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Some conditions for commutativity of periodic rings (**)

1 - Introduction

Throughout, R will represent an associative ring with center C. If $(x_i)_{i \in N}$ is a sequence of elements of R and k is a positive integer we define $[x_1, \ldots, x_{k+1}]$ inductively as follows:

$$[x_1, x_2] = x_1 x_2 - x_2 x_1$$

$$[x_1, \ldots, x_k, x_{k+1}] = [[x_1, \ldots, x_k], x_{k+1}].$$

If $x_1 = x$ and $x_2 = \ldots = x_{k+1} = y$, we write $[x_1, \ldots, x_{k+1}] = [x, y]_k$. Also for $k = 0$ we define $[x, y]_k = x$.

By a ring R *with torsion-free commutators*, we mean that $m[x, y] = 0$ implies $[x, y] = 0$ for all $m \geq 1$, $x, y \in R$.

Following [2] a ring R is said to be *period* if for any x in R there exist two distinct positive integers m, n such that $x^m = x^n$.

A ring R is called *left* (resp. *right*) *s-unital* [5] if for each $x \in R$ we have $x \in Rx$ (resp. $x \in xR$). A ring R is called *s-unital* if for each x in R, $x \in Rx \cap xR$. If R is an s-unital ring, then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ for all x in F (see [5]).

Recently A. Giambruno, J. Z. Goncalves and A. Mandel [3] proved that, if D is a division ring with center C such that there exists a fixed integer r_0, so that for each pair x, y in D there are integers $k = k(x, y) \geq 1$, $m = m(x, y) \leq r_0$ and $n = n(x, y) > 1$ satisfying $[x, y^m]_k^n - [x, y^m]_k \in C$, then D is commutative.

The objective of this paper is to prove a commutativity theorem for periodic rings which satisfy an analogous property. Indeed we generalize a well-known result of N. Jacobson [4] by proving

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Theorem 1. *Let R be a periodic left s-unital ring with torsion-free commutators. If for all x, y in R there exist positive integers k and $n > 1$, depending on x and y, such that $[x, y]_k^n - [x, y]_k \in C$, then R is commutative and every prime ideal in R is maximal.*

2 - Preliminary lemmas

In preparation for the proof of our main theorem we first recall the following lemmas.

Lemma 1 [5]. *Let R be a left s-unital ring. If for e, x, y in R, $ex = x$, $ey = y$ and $x^m y = (x + e)^m y = 0$, then $y = 0$.*

Lemma 2 [1]. *Let R be a ring such that $[x,[x, y]] = 0$ for all x, $y \in R$, then $[x^k, y] = kx^{k-1}[x, y]$ for any positive integer k.*

The following lemma is proved in [2], however for the sake of the self-containedness, we shall sketch the proof.

Lemma 3. *If R is a periodic ring, then for each $x \in R$ some power of x is idempotent.*

Proof. If $x^n = x^m$ with $n > m$, then $x^{n-m+1} = (x^{n-m+1})^{n-m+1}$ and hence $(x^{n-m+1})^{n-m}$ is idempotent.

Lemma 4. *Let R be a periodic left s-unital ring. Then for each x in R there exists an element e in R such that $e^2 = e$, $ex = x$.*

Proof. Let $x \in R$, then $cx = x$ for some $c \in R$. But by Lemma 3, there exists an integer $k \geqslant 1$ such that $c^{2k} = c^k$. Setting $c^k = e$, we have $ex = x$, $e^2 = e$.

Lemma 5. *Let R be a ring with the property*

P. *For all x, $y \in R$ there exist positive integers $m = m(x, y) \geqslant 1$, $k = k(x, y) \geqslant 1$ and $n = n(x, y) \geqslant 2$ such that $[x, y^m]_k^n - [x, y^m]_k \in C$.*

Then every idempotent element is central.

Proof. Let $e, x \in R$, $e^2 = e$. Then it is easy to see that for any $k \geqslant 1$ we have $[xe - exe, e]_k = xe - exe$. Thus **P** implies that $(xe - exe)^n - (xe - exe) \in C$ for some $n > 1$. But obviously $(xe - exe)^2 = 0$, hence $exe - xe \in C$. Therefore

$exe - xe = [exe - xe, e] = 0$, i.e. $exe = xe$. Similarly $exe = ex$. Whence $xe = ex$, i.e. $e \in C$ as desired.

