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On the Selberg integral via Heath-Brown's identity (**)

1 - Introduction

The object of this paper is to prove a well-known result of prime number theory by means of Heath-Brown's identity. We deal with the Selberg integral

$$J(X, h) = \int_X^{2X} |\psi(t) - \psi(t-h) - h|^2 \, dt .$$

It is well-known that the current density estimates yield the following

Theorem. *If* $h \geqslant X^{\frac{1}{6}+\varepsilon}$ *for some fixed* $\varepsilon > 0$, *then* $J(X, h) = o(Xh^2)$.

For this, see for instance B. Saffari and R. C. Vaughan [4], Lemma 5. D. R. Heath-Brown [1] proved Huxley's Theorem [2] that one has $\psi(X) - \psi(X-h) \sim h$ provided that $h \geqslant X^{\frac{7}{12}+\varepsilon}$, by means of his identity (see Lemma 1 of [1], or Lemma 2 below), thereby avoiding a direct appeal to the properties of the zeros of the Riemann zeta-function, except for Vinogradov's zero-free region.

We extend this approach to the above integral. It will be apparent from the proof that the same result holds provided that $\varepsilon = \varepsilon(X) \geqslant A(\log\log X)^{\frac{4}{3}} (\log X)^{-\frac{1}{3}}$ for a sufficiently large $A > 0$.

2 - Preliminaries

In what follows we shall assume that X is sufficiently large. Our implicit constants may depend on the parameter k (see Lemma 2 below), and we shall eventually choose $k = 5$. For the notations see D. R. Heath-Brown [1]. $\mathcal{L}$ denotes $\log X$.

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Lemma 1. *The Theorem follows from the estimate*

$$\int_X^{2X} |\psi(t) - \psi(t-\theta t) - \theta t|^2 \, dt = o(\theta^2 X^3)$$

uniformly for $X^{-\frac{5}{6}+\varepsilon} \leqslant \theta \leqslant 1$.

Proof. See the proof of Lemma 6 of B. Saffari and R. C. Vaughan [4].

Lemma 2 (Heath-Brown's identity). *For any integer* $k \geqslant 1$ *we have*

$$-\frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{k} (-1)^j \binom{k}{j} \zeta'(s)\zeta^{j-1}(s) M^j(s) - \frac{\zeta'}{\zeta}(s)(1 - M(s)\zeta(s))^k . \tag{1}$$

This is Lemma 1 of [1]. In Heath-Brown's identity we choose

$$M(s) = \sum_{n \leqslant y} \frac{\mu(n)}{n^s} \quad \text{where } y^k = 2X.$$

Obviously, all coefficients of terms $n \leqslant 2X$ of the last summand in (1) vanish, because

$$\zeta(s)M(s) = 1 + \sum_{n \geqslant 2} n^{-s} \sum_{\substack{d \mid n \\ d \leqslant y}} \mu(d).$$

This means that when $t \in [X, 2X]$ the sum

$$S(t, \theta) = \psi(t) - \psi(t - \theta t) = \sum_{t - \theta t < n \leqslant t} \Lambda(n)$$

is equal to the sum of the coefficients with $n \in]t - \theta t, t]$ of the sum over j in (1).

For $j \in \{1, \dots, k\}$ we define $\mathfrak{A}^j$ to be the set of all $2j$-tuples $N^j = (N_1, \dots, N_{2j})$ such that $N_i \geqslant \frac{1}{2}$ for all $i = 1, \dots, 2j$, and $2^r N_i = X$ for a suitable non-negative integer r if $i \leqslant j$, and $2^r N_i = y$ if $i > j$. Denote by $\mathfrak{A}$ the union of all $\mathfrak{A}^j$, for $j \in \{1, \dots, k\}$. Since $|\mathfrak{A}| \ll \mathscr{L}^{2k}$, we see that $S(t, \theta)$ is a linear combination of $O(\mathscr{L}^{2k})$ sums of the type

