in [5]. There, a purely transverse electromechanic field $\mathbf{v}_1$ is studied which decouples from a four-component field $\mathbf{v}_2$ with polarization normal to $\mathbf{v}_1$. Under suitable assumptions on the material symmetry of the solid and accounting for normal incidence, the governing equation for $\mathbf{v}_2$ has the form (1.1) with $n = 4$.

From a mathematical point of view, equation (1.1) deserves some interest in connection with both the direct and the inverse problems. In particular, the wave splitting technique, which is performed in Section 2, allows us to extend in a natural way the usual analysis of the direct problem for a homogeneous layer. The resulting equation can be put in a form which is susceptible of an iterative solution. The convergence of such a solution is shown in Section 3. The invariant imbedding approach is also exploited in Section 4 to suggest a suitable formulation of the inverse problem.

2 - Wave splitting

It has been recently shown how a diagonalizing procedure of the system of linear differential equations arising in layered media can give more insights about the physical meaning of the problem at hand [3], [5]. We develop here the general procedure which applies to the time-harmonic problem (1.1) for a n-component field $\mathbf{v}$.

As a first step we introduce the quantity $\mathbf{w} = \mathbf{G}\mathbf{v}'$ in order to reduce (1.1) to a first order equation. We have $\mathbf{v}' = \mathbf{G}^{-1}\mathbf{w}$ and, after substitution, we get

$$\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix}' = \begin{pmatrix}0 & \mathbf{G}^{-1}\\ -\omega^2\,\mathbf{S} + \omega\mathbf{C}' & 0\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix}. \tag{2.1}$$

The wave splitting technique consists in defining a couple of n-dimensional fields $\mathbf{v}^+$ and $\mathbf{v}^-$ such that the $2n \times 2n$ matrix in the right hand side of (2.1) splits into a homogeneous-like diagonal part plus terms due to inhomogeneity. To this end we pose

$$\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix} = \mathbf{D}\begin{pmatrix}\mathbf{v}^+\\ \mathbf{v}^-\end{pmatrix}. \tag{2.2}$$

where

$$\mathbf{D} = \begin{pmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{N} & -\mathbf{N}\end{pmatrix} \tag{2.3}$$

and look for a $n \times n$ matrix $\mathbf{N}$ which meets the diagonalization requirement. Substitution of (2.3) into (2.2) and, in turn, into (2.1), yields

$$\begin{aligned}\begin{pmatrix}\mathbf{v}^+\\ \mathbf{v}^-\end{pmatrix}' &= -\frac{1}{2}\begin{pmatrix}\mathbf{N}^{-1}\mathbf{N}' & -\mathbf{N}^{-1}\mathbf{N}'\\ -\mathbf{N}^{-1}\mathbf{N}' & \mathbf{N}^{-1}\mathbf{N}'\end{pmatrix}\begin{pmatrix}\mathbf{v}^+\\ \mathbf{v}^-\end{pmatrix}\\ &+\frac{1}{2}\begin{pmatrix}\mathbf{N}^{-1}(-\omega^2\,\mathbf{S}+\omega\mathbf{C}')+\mathbf{G}^{-1}\mathbf{N} & \mathbf{N}^{-1}(-\omega^2\,\mathbf{S}+\omega\mathbf{C}')-\mathbf{G}^{-1}\mathbf{N}\\ -\mathbf{N}^{-1}(-\omega^2\mathbf{S}+\omega\mathbf{C}')+\mathbf{G}^{-1}\mathbf{N} & -\mathbf{N}^{-1}(-\omega^2\mathbf{S}+\omega\mathbf{C}')-\mathbf{G}^{-1}\mathbf{N}\end{pmatrix}\begin{pmatrix}\mathbf{v}^+\\ \mathbf{v}^-\end{pmatrix}.\end{aligned} \tag{2.4}$$

The only purely inhomogeneous term of the second matrix in the right hand side of (2.4) is $\omega\mathbf{N}^{-1}\,\mathbf{C}'$. The remaining terms can be diagonalized by imposing $\omega^2\mathbf{N}^{-1}\mathbf{S} + \mathbf{G}^{-1}\mathbf{N} = 0$, which amounts to set

$$\mathbf{N} = i\omega\mathbf{G}^{\frac{1}{2}}\,\mathbf{S}^{\frac{1}{2}}\,. \tag{2.5}$$