Lemma 6. *Let R be a periodic left s-unital ring satisfying the hypothesis of Lemma 5; then*

i. *the commutator ideal is nil*

ii. *for all x, y in R, $[x, y^m]_k \in C$ for the same $m \geqslant 1$, $k \geqslant 1$*

iii. *R is an s-unital ring.*

Proof.

i. Let $x \in R$. By Lemma 4, $x^{2k} = x^k$ for some $k \geqslant 1$. In view of Lemma 5, this implies that $x^k \in C$ and therefore by a well-known theorem of I. N. Herstein [4] the commutator ideal is nil.

ii. Let $x, y \in R$, then $[x, y^m]_k^n - [x, y^m]_k \in C$ for some $m \geqslant 1$, $k \geqslant 1$, $n > 1$, by the hypothesis **P** of Lemma 5. On the other hand, **i** yields that $[x, y^m]_k^{n_0} = 0$ for some $n_0 \geqslant 1$. If $n \geqslant n_0$ then $[x, y^m]_k \in C$, as desired. So let $1 < n < n_0$, and set $a = [x, y^m]_k$, then $a^{n_0 - 1} = \pm(a^n - a)^{n_0 - 1} \in C$; and therefore from the fact that $(a^n - a)^{n_0 - 2} \in C$ we deduce that $a^{n_0 - 2} \in C$. Now it is easy to see that $a \in C$.

iii. By Lemma 4, for each $x \in R$ there exists an idempotent element $e \in R$ such that $x = ex$. But by Lemma 5 we have $e \in C$; hence R is an s-unital ring.

With the above lemmas established, we are able to complete the proof of Theorem 1.

Proof of Theorem 1. Let $x, y \in R$, then $[x, y]_k \in C$, by Lemma 6 **ii** (note that here $m = 1$). Setting $a = [x, y]_{k-1}$, we have $[[a, y], y] = 0$. But since R is periodic hence $y^{2m} = y^m$ for some $m \geqslant 1$ and therefore $[a, y^m] = 0$, by Lemma 5. Thus by Lemma 2, $my^{m-1}[a, y] = 0$. Replacing y by $y + e$ we conclude that, for some $m' \geqslant 1$, we can write

$$0 = m'(y + e)^{m'-1}[a, y + e] = m'(y + e)^{m'-1}[a, y].$$

Put $m_0 = \max\{m - 1, m' - 1\}$ and $p = mm'$. Then $y^{m_0}p[a, y] = 0$ and also $(y + e)^{m_0}p[a, y] = 0$. Thus, by Lemma 1, $p[a, y] = 0$, i.e. $p[x, y]_k = 0$. This implies that $[x, y]_k = 0$, by the hypothesis.

We have proved that if $[x, y]_{k+1} = 0$, then $[x, y]_k = 0$. Now by induction on k we get $[x, y] = 0$. Thus R is commutative.

To complete the proof, let P be a prime ideal in R, and M be an ideal of R such that $P \subset M \subseteq R$. Let $x \in M \setminus P$ and y be an arbitrary element in R. Since R is a commutative s-unital periodic ring, hence $x^m = x^n$ for some $n > m$, $xe = x$, $ye = y$ for some $e \in R$. Thus $x^m(e - x^{n-m}) = 0 \in P$. But P is a prime ideal and $x \in M \setminus P$, therefore $e - x^{n-m} \in P$. This yields that $e \in M$, and so $y \in M$. Which means that $R = M$. Thus P is a maximal ideal.

Remarks. Let F be a field. Consider the non-commutative ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \text{ with } a, b, c, d \in F \right\}.$$

The case $F = GF(2)$ shows that the torsion-freeness of the commutators is essential in Theorem 1. The case $F = Z$ shows that if R is not periodic the result of Theorem 1 does not hold.

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Sommario

In questo lavoro, per gli anelli a commutatori liberi da torsione, si stabilisce un teorema che generalizza un noto risultato di N. Jacobson.

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