$$\Sigma(N^j, t, \theta) = \sum_{\substack{n_i \in]N_i, 2N_i] \forall i = 1, \dots, 2j \\ t - \theta t < n_1 \dots n_{2j} \leqslant t}} (\log n_1) \mu(n_{j+1}) \dots \mu(n_{2j})$$

where N^j rangers over $\mathfrak{A}$. For the sake of definiteness, set

$$S(t, \theta) = \sum_{j=1}^{k} \alpha(j, k) \sum_{N^j \in \mathfrak{A}^j} \Sigma(N^j, t, \theta)$$

where $\alpha(j, k) \ll 1$.

Our aim is to prove that each Σ can be written in the following way

$$\Sigma(N^j, t, \theta) = \theta \mathfrak{M}(N^j, t) + \mathfrak{R}(N^j, t, \theta) \tag{2}$$

where $\mathfrak{M}(N^j, t)$ is independent of θ. In fact, if (2) holds for suitable $\mathfrak{M}$ anf $\mathfrak{R}$, we let

$$\mathfrak{M}(t) = \sum_{j=1}^{k} \alpha(j, k) \sum_{N^j \in \mathfrak{N}^j} \mathfrak{M}(N^j, t) \quad \text{and} \quad \mathfrak{R}(t, \theta) = \sum_{j=1}^{k} \alpha(j, k) \sum_{N^j \in \mathfrak{N}^j} \mathfrak{R}(N^j, t, \theta)$$

so that $S(t, \theta) = \psi(t) - \psi(t - \theta t) = \theta \mathfrak{M}(t) + \mathfrak{R}(t, \theta)$. Then we have

$$\int_X^{2X} |S(t, \theta) - \theta t|^2 \, dt = \int_X^{2X} (\theta^2 (\mathfrak{M}(t) - t)^2 + 2(S(t, \theta) - \theta t) \mathfrak{R}(t, \theta) - \mathfrak{R}(t, \theta)^2) \, dt .$$

We set $H(X, \theta) = \int_X^{2X} |\mathfrak{R}(t, \theta)|^2 \, dt$. Applying the Brun-Titchmarsh and the Cauchy inequalities to the second summand on the right, we have

$$(3) \quad \int_X^{2X} |S(t, \theta) - \theta t|^2 \, dt = \theta^2 \int_X^{2X} (\mathfrak{M}(t) - t)^2 \, dt + O(H(X, \theta) + X^{\frac{3}{2}} \theta H(X, \theta)^{\frac{1}{2}}) .$$

Hence, recalling Lemma 1 and the fact that $|\mathfrak{N}| \ll \mathcal{L}^{2k}$, we have proved

Lemma 3. *The Theorem follows from the estimates*

$$(4) \qquad \int_X^{2X} (\mathfrak{M}(t) - t)^2 \, dt = o(X^3)$$

$$(5) \qquad \max_{N \in \mathfrak{N}} \int_X^{2X} |\mathfrak{R}(N, t, \theta)|^2 \, dt = o(X^3 \theta^2 \mathcal{L}^{-2k})$$

uniformly for $X^{-\frac{5}{6} + \varepsilon} \leqslant \theta \leqslant 1$.

We shall prove the first part of Lemma 3 by taking θ *large*, whereas the proof of the second estimate is achieved by means of a mean value estimate as described below.

We denote by $d_m(n)$ the coefficient of n^{-s} in $\zeta^m(s)$. The following result is a consequence of Theorem 2 of P. Shiu [5].

Lemma 4. *For fixed* $\delta > 0$ *and any non-negative integer* m *we have*

$$\sum_{x \leqslant n \leqslant x + y} d_m(n) \ll_{\delta, m} y (\log x)^{m-1}$$

uniformly for $x^{\delta} \leqslant y \leqslant x$.