The sign in the right hand side of (2.5) is taken in such a way that $\mathbf{G}^{-1}\,\mathbf{N} - \omega^2\,\mathbf{N}^{-1}\mathbf{S}$ have positive imaginary part, according to the notation of the splitted field $(\mathbf{v}^+, \mathbf{v}^-)^T$. Finally, after the positions

$$\Lambda = \mathbf{G}^{-\frac{1}{2}}\,\mathbf{S}^{\frac{1}{2}} \qquad \mathbf{P} = \mathbf{S}^{-\frac{1}{2}}\,\mathbf{G}^{-\frac{1}{2}}$$

we write equation (2.4) in the form

$$\begin{aligned}\begin{pmatrix}\mathbf{v}^+\\ \mathbf{v}^-\end{pmatrix}' &= i\omega\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}\begin{pmatrix}\mathbf{v}^+\\ \mathbf{v}^-\end{pmatrix}\\ &+\frac{1}{2}\left[\begin{pmatrix}\mathbf{P}'\,\mathbf{P}^{-1} & -\mathbf{P}'\,\mathbf{P}^{-1}\\ -\mathbf{P}'\,\mathbf{P}^{-1} & \mathbf{P}'\mathbf{P}^{-1}\end{pmatrix} - \frac{i}{2}\begin{pmatrix}\mathbf{P} & \mathbf{P}\\ -\mathbf{P} & -\mathbf{P}\end{pmatrix}\begin{pmatrix}\mathbf{C}' & 0\\ 0 & \mathbf{C}'\end{pmatrix}\right]\begin{pmatrix}\mathbf{v}^+\\ \mathbf{v}^-\end{pmatrix}.\end{aligned} \tag{2.6}$$

The effectiveness of the present formulation rests upon the physical meaning of the three contributions in the right hand side of (2.6). The first term is the counterpart of that occurring in the homogeneous layer's problem where $\mathbf{v}$ splits into a forward and a backward propagating plane waves

$$v_j^+ = v_j^+(0)\exp(i\omega\Lambda_j) \qquad v_j^- = v_j^-(0)\exp(-i\omega\Lambda_j)$$

where Λ_j ($j = 1, \dots, n$) are the diagonal entries of Λ. The second term specifically takes into account the inhomogeneity of the layer and the third term characterizes the coupling effects. These results are obvious consequences of the fact that Λ and $\mathbf{P}$ are diagonal and $\mathbf{C}'$ has null diagonal entries.

3 - The convergence theorem

In terms of the components of $\mathbf{v}^+$ and $\mathbf{v}^-$, equation (2.6) reads

$$\text{(3.1)} \qquad \begin{aligned} v_j^{+\prime} &= i\omega\Lambda_j v_j^+ + \tfrac{1}{2}(\overline{A}_{jk} v_k^+ - A_{jk} v_k^-) \\ v_j^{-\prime} &= -i\omega\Lambda_j v_j^- - \tfrac{1}{2}(\overline{A}_{jk} v_k^+ - A_{jk} v_k^-) \end{aligned}$$

where

$$\text{(3.2)} \qquad \mathbf{A} = \mathbf{P}'\mathbf{P}^{-1} + i\mathbf{P}\mathbf{C}'$$

and the bar denotes complex conjugate.

In view of an integral formulation of equations (3.1) we introduce the following new variables

$$\text{(3.3)} \qquad \sigma_j = \int_0^z \Lambda_j \, d\tau \qquad j = 1, \dots, n$$

which play the role of characteristic travelling times. Their regularity is ensured by the positiveness of Λ_j. Denoting by $(\cdot)_{,j}$ the derivative with respect to σ_j and multiplying equations (3.1) respectively by $\exp(-i\omega\sigma_j)$ and $\exp(i\omega\sigma_j)$ we obtain

$$\text{(3.4)} \qquad \begin{aligned} [v_j^+ \exp(-i\omega\sigma_j)]_{,j} &= \tfrac{1}{2}(\overline{\mathcal{A}}^j_{jk} v_k^+ - \mathcal{A}^j_{jk} v_k^-)\exp(-i\omega\sigma_j) \\ [v_j^- \exp(+i\omega\sigma_j)]_{,j} &= -\tfrac{1}{2}(\overline{\mathcal{A}}^j_{jk} v_k^+ - \mathcal{A}^j_{jk} v_k^-)\exp(+i\omega\sigma_j) \end{aligned}$$

where $\qquad \mathcal{A}^j_{jk} = (\mathbf{P}_{,j}\mathbf{P}^{-1} + i\mathbf{P}\mathbf{C}_{,j})_{jk} \qquad$ (j not summed).

