Riv. Mat. Univ. Parma (5) **5** (1996), 217-222

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Homogeneous product-sum polynomials and combinatorial identities (**)

Symmetric functions have important applications to algebraic computation and combinatorial enumeration (cf. [1], [5]). Two fundamental bases may be stated in

Definition. For complex indeterminates $\{x_k\}_{k>0}$, a pair of symmetric polynomials are defined as follows:

Complete symmetric function (homogeneous product-sum polynomial)

(1) $$H_n^m(x_1, x_2, \ldots, x_m) = \sum_{\substack{k_1+k_2+\ldots+k_m=n \\ 0\le k_i<\infty, (i=1,2,\ldots,m)}} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}.$$

Elementary symmetric function

(2) $$\Lambda_n^m(x_1, x_2, \ldots, x_m) = \sum_{\substack{k_1+k_2+\ldots+k_m=n \\ 0\le k_i\le 1, (i=1,2,\ldots,m)}} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}.$$

The purpose of this note is to introduce their properties and basic relations. Applications to combinatorial identities will be sketched.

Theorem 1 (L. C. Biedenharn and J. D. Louck [2]). *Let $\{x_k\}$ be distinct complex numbers. H_n^m can be expressed as divided differences.*

(3) $$H_n^m(x_1 x_2, \ldots, x_m) = \sum_{k=1}^{m} \frac{x_k^{m+n-1}}{\prod_{\substack{i=1 \\ i\ne k}}^{m} (x_k - x_i)}.$$

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(**) Received November 25, 1996. AMS classification 05 E 05. Partially supported by IAMI-CNR.

Proof. Recall the generating function for complete symmetric functions and the expansion in partial fractions

$$\sum_{n \geqslant 0} H_n^m(x_1, x_2, \ldots, x_m)\, t^n = \prod_{k=1}^{m} \frac{1}{1 - tx_k} = \sum_{k=1}^{m} \frac{x_k^{m-1}}{1 - tx_k} \prod_{\substack{i=1 \\ i \neq k}}^{m} (x_k - x_i)^{-1}.$$

The coefficient of t^n in the expansion reads as the divided differences for H_n^m stated in the theorem.

This derivation is simpler than the analytic proof due to L. C. Biedenharn and J. D. Louck [2].

Theorem 2 (Shifted parameters). *For complex numbers $\{x_k\}$ and c, there holds*

$$(4) \qquad H_n^m(x_1 + c, x_2 + c, \ldots, x_m + c) = \sum_{k=0}^{n} c^{n-k} \binom{m+n-1}{n-k} H_k^m(x_1, x_2, \ldots, x_m).$$

Proof. Using the binomial expansion for the definition of H_n^m, we have

$$H_n^m(x_1 + c, x_2 + c, \ldots, x_m + c)$$

$$= \sum_{\substack{k_1 + k_2 + \ldots + k_m = n \\ 0 \leqslant k_i < \infty, (i = 1, 2, \ldots, m)}} (x_1 + c)^{k_1} (x_2 + c)^{k_2} \ldots (x_m + c)^{k_m}$$

$$= \sum_{\substack{k_1 + k_2 + \ldots + k_m = n \\ 0 \leqslant k_i < \infty, (i = 1, 2, \ldots, m)}} \sum_{\substack{j_i = 0 \\ (i = 1, 2, \ldots, m)}}^{k_i} \prod_{i=1}^{m} \binom{k_i}{j_i} x_i^{j_i} c_i^{k_i - j_i}.$$

Changing the summation order and noticing that

$$\sum_{\substack{k_1 + k_2 + \ldots + k_m = n \\ 0 \leqslant k_i < \infty, (i = 1, 2, \ldots, m)}} \prod_{i=1}^{m} \binom{k_i}{j_i} = \binom{m+n-1}{n - \sum_{i=1}^{m} j_i}$$

we get

$$H_n^m(x_1 + c, x_2 + c, \ldots, x_m + c) = \sum_{\substack{j_i \geqslant 0 \\ (i = 1, 2, \ldots, m)}} \binom{m+n-1}{n - \sum_{i=1}^{m} j_i} c^{n - \sum_{i=1}^{m} j_i} \prod_{i=1}^{m} x_i^{j_i}$$

which becomes the desired formula (4) with the shifted parameters after replacement $k = \sum_{i=1}^{m} j_i$ is performed.

Proposition 1 (Recurrence relation). *For complex numbers* $\{x_k\}$, *we have*

$$H_n^{m+1}(x_0, x_1, \ldots, x_m) = \sum_{k=0}^{n} x_0^k H_{n-k}^m(x_1, x_2, \ldots, x_m). \tag{5}$$

Proof. It is an immediate consequence of the definition.

Proposition 2 (Convolution formula). *For complete symmetric functions, we have*

$$\begin{aligned} &H_n^{m+p}(x_1, x_2, \ldots, x_{m+p}) \\ &= \sum_{k=0}^{n} H_k^m(x_1, x_2, \ldots, x_m) H_{n-k}^p(x_{m+1}, x_{m+2}, \ldots, x_{m+p}). \end{aligned} \tag{6}$$

Proof. It is also a direct consequence of the definition, which may be considered as an extension of Proposition 1.