3 - Reduction to mean-value estimates

Our approximation of the type (2) shall yield three error terms: obviously, it is enough to prove that (5) holds for each one separately. Actually, for all but the last error term we shall prove the stronger inequality

(6) $$\max_{X \leqslant t \leqslant 2X} \max_{N \in \mathfrak{N}} |\Re(N, t, \theta)| = o(X\theta \mathcal{L}^{-k}).$$

For $N \in \mathfrak{N}^j$ we put $f_r(s) = \sum_{n \in]N_r, 2N_r]} a_r(n) n^{-s}$, where $a_1(n) = \log n$, $a_r(n) = 1$ for $r = 2, \ldots, j$ and $a_r(n) = \mu(n)$ otherwise. Now set

$$F(s) = F_N(s) = \prod_{r=1}^{2j} f_r(s) = \sum_{n \leqslant X} \frac{c_n}{n^s}$$

say, where $|c_n| \leqslant \mathrm{d}_{2j}(n)\mathcal{L}$. We remark that we may assume that $\prod_i N_i \geqslant 2^{-(2j+1)}X$, since otherwise $\Sigma = 0$ and (2) is trivial. Thus $c_n \neq 0$ only for $n \in I(X) = [2^{-(2j+1)}X, 2^{2j}X]$.

Observe that for $s = \frac{1}{2} + i\tau$ and $2 \leqslant r \leqslant 2j$ we have

(7) $$|f_1(s)| \ll N_1^{\frac{1}{2}} \mathcal{L} \qquad |f_r(s)| \ll N_r^{\frac{1}{2}} \qquad \text{so that} \quad |F(s)| \ll X^{\frac{1}{2}} \mathcal{L}.$$

By Perron's formula we have

$$\Sigma(N) = \frac{1}{2\pi i} \int_{\frac{1}{2} - iT_1}^{\frac{1}{2} + iT_1} F(s) \frac{t^s - (t - \theta t)^s}{s} \mathrm{d}s$$

$$+ O\Big(\sum_{n \in I(X)} |c_n| \Big[\min\Big(1, T_1^{-1} \Big|\log \frac{t}{n}\Big|^{-1}\Big) + \min\Big(1, T_1^{-1} \Big|\log \frac{t - \theta t}{n}\Big|^{-1}\Big)\Big]\Big).$$

We choose $\Delta = \dfrac{\varepsilon}{6k}$ and $T_1 = X^{\frac{5}{6} - 5k\Delta}$ and deal with the first summand in the error term. For the sake of brevity, for any non-negative interger r let $I_r = \{n \in I(X) : rT_1^{-1} \leqslant |\log \frac{t}{n}| < (r+1)T_1^{-1}\}$. Observe that $I_r \neq \emptyset$ only for $0 \leqslant r \leqslant M$, say, with $M \ll T$. We then have

$$\sum_{n \in I(X)} |c_n| \min\Big(1, T_1^{-1} \Big|\log \frac{t}{n}\Big|^{-1}\Big) \ll \sum_{n \in I_0} |c_n| + \sum_{1 \leqslant r \leqslant M} \sum_{n \in I_r} T_1^{-1} |c_n| \Big|\log \frac{t}{n}\Big|^{-1}$$

$$\ll \sum_{n \in I_0} |c_n| + \sum_{1 \leqslant r \leqslant M} \sum_{n \in I_r} T_1^{-1} |c_n| (rT_1^{-1})^{-1} \ll \sum_{n \in I_0} |c_n| + \sum_{1 \leqslant r \leqslant M} r^{-1} \sum_{n \in I_r} |c_n|.$$

Furthermore $tT_1^{-1} \ll |I_r| \ll tT_1^{-1}$ for all $r \leqslant M$, and (6) follows using Lemma 4.

The other summand in the error term is dealt with in the same way.