After integration and rearrangement equations (3.4) can be transformed into the following system of linear integral equations

$$\text{(3.5)} \qquad \begin{aligned} v_j^+ &= v_j^+(0)\exp(+i\omega\sigma_j) + \tfrac{1}{2}\int_0^{\sigma_j}(\overline{\mathcal{A}}^j_{jk} v_k^+ - \mathcal{A}^j_{jk} v_k^-)\exp[-i\omega(\xi - \sigma_j)]\, d\xi \\ v_j^- &= v_j^-(0)\exp(-i\omega\sigma_j) - \tfrac{1}{2}\int_0^{\sigma_j}(\overline{\mathcal{A}}^j_{jk} v_k^+ - \mathcal{A}^j_{jk} v_k^-)\exp[+i\omega(\xi - \sigma_j)]\, d\xi . \end{aligned}$$

If the data $v_j^{\pm}(0)$, $(j = 1, \dots, n)$ at the edge of the layer are given, the integration of system (3.5) gives the solution of our problem in terms of the components of the forward and backward splitted wave fields within the layer.

We introduce the quantity $\mathbf{V} = (v_1^+, v_2^+, \dots, v_n^+, v_1^-, v_2^-, \dots, v_n^-)^T$ which can be viewed as a composition of z via the σ_j's with values in a $2n$-dimensional vector space $B_\omega(0, d)$. We also introduce

$$\mathbf{V}^{(0)} = (v_1^+(0)\, e^{i\omega\sigma_1}, \dots, v_n^+(0)\, e^{i\omega\sigma_n}, v_1^-(0)\, e^{-i\omega\sigma_1}, \dots, v_n^-(0)\, e^{-i\omega\sigma_n})^T$$

and look for a solution to equations (3.5) in the form of the series expansion

$$\mathbf{V} = \sum_p \mathbf{V}^{(p)} \qquad \mathbf{V}^{(p+1)} = \mathbf{L}_\omega \mathbf{V}^{(p)} \tag{3.6}$$

with $p \geqslant 0$ and where the action of the linear operator $\mathbf{L}_\omega$ on $B_\omega(0, d)$ is defined as

$$\mathbf{L}_\omega \mathbf{V}^{(p)} = \begin{pmatrix} \frac{1}{2} \int_0^{\sigma_j} d\xi\, E^{(j)}\, \overline{\mathfrak{a}}_{jk}^j & -\frac{1}{2} \int_0^{\sigma_j} d\xi\, E^{(j)}\, \mathfrak{a}_{jk}^j \\ -\frac{1}{2} \int_0^{\sigma_j} d\xi\, \overline{E}^{(j)}\, \overline{\mathfrak{a}}_{jk}^j & \frac{1}{2} \int_0^{\sigma_j} d\xi\, \overline{E}^{(j)}\, \mathfrak{a}_{jk}^j \end{pmatrix} \begin{pmatrix} v_k^{+(p)} \\ v_k^{-(p)} \end{pmatrix} \tag{3.7}$$

$(j = 1, \dots, n)$, with

$$E^{(j)} = \exp[-i\omega(\xi - \sigma_j)]. \tag{3.8}$$

In order to state a convergence theorem for the series (3.6), we introduce the norm in $B_\omega(0, d)$

$$\|\mathbf{V}\|_\omega = \sup_{0<z<d} \{\sum_{j=1}^n [\,|v_j^+(\sigma_j(z), \omega)|^2 + |v_j^-(\sigma_j(z), \omega)|^2]\}^{\frac{1}{2}} \tag{3.9}$$

and define the quantity

$$\sigma = \max_j \{\sigma_j(d)\}. \tag{3.10}$$

The following result holds.