Proposition 3 (Alternating summation). *Complete symmetric and elementary symmetric functions are connected by relation*

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k \Lambda_k^m(x_1, x_2, \ldots, x_m) H_{n-k}^p(x_1, x_2, \ldots, x_p) \\ &= \begin{cases} \Lambda_n^{m-p}(x_{p+1}, x_{p+2}, \ldots, x_m)(-1)^n & m > p \\ H_n^{p-m}(x_{m+1}, x_{m+2}, \ldots, x_p) & m \leqslant p. \end{cases} \end{aligned} \tag{7}$$

Proof. Notice the generating functions:

$$\sum_{n \geqslant 0} (-1)^n \Lambda_n^m(x_1, x_2, \ldots, x_m) t^n = \prod_{i=1}^{m} (1 - tx_i)$$

$$\sum_{n \geqslant 0} H_n^p(x_1, x_2, \ldots, x_p) t^n = \prod_{i=1}^{p} \frac{1}{1 - tx_i}.$$

The coefficient of t^n in the expansion of their product results in the convolution between $\{\Lambda_i^m\}_i$ and $\{H_j^p\}_j$ from the first member of (7). It reduces to the second member in view of the definition according to $m > p$ or $m \leqslant p$.

Symmetric functions are symbolic generalizations of many classical numbers. Some typical instances are displayed from the following specific settings:

Example 1 (Binomial coefficients).

$$\binom{m}{n} = \Lambda_n^m(1, 1, \dots, 1) = H_{m-n}^{1+n}(1, 1, \dots, 1). \tag{8}$$

Example 2 (Stirling numbers [4]). For two kinds of Stirling numbers, we have

$$\Lambda_{m-n}^{m-1}(1, 2, \dots, m-1) = S_1(m, n)(-1)^{m+n} \tag{9}$$

$$H_{m-n}^{n}(1, 2, \dots, n) = S_2(m, n). \tag{10}$$

For $q \neq 1$, the Gaussian binomial coefficient is defined by

$$\begin{bmatrix} x \\ n \end{bmatrix} = \begin{cases} 1 & n = 0 \\ \dfrac{(1-q^x)(1-q^{x-1})\dots(1-q^{x-n+1})}{(1-q)(1-q^2)(1-q^n)} & n = 1, 2, \dots \end{cases}$$

which reduces to the binomial coefficient $\binom{x}{n}$ when $q \to 1$.

Example 3 (Gaussian binomial coefficients [1]).

$$\Lambda_n^m(1, q, \dots, q^{m-1}) = \begin{bmatrix} m \\ n \end{bmatrix} q^{\binom{n}{2}} \tag{11}$$

$$H_{m-n}^{n+1}(1, q, \dots, q^n) = \begin{bmatrix} m \\ n \end{bmatrix} \tag{12}$$

where $\Lambda_n^m(1, q, \dots, q^{m-1})$ may be interpreted as the generating function for partitions of unequal parts with the number of parts less or equal to n and each part less than m. While $H_{m-n}^{n+1}(1, q, \dots, q^n)$ is the generating function for the partitions with the number of parts less or equal to $m - n$ and each part less or equal to n.

Substituting these examples into the propositions, we get three groups of *combinatorial identities*:

Corollary 1. *The following binomial identities hold true*:

$$\binom{m+n}{n} = \sum_{k=0}^{n} \binom{m+n-k-1}{n-k} \tag{13}$$

$$\binom{m+n+p-1}{n} = \sum_{k=0}^{n} \binom{m+k-1}{k} \binom{n+p-k-1}{n-k} \tag{14}$$

$$\binom{n+p-m-1}{n} = \sum_{k=0}^{n} (-1)^k \binom{m}{k} \binom{n+p-k-1}{n-k}. \tag{15}$$

Proof. The identities follow from setting $x_k = 1$, respectively, in Propositions 1, 2 and 3 in view of Example 1.

Corollary 2. *For Stirling numbers we have identities:*

$$S_2(m+n, n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}k^{m+n} \tag{16}$$

$$S_2(m+n+1, m+1) = \sum_{k=0}^{n}(m+1)^k S_2(m+n-k, m) \tag{17}$$

$$\sum_{k=0}^{n} S_1(m+1, 1+m-k)S_2(n+p-k, p) = \begin{cases} \Lambda_n^{m-p}(p+1, \ldots, m)(-1)^n & m > p \\ H_n^{p-m}(m+1, \ldots, p) & m \leqslant p\,. \end{cases} \tag{18}$$

Proof. The expression (16) in terms of divided differences follows from setting $x_k = k$ in Theorem 1. The recurrence relation (17) is derived by putting $x_0 = m+1$ and $x_k = k$ for $k = 1, 2, \ldots, m$ in Proposition 1 in view of Example 2. The same example enables us to get (18) from setting $x_k = k$ in Proposition 3. If we replace m and n with $m-1$ and $m-p$, then the last result reduces to the orthogonal relation [4]

$$\delta_{m,p} = \sum_{k=p}^{m} S_1(m, k)S_2(k, p)\,.$$

Corollary 3. *The following q-binomial identities hold true:*

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \sum_{k=0}^{n} q^{n-k}\begin{bmatrix} m+n-k-1 \\ n-k \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} m+n+p-1 \\ n \end{bmatrix} = \sum_{k=0}^{n}\begin{bmatrix} m+k-1 \\ k \end{bmatrix}\begin{bmatrix} n+p-k-1 \\ n-k \end{bmatrix} q^{m(n-k)} \tag{20}$$

$$q^{mn}\begin{bmatrix} n+p-m-1 \\ n \end{bmatrix} = \sum_{k=0}^{n}(-1)^k\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n+p-k-1 \\ n-k \end{bmatrix} q^{\binom{n}{2}}\,. \tag{21}$$

Proof. The identities follow from setting $x_k = q^k$, respectively, in Propositions 1, 2 and 3 in view of Example 3. When $q \to 1$, they reduce to the binomial formulas displayed in Corollary 2.

References

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Sommario

Vengono studiate le funzioni simmetriche complete. Il calcolo algebrico viene utilizzato per stabilire alcune formule fondamentali. Applicazioni alle identità combinatorie sono dimostrate come conseguenza.

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