The main term of $\Sigma(N)$ will come from a short interval: for $s = \frac{1}{2} + i\tau$ and $|\tau| \leqslant T_0$ we have

$$\frac{t^s - (t - \theta t)^s}{s} = \theta t^s + O(|s| \theta^2 t^{\frac{1}{2}}).$$

Hence by (7)

$$\frac{1}{2\pi i}\int_{\frac12 - iT_0}^{\frac12 + iT_0} F(s)\,\frac{t^s - (t-\theta t)^s}{s}\,ds = \theta\,\frac{1}{2\pi i}\int_{\frac12 - iT_0}^{\frac12 + iT_0} F(s)\,t^s\,ds + O(T_0^2\theta^2 X\mathcal{L}).$$

We can obviously take as $\mathfrak{M}(N, t)$ the integral on the right hand side, and the error term will satisfy (6) provided that T_0 depends only on X and

(8) $$\theta = o(T_0^{-2}\mathcal{L}^{-(k+1)})$$

which we now assume. Then (5) will follow from the estimate

(9) $$\max_{N\in\mathfrak{A}}\int_X^{2X}\Big|\int_{T_0}^{T_1} F_N\big(\tfrac12 + i\tau\big)\,\frac{t^{\frac12 + i\tau} - (t-\theta t)^{\frac12 + i\tau}}{\frac12 + i\tau}\,d\tau\Big|^2 dt = o(X^3\theta^2\mathcal{L}^{-2k}).$$

For the sake of brevity write $s_r = \dfrac12 + i\tau_r$ for $r = 1, 2$, and similarly for s. We set

$$J(N) = \int_X^{2X}\Big|\int_{T_0}^{T_1} F(s)\,\frac{t^s - (t-\theta t)^s}{s}\,d\tau\Big|^2 dt = \iint_{[T_0,\,T_1]^2} F(s_1)\,\overline{F(s_2)}\,c(X, \theta, \tau_1, \tau_2)\,d\tau_1\,d\tau_2$$

where

$$c(X, \theta, \tau_1, \tau_2) = \frac{1-(1-\theta)^{\frac12 + i\tau_1}}{\frac12 + i\tau_1}\,\frac{1-(1-\theta)^{\frac12 - i\tau_2}}{\frac12 - i\tau_2}\int_X^{2X} t^{1 + i(\tau_1 - \tau_2)}\,dt \ll \frac{(X\theta)^2}{1 + |\tau_1 - \tau_2|}.$$

By the Cauchy-Schwarz inequality we have

$$J(N) \ll (X\theta)^2 \iint_{[T_0,\,T_1]^2} \frac{|F(s_1)F(s_2)|}{1 + |\tau_1 - \tau_2|}\,d\tau_1\,d\tau_2$$

(10) $$\ll (X\theta)^2\Big\{\iint_{[T_0,\,T_1]^2}\frac{|F(s_1)|^2\,d\tau_1\,d\tau_2}{1 + |\tau_1 - \tau_2|}\iint_{[T_0,\,T_1]^2}\frac{|F(s_2)|^2\,d\tau_1\,d\tau_2}{1 + |\tau_1 - \tau_2|}\Big\}^{\frac12}$$

$$\ll (X\theta)^2\int_{T_0}^{T_1}|F(s_1)|^2\,d\tau_1\int_{T_0}^{T_1}\frac{d\tau_2}{1 + |\tau_1 - \tau_2|} \ll (X\theta)^2\,\mathcal{L}\int_{T_0}^{T_1}|F(s)|^2\,d\tau$$

$$\ll (X\theta)^2\mathcal{L}^2 \max_{T_0 \le T \le T_1}\int_T^{2T}|F(s)|^2\,d\tau.$$

Thus (9) is a consequence of (10) and of the following result, whose proof is deferred to the next section.

Lemma 5. *If* (8) *holds for* $T_0 = \exp(\mathcal{L}^{\frac{1}{3}})$ *and* $\theta \geq X^{-\frac{5}{6}+\varepsilon}$, *we have*

$$\max_{N \in \mathfrak{A}} \int_T^{2T} |F_N(\frac{1}{2} + i\tau)|^2 \, d\tau = o(X\mathcal{L}^{-(2k+2)})$$

uniformly for $T_0 \leq T \leq T_1$.