Theorem. *A sufficient condition for the series* (3.6) *to converge in norm on* $B_\omega(0, d)$ *is*

$$\frac{1}{2} \int_0^\sigma \sum_{j,k}^n [\,|(\mathbf{P}_{,j}\mathbf{P}^{-1})_{jk}|^2 + |(\mathbf{P}\mathbf{C}_{,j})_{jk}|^2]\, d\xi < 1. \tag{3.11}$$

Proof. In view of the regularity of σ_j, the continuity of $\mathbf{V}$ and the definition (3.9), $B_\omega(0, d)$ turns out to be a Banach space. Owing to (3.7) we have, for an arbitrary $\mathbf{V}$, that $\|\mathbf{L}_\omega \mathbf{V}\|_\omega$ is equal to

$$\sup_{0<z<d} \{\frac{1}{2}\sum_j [\,|\int_0^{\sigma_j} E^{(j)}(\bar{\mathfrak{a}}^j_{jk}\, v_k^+ - \mathfrak{a}^j_{jk}\, v_k^-)\,\mathrm{d}\xi|^2 + |\int_0^{\sigma_j} \bar{E}^{(j)}(\bar{\mathfrak{a}}^j_{jk}\, v_k^+ - \mathfrak{a}^j_{jk}\, v_k^-)\,\mathrm{d}\xi|^2]\}^{\frac{1}{2}}\,.$$

Accounting for (3.8) and using the Hölder's inequality we obtain

$$\begin{aligned}\|\mathbf{L}^{(\omega)}\mathbf{V}^{(p)}\|_\omega &\leqslant \sup_{0<z<d}\{\frac{1}{2}\sum_j \int_0^{\sigma_j} |\bar{\mathfrak{a}}^j_{jk}\, v_k^{+(p)} - \mathfrak{a}^j_{jk}\, v_k^{-(p)}|^2\,\mathrm{d}\xi\}^{\frac{1}{2}}\\ &\leqslant \sup_{0<z<d}\{\frac{1}{2}\sum_j \int_0^{\sigma_j}\sum_k |\mathfrak{a}^j_{jk}|^2[(\sum_k |v_k^{+(p)}|^2)^{\frac{1}{2}} - (\sum_k |v_k^{-(p)}|^2)^{\frac{1}{2}}\,]^2\mathrm{d}\xi\}^{\frac{1}{2}}\\ &\leqslant \sup_{0<z<d}\{\frac{1}{2}\sum_{j,k}\int_0^{\sigma_j} |\mathfrak{a}^j_{jk}|^2\,\mathrm{d}\xi\}^{\frac{1}{2}}\|\mathbf{V}^{(p)}\|_\omega\end{aligned}$$

and finally, by equations (3.6) and the definition (3.10)

$$\|\mathbf{V}^{(p+1)}\|_\omega \leqslant \{\frac{1}{2}\int_0^{\sigma}\sum_{j,k} |\mathfrak{a}^j_{jk}|^2\,\mathrm{d}\xi\}^{\frac{1}{2}}\|\mathbf{V}^{(p)}\|_\omega\,.$$

In view of (3.2), the definition of $\mathfrak{a}^j_{jk}$ and the completeness of $B_\omega(0, d)$, the condition (3.11) turns out to be sufficient for the convergence of the series (3.6).

We remark that the present theorem can be extended to the case of a layered half-space ($z \in (0, \infty)$). In this case, of course, the convergence of the series (3.6) requires stronger conditions on the matrices $\mathbf{P}'\,\mathbf{P}^{-1}$ and $\mathbf{P}\mathbf{C}'$ since they must be $L^2(\boldsymbol{R}^+)$. Essentially, this fact restricts our result to asymptotically homogeneous half-spaces.

4 - The invariant imbedding formulation

The solution (3.6) to the integral equation (3.5) of the wave-splitting problem requires the knowledge of $\mathbf{V}^{(0)}$ and, in turn, of the boundary values $v_j^{\pm}(0)$. These quantities can be obtained from the data on $\mathbf{v}$ and $\mathbf{w}$ at the edge of the layer. For practical purposes, a more convenient approach to the solution of equation (3.1) consists in restating the problem in terms of reflection and transmission matrices $\mathbf{R}$ and $\mathbf{T}$.

The object of the invariant imbedding formulation (see [1]) is the conversion of a system of linear differential equations for $\mathbf{v}^+$ and $\mathbf{v}^-$ with assigned boundary values, into a system of non linear differential equations for the matrices **R** and **T** which is independent on the boundary conditions at the edges of the layer.