4 - Proof of Lemma 5

The proof is very similar to the proof of Lemma 3 in [1]. For the sake of brevity we do not duplicate the whole argument, but merely outline it, giving the needed modifications.

We shall say that a set $\mathcal{S}$ of points in $[T, 2T]$ is *well spaced* if $|\tau_m - \tau_n| \geq 1$ for every $\tau_m, \tau_n \in \mathcal{S}$ with $n \neq m$. For brevity, we write $s = s(\tau) = \frac{1}{2} + i\tau$ and similarly $s_r = \frac{1}{2} + i\tau_r$. We first write F as the product of F_1 and F_2, where $F_2(s)$ is the product of all factors f_h of F with $N_h \leq X^{\Delta}$. Let Y denote the product of all N_h when f_h is a factor of F_1 and Z denote the product of all N_h when f_h is a factor of F_2, so that $Z \leq X^{2k\Delta}$ and $YZ \ll X$. Hence, by (7) we have $|F_2(s(\tau))| \ll Z^{\frac{1}{2}} \mathcal{L}$ and

$$\int_T^{2T} |F_N(s(\tau))|^2 \, d\tau \ll Z\mathcal{L}^2 \int_T^{2T} |F_1(s(\tau))|^2 \, d\tau = Z\mathcal{L}^2 J(X, T) \tag{11}$$

say. Now there exists a set $\mathcal{S}$ with $|\mathcal{S}| \ll T$ of well-spaced points τ_n in $[T, 2T]$ such that

$$J(X, T) \ll \sum_{\tau_n \in \mathcal{S}} |F_1(s_n)|^2.$$

For each factor f_h of F_1 set

$$|f_h(s_n)| = N_h^{\sigma(h, n) - \frac{1}{2}}.$$

Then Heath-Brown proves that

$$\sigma(h, n) \leq 1 - \eta(T_1), \qquad \text{where} \qquad \eta(T_1) = 2C(\log T_1)^{-\frac{2}{3}}(\log\log T_1)^{-\frac{1}{3}} \tag{12}$$

and C is a suitable absolute constant. This follows from Richert's form of the Korobov-Vinogradov zero-free region for the Riemann zeta-function. We then set $L = [\mathcal{L}]$ and split the range for σ into subranges I_l for $0 \leq l \leq L$ as follows

$$I_0 =]-\infty, \frac{1}{2}] \qquad I_l =]\frac{1}{2} + \frac{l-1}{L}, \frac{1}{2} + \frac{l}{L}] \quad \text{for } 1 \leq l \leq L.$$

We divide the points τ_n into classes $C(h, l)$ in the following way

$$C(h, l) = \{\tau_n \in \mathcal{S} \colon \sigma(h, n) = \max_{1 \leq r \leq 2j} \sigma(r, n) \text{ and } \sigma(h, n) \in I_l\},$$

so that the classes $C(h, l)$ are not necessarily disjoint. Since there are $O(\mathcal{L})$ classes, there is a class, $C(h', l')$ say, such that

$$J(X, T) \ll \mathcal{L} \sum_{\tau \in C(h', l')} |F_1(s(\tau))|^2 .$$

Note that if $\tau_n \in C(h', l')$ then

$$|F_1(s(\tau_n))|^2 = \prod_{h=1}^{2j} N_h^{2\sigma(h,n)-1} \leqslant \prod_{h=1}^{2j} N_h^{2l'L^{-1}} \leqslant Y^{2l'L^{-1}} .$$

If $l' = 0$ then $J(X, T) \ll T\mathcal{L}$, and Lemma 5 follows from (11) since $Z \leqslant X^{2k\Delta}$.