If we assume that an incident wave impinges on the layer at $z = 0$, we can introduce the following Riccati's transformation

$$\mathbf{v}^-(z) = \mathbf{R}(z)\mathbf{v}^+(z) \qquad \mathbf{v}^+(z) = \mathbf{T}^{-1}(z)\mathbf{v}^+(d) \tag{4.1}$$

together with the assumption

$$\mathbf{v}^-(d) = 0\,. \tag{4.2}$$

Substitution of (4.1) into (3.1) yields,

$$\mathbf{R}' = -\tfrac{1}{2}\,\overline{\mathbf{A}} - i\omega(\Lambda\mathbf{R} + \mathbf{R}\Lambda) + \tfrac{1}{2}(\mathbf{A}\mathbf{R} - \mathbf{R}\overline{\mathbf{A}}) + \tfrac{1}{2}\,\mathbf{R}\mathbf{A}\mathbf{R} \tag{4.3}$$

$$\mathbf{T}' = -i\omega\,\mathbf{T}\Lambda - \tfrac{1}{2}\,\mathbf{T}\overline{\mathbf{A}} + \tfrac{1}{2}\,\mathbf{T}\mathbf{A}\mathbf{R}\,. \tag{4.4}$$

Equations (4.3) and (4.4) can be used to solve the direct problem as follows. Owing to (4.2) we have $\mathbf{R}(d) = 0$. This condition is exploited as an initial datum to solve (4.3) and to obtain $\mathbf{R}(0)$. Anagolously, since $\mathbf{T}(d) = \mathbf{1}$ we can integrate (4.4) to obtain $\mathbf{T}(0)$. Then equations (4.1) give the reflected field $\mathbf{v}^-(0)$ and the transmitted field $\mathbf{v}^+(d)$ in terms of the incident field $\mathbf{v}^+(0)$.

The advantage of equations (4.3) and (4.4) also relies on their usefulness in stating the inverse scattering problem for a layer. To this end we write (4.3) and (4.4) in components

$$R'_{jk} = -\tfrac{1}{2}\,\overline{A}_{jk} - i\omega(\Lambda_j + \Lambda_k)R_{jk} + \tfrac{1}{2}(A_{jl}R_{lk} - R_{jl}\overline{A}_{lk}) + \tfrac{1}{2}\,R_{jl}A_{lh}R_{hk}$$
$$T'_{jk} = -i\omega\Lambda_k T_{jk} - \tfrac{1}{2}\,T_{jl}\overline{A}_{lk} + \tfrac{1}{2}\,T_{jl}A_{lh}R_{hk}\,.$$

Then we apply the change of variable (3.3) and pose $\sigma_{jk} = \sigma_j + \sigma_k$.

Denoting by $(\cdot)_{,jk}$ the derivative with respect to σ_{jk} we get:

$$\begin{aligned}(R_{jk})_{,jk} = {} & -\tfrac{1}{2}\,\overline{\mathfrak{a}}_{jk} - i\omega R_{jk} + \tfrac{1}{2}\Big(\mathfrak{a}_{jl}R_{lk}\,\frac{\Lambda_j + \Lambda_l}{\Lambda_j + \Lambda_k} - R_{jl}\,\overline{\mathfrak{a}}_{lk}\,\frac{\Lambda_l + \Lambda_k}{\Lambda_j + \Lambda_k}\Big) \\ & + \tfrac{1}{2}\,R_{jl}\mathfrak{a}_{lh}R_{hk}\,\frac{\Lambda_l + \Lambda_h}{\Lambda_j + \Lambda_k}\end{aligned} \tag{4.5}$$

$$(T_{jk})_{,k} = -i\omega\,T_{jk} - \tfrac{1}{2}\,T_{jl}\overline{\mathfrak{a}}^k_{lk} + \tfrac{1}{2}\,T_{jl}\overline{\mathfrak{a}}^h_{lh}\,\frac{\Lambda_h}{\Lambda_k}\,R_{hk} \tag{4.6}$$

with $\mathcal{A}_{jk} = (\mathbf{P}_{,jk}\mathbf{P}^{-1} + i\mathbf{P}\mathbf{C}_{,jk})_{jk}$ and where $R_{jk} = R_{jk}(\sigma_{jk}, \omega)$, $T_{jk} = T_{jk}(\sigma_k, \omega)$, $\mathcal{A}_{jk} = \mathcal{A}_{jk}(\sigma_{jk})$.

After multiplication of (4.5) by $\exp(i\omega\sigma_{jk})$ and (4.6) by $\exp(i\omega\sigma_k)$, we integrate to obtain:

$$
\begin{aligned}
R_{jk}(\sigma_{jk}, \omega) = {} & \frac{1}{2}\int_{\sigma_{jk}}^{\sigma_{jk}^d} \overline{\mathcal{A}}_{jk}(\tau)\exp[i\omega(\tau - \sigma_{jk})]\,d\tau \\
& - \frac{1}{2}\int_{\sigma_{jk}}^{\sigma_{jk}^d} \mathcal{A}_{jl}(\tau)R_{lk}(\tau - \sigma_j + \sigma_k, \omega)e^{-i\omega\sigma_{jl}}\exp(i\omega t)\,d\tau \\
& + \frac{1}{2}\int_{\sigma_{jk}}^{\sigma_{jk}^d} R_{jl}(\tau + \sigma_j - \sigma_k, \omega)\overline{\mathcal{A}}_{lk}(\tau)e^{-i\omega\sigma_{lk}}\exp(i\omega\tau)\,d\tau \\
& - \frac{1}{2}\int_{\sigma_{jk}}^{\sigma_{jk}^d} R_{jl}(\tau + \sigma_j - \sigma_h, \omega)\mathcal{A}_{lh}(\tau)R_{hk}(\tau - \sigma_l + \sigma_k, \omega)\exp[i\omega(\tau - \sigma_{lh})]\,d\tau
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
T_{jk}(\sigma_k, \omega) = {} & \delta_{jk}\exp[i\omega(\sigma_k^d - \sigma_k)] \\
& + \frac{1}{2}\int_{\sigma_k}^{\sigma_k^d} T_{jl}(\tau + \sigma_l - \sigma_k, \omega)\overline{\mathcal{A}}_{lk}^{k}(\tau)\exp[i\omega(\tau - \sigma_k)]\,d\tau \\
& - \frac{1}{2}\int_{\sigma_k}^{\sigma_k^d} T_{jl}(\tau + \sigma_l - \sigma_h, \omega)\mathcal{A}_{lh}^{h}(\tau)R_{hk}(\tau + \sigma_k, \omega)\exp[i\omega(\tau - \sigma_h)]\,d\tau
\end{aligned}
\tag{4.8}
$$

where $\sigma^d = \sigma(d)$ and where the conditions $\mathbf{R}(\sigma^d) = 0$, $\mathbf{T}(\sigma^d) = \mathbf{1}$ have been exploited.

Equations (4.7) and (4.8) are non linear integral equations for the reflection and transmission matrices. In order to obtain $\mathcal{A}_{jk}$ from the scattering data $\mathbf{R}(0, \omega)$ and $\mathbf{T}(0, \omega)$ we firstly consider (4.7). For $\sigma_j = 0$ $(j = 1, \ldots, n)$, we have

$$
\begin{aligned}
R_{jk}(0, \omega) = {} & \frac{1}{2}\int_0^{\sigma_{jk}^d} \overline{\mathcal{A}}_{jk}(\tau)\exp(i\omega\tau)\,d\tau \\
& - \frac{1}{2}\int_0^{\sigma_{jk}^d} [\mathcal{A}_{jl}(\tau)R_{lk}(\tau, \omega) - R_{jl}(\tau, \omega)\overline{\mathcal{A}}_{lk}(\tau)]\exp(i\omega\tau)\,d\tau \\
& - \frac{1}{2}\int_0^{\sigma_{jk}^d} R_{jl}(\tau, \omega)\mathcal{A}_{lh}(\tau)R_{hk}(\tau, \omega)\exp(i\omega\tau)\,d\tau .
\end{aligned}
\tag{4.9}
$$

Taking the Fourier transform of both sides in (4.9) and taking into account that $\mathcal{a}_{jk}(\tau) = 0$ for $\tau \notin (0, \sigma_{jk}^d)$ we arrive at the inversion formula

$$
\begin{aligned}
\overline{\mathcal{a}}_{jk}(\xi) &= \frac{1}{\pi} \hat{R}_{jk}(0, \xi) \\
&+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\sigma_{jk}^d} [\mathcal{a}_{jl}(\tau) R_{lk}(\tau, \omega) - R_{jl}(\tau, \omega) \overline{\mathcal{a}}_{lk}(\tau)] \exp[i\omega(\tau - \xi)] \, d\tau \, d\omega \\
&+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\sigma_{jk}^d} [R_{jl}(\tau, \omega) \mathcal{a}_{lh}(\tau) R_{hk}(\tau, \omega)] \exp[i\omega(\tau - \xi)] \, d\tau \, d\omega
\end{aligned} \tag{4.10}
$$

where the hat denotes the Fourier transform.