If $l' \geqslant 1$ we set

$$\sigma = \frac{1}{2} + \frac{l'-1}{L} \qquad f(s) = f_{h'}(s) \qquad N = N_{h'} \qquad R = |C(h', l')| .$$

We relabel the points τ_n so that $C(h', l') = \{\tau_n : 1 \leqslant n \leqslant R\}$. Thus

(13) $\quad J(X, T) \ll Y^{2\sigma-1} R\mathcal{L} \quad$ while $\quad |f(s_n)| \gg N^{\sigma - \frac{1}{2}}$ for $1 \leqslant n \leqslant R$.

The mean and large values technique of Montgomery [3] and Lemma 4 yield

$$R \ll T^{\frac{12}{5}(1-\sigma)} \mathcal{L}^A \qquad \text{for } N \leqslant T^{\frac{3}{5}}$$

for some fixed $A > 0$. This is proved by means of mean-value estimates for Dirichlet polynomials (see e.g. Theorem 7.3 of [3]) for the range $[\frac{1}{2}, \frac{3}{4}]$, and the Halász Lemma (see e.g. 2.9 of [2]) for the range $[\frac{3}{4}, 1]$. This implies that for $N \leqslant T^{\frac{3}{5}}$ we have

$$\int_T^{2T} |F_N(s(\tau))|^2 \, d\tau \ll Z\mathcal{L}^{A+3} Y^{2\sigma-1} T^{\frac{12}{5}(1-\sigma)} \ll X^{1-8k\Delta(1-\sigma)} \mathcal{L}^{A+3}$$

using (11) and (13). Now (12) easily implies that

$$X^{-8k\Delta(1-\sigma)} \mathcal{L}^{A+3} \ll \mathcal{L}^{-(2k+2)}$$

and Lemma 5 follows.

In the case $N \geqslant T_1^{\frac{3}{5}}$, we assume that $\Delta \leqslant \frac{1}{4k}$, so that $X \leqslant T_1^{\frac{12}{5}}$ and $N \geqslant X^{\frac{1}{4}}$. This means that $f = f_h$ for some $h \leqslant j$, provided that we choose $k = 5$. Then Hölder's inequality and Ingham's fourth power moment estimate for $\zeta(\frac{1}{2} + it)$ (see e.g. Titchmarsh [6], (7.6.1)) yield

$$RN^{4\sigma-2} \ll (T + R + RN^2 T^{-4}) \mathcal{L}^{13} \ll (T + RN^2 T^{-4}) \mathcal{L}^{13}$$

since $R \ll T$. From this we deduce at once that either

$$R \ll TN^{2-4\sigma} \mathcal{L}^{13} \ll T_1^{\frac{12}{5}(1-\sigma)}$$

and Lemma 5 follows as above, or

$$N^{4\sigma-2} \ll N^2 T^{-4} \mathcal{L}^{13}$$

which means that $T \ll N^{1-\sigma} \mathcal{L}^4$, and finally

$$R \ll T \ll N^{1-\sigma} \mathcal{L}^4 \ll X^{1-\sigma} \mathcal{L}^4 \ll T_1^{\frac{12}{5}(1-\sigma)} \mathcal{L}^4$$

and Lemma 5 follows in this case as well.

6 - Completion of the proof of Lemma 3: the main term

Our estimates thus far have been uniform for $\theta \geqslant X^{-\frac{5}{6}+\varepsilon}$, provided that (8) holds. In particular, we may take $\theta_0 = \exp\{-\sqrt{\mathcal{L}}\}$. In this case, we see that the left hand side of (3) is $o(X^3 \theta_0^2)$ by the Prime Number Theorem, and the remainder term on the right is also $o(X^3 \theta_0^2)$ by (5). Thus (4) follows and the proof of the Theorem is complete.

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Sommario

Diamo una dimostrazione alternativa della stima $J(X, h) = o(Xh^2)$ *per l'integrale di Selberg*

$$J(X, h) = \int_X^{2X} |\psi(t) - \psi(t-h) - h|^2 \, dt$$

quando $h \geqslant X^{\frac{1}{6}+\varepsilon}$, *per mezzo di una identità di Heath-Brown.*

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