Analogous derivations of the inverse formula (4.10) have been developed in the scalar case for a layer of infinite depth (an inhomogeneous half-space) (see [4]). In that case the travelling time σ ranges throughout $\boldsymbol{R}^+$ and, as noted in Section 3, the solution of the problem applies only to asymptotically homogeneous half-spaces.

For a layer of finite depth the quantities $\sigma_j(d)$ are unknown as well as the material parameters $\mathcal{a}(\sigma)$, and the effective inversion of the scattering problem also requires the integration of (4.8). Hence the actual procedure in the present case is as follows. In view of (4.7) and (4.10) we pose:

$$\mathcal{a}_{jk}(\sigma_{jk}) = \sum_n \mathcal{a}_{jk}^{(n)}(\sigma_{jk}) \tag{4.11}$$

$$R_{jk}(\sigma_{jk}, \omega) = \sum_n R_{jk}^{(n)}(\sigma_{jk}, \omega) \tag{4.12}$$

with:

$$
\begin{aligned}
\mathcal{a}_{jk}^{(0)}(\sigma_{jk}) &= \frac{1}{\pi} \hat{R}_{jk}(0, \sigma_{jk}) \\
R_{jk}^{(0)}(\sigma_{jk}, \omega) &= \frac{1}{2} \int_{\sigma_{jk}}^{\sigma_{jk}^d} \mathcal{a}_{jk}^{(0)}(\tau) \exp[i\omega(\tau - \sigma_{jk})] \, d\tau \\
&= \frac{i}{2\pi} \exp(-i\omega\sigma_{jk}) \int_{-\infty}^{\infty} \frac{R_{jk}(0, \omega')}{\omega' - \omega} [e^{-i(\omega' - \omega)\sigma_{jk}^d} - e^{-i(\omega' - \omega)\sigma_{jk}}] \, d\omega'
\end{aligned} \tag{4.13}
$$

and:

$$
\begin{aligned}
\mathcal{a}_{jk}^{(n+1)}(\sigma_{jk}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\sigma_{jk}^d} \sum_{p=0}^{n} [\mathcal{a}_{jl}^{(p)} R_{lk}^{(n-p)} - R_{jl}^{(n-p)} \mathcal{a}_{lk}^{(p)} \\
+ \sum_{q=0}^{n-p} R_{jl}^{(n-p-q)} \mathcal{a}_{lh}^{(p)} R_{hk}^{(q)}] \exp[i\omega(\tau - \sigma_{jk})] \, d\tau \, d\omega
\end{aligned} \tag{4.14}
$$

$$R_{jk}^{(n+1)}(\sigma_{jk}, \omega) = \frac{1}{2} \int_{\sigma_{jk}}^{\sigma_{jk}^d} \mathcal{A}_{jk}^{(n)}(\tau) \exp[i\omega(\tau - \sigma_{jk})] \, d\tau$$

$$(4.15) \qquad - \frac{1}{2} \int_{\sigma_{jk}}^{\sigma_{jk}^d} \sum_{p=0}^{n} [\mathcal{A}_{jl}^{(p)} R_{lk}^{(n-p)} e^{-i\omega\sigma_{jl}} - R_{jl}^{(n-p)} \mathcal{A}_{lk}^{(p)} e^{-i\omega\sigma_{lk}}$$

$$+ \sum_{q=0}^{n-p} R_{jl}^{(n-p-q)} \mathcal{A}_{lh}^{(p)} R_{hk}^{(q)} e^{-i\omega\sigma_{lh}}] \exp(i\omega\tau) \, d\tau .$$

Equations (4.11)-(4.15) allow us to obtain $\mathcal{A}$ and $\mathbf{R}$ in terms of the quantities σ_j^d. Successive substitution into equation (4.8) yields a linear integral equation for $\mathbf{T}$, that can be solved iteratively as

$$(4.16) \qquad T_{jk}(\sigma_k, \omega) = \sum_n T_{jk}^{(n)}(\sigma_k, \omega)$$

with:

$$T_{jk}^{(0)}(\sigma_k, \omega) = \delta_{jk} \exp[i\omega(\sigma_k^d - \sigma_k)]$$

$$(4.17) \qquad T_{jk}^{(n+1)}(\sigma_k, \omega) = \frac{1}{2} \int_{\sigma_k}^{\sigma_k^d} T_{jl}^{(n)} [\overline{\mathcal{A}}_{lk}^k e^{-i\omega\sigma_k} - \mathcal{A}_{lh}^h R_{hk} e^{-i\omega\sigma_h}] \exp(i\omega\tau) \, d\tau .$$

In particular, equations (4.16) and (4.17), evaluated at $\sigma_j = 0$, $(j = 1, \dots, n)$ enable us to obtain σ_j^d, $(j = 1, \dots, n)$ once the transmission data $T_{jk}(0, \omega)$ are given.

Summarizing the results of this section, we have obtained a recipe to evaluate the material parameters $\mathcal{A}_{jk}(\sigma_{jk})$ by the knowledge of the reflection and transmission matrices $\mathbf{R}(0, \omega)$ and $\mathbf{T}(0, \omega)$, which can be extracted from the scattering data.

The convergence of the iterative solution (4.11) remains an open question also for the scalar case in a half-space, owing to the non linearity of (4.7) (see [4]). However, in connection to this point, a partial result can be obtained.

Theorem. *Let* $\mathbf{P}'\mathbf{P}^{-1}$ *and* $\mathbf{P}\mathbf{C}'$ *satisfy inequality* (3.11) *and let* $R_{jk}(\sigma_{jk}, \omega) \in L^2(\boldsymbol{R}^{++})$ *for any* $\sigma_{jk} \in (0, \sigma_{jk}^d)$. *If the series in* (4.12) *converge in the norm of* $L^2(\boldsymbol{R}^{++})$ *for* $\sigma_j = 0$, $(j = 1, \dots, n)$, *then the series in* (4.11) *converge in norm to* $\mathcal{A}_{jk}(\sigma_{jk})$.

Proof. We firstly note that the hypothesis (3.11) implies $\mathcal{A}_{jk} \in L^2(0, \sigma_{jk}^d)$ for any pair jk. Then, by evaluating (4.15) at $\sigma_{jk} = 0$ we get

$$\frac{1}{2} \int_0^{\sigma_{jk}^d} \sum_{p=0}^{n} [\mathcal{A}_{jl}^{(p)} R_{lk}^{(n-p)} - R_{jl}^{(n-p)} \mathcal{A}_{lk}^{(p)} + \sum_{q=0}^{n-p} R_{jl}^{(n-p-q)} \mathcal{A}_{lh}^{(p)} R_{hk}^{(q)}] \exp(i\omega\tau) \, d\tau$$

$$= \frac{1}{2} \int_0^{\sigma_{jk}^d} \mathcal{A}_{jk}^{(n)}(\tau) \exp(i\omega\tau) \, dt - R_{jk}^{(n+1)}(0, \omega) .$$

After substitution into (4.14), we obtain

$$(4.18)\quad \mathfrak{a}_{jk}^{(n+1)}(\sigma_{jk}) - \mathfrak{a}_{jk}^{(n)}(\sigma_{jk}) = -\frac{1}{\pi}\int_{-\infty}^{\infty} R_{jk}^{(n+1)}(0,\omega)\exp(-i\omega\sigma_{jk})\,d\omega\,.$$

Taking the norm of both sides in (4.18) and using the Parseval's formula for the Fourier transform, we arrive to

$$\|\mathfrak{a}_{jk}^{(n+1)} - \mathfrak{a}_{jk}^{(n)}\| = \sqrt{\frac{2}{\pi}}\,\|R_{jk}^{(n+1)}(0,\omega)\|\,.$$

Hence, by the convergence of $\Sigma_n R_{jk}^{(n)}(0,\omega)$ and the completeness of $L^2(0,\sigma_{jk}^d)$ we obtain the convergence of $\Sigma_n \mathfrak{a}_{jk}^{(n)}$ in $(0,\sigma_{jk}^d)$.

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Sommario

Si studiano le onde armoniche che si propagano in un materiale disomogeneo perpendicolarmente ai suoi piani di stratificazione. Il problema viene formulato per un campo ad n componenti generalizzando alcuni risultati precedentemente ottenuti in ambito elettromeccanico. Mediante la tecnica di wave splitting *si ottiene un sistema di equazioni integrali lineari e si stabilisce un teorema di convergenza per la sua soluzione. Si considera poi l'approccio di* invariant imbedding *al sistema di equazioni differenziali di partenza per una possibile formulazione del problema inverso.*

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