Riv. Mat. Univ. Parma (5) 5 (1996), 223-234

L. MASTROENI and M. MATZEU (*)

Parabolic variational inequalities with degenerate elliptic part (**)

1 - Introduction

The theory of degenerate elliptic operators related to the consideration of a *weight function* in the elliptic condition was introduced by S. N. Kruskov in [11]. Precisely the second order terms $a_{ij}(i, j = 1, \dots, N)$ of the elliptic operator are supposed to satisfy a condition of the type

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i \xi_j \geqslant w(x)\,|\xi^2| \qquad \text{almost everywhere } x \in \Omega, \ \ \forall \xi \in \boldsymbol{R}^N$$

where w (the *weight function*) is a measurable function on an open subset Ω of $\boldsymbol{R}^N$ with $w(x) > 0$ a.e. $x \in \Omega$. In the following, M. K. V. Murthy and G. Stampacchia, in a celebrated paper [18] developed a basic methodology in order to deal with general boundary value problems associated with this kind of operators, through a deep investigation of the properties of the *weighted* Sobolev spaces which are naturally connected to the weighted ellipticity condition. In [18] a particular attention is devoted to the variational inequalities.

In the following, many authors obtained various results in the framework of the local regularity for equations. We mention, among the others, [2], [7], [8], [19], and, for the parabolic case, [5], [9]. As for the variational inequality problem, we mention a recent paper by M. A. Vivaldi [20], where a suitable further integral term is added to the degenerate elliptic operator.

The aim of this paper is to state two existence and uniqueness results for parabolic variational inequalities with this kind of degeneracy for the elliptic part,

---

(*) Dip. Studi Economico-Finanziari e Metodi Quantitativi, Univ. Tor Vergata, Via di Tor Vergata, 00133 Roma. Dip. di Matem., Univ. Tor Vergata, V.le della Ricerca Scientifica, 00133 Roma.

(**) Received December 9, 1996. AMS classification 35 K 85.

through the use of an elliptic regularization method based on the lattice structure of the suitable Soboles spaces for this problem and the related so called inequalities of Lewy-Stampacchia's type which hold for elliptic problems. This method in dealing with parabolic variational inequalities was introduced in [4] for the uniform elliptic case and was developed in a very abstract formulation, also concerning nonlinear elliptic parts in [6]. Here the arguments of [6], which rely on the strong coercive character of the problem, are suitably adapted to this degenerate situation.

The authors have decided to present the results in case that the space variable varies into the whole space $\boldsymbol{R}^N$, rather than, as usual, in a bounded open domain of $\boldsymbol{R}^N$, having in mind a suitable application to the financial markets' theory. Indeed a recent literature has been developed in these last years, concerning the use of evolutionary variational inequalities in the formulation of the American option pricing problem: here the natural environment for the evolution of stock prices is indeed the whole space $\boldsymbol{R}^N$ rather than a fixed bounded set (see e.g. [10], [13], [14], [15], [21] for various results in this framework). However, we point out that the results obtained in the present paper can be easily adapted to the case that $\boldsymbol{R}^N$ is replaced by an open bounded set with a sufficiently smooth boundary. At the moment the authors are investigating for some suitable regularity results, which should be obtained through the use of the Lewy-Stampacchia's inequalities, in order to allow the precise financial interpretation of the solution to a variational inequality of a similar type in a suitable framework of *incomplete* markets.

Finally, let us mention that, in case where the degenerate ellipticity is given by the simple requirement of positive *semidefiniteness* of the matrix associated with the second order part of the operator, many interesting results were obtained in several papers by J. L. Menaldi, for variational inequalities in the framework of optimal control problems (see e.g. [16] and the related references).

2 - The existence results

Let $T > 0$, $N \in \boldsymbol{N}$ and consider the *variational inequality*

$$(1)\qquad \begin{aligned} & u \in X,\ \frac{\partial u}{\partial t} \in X',\ u(t, x) \geq \psi(t, x) \text{ a.e. } x \in \boldsymbol{R}^N,\ \forall t \in [0, T],\ u(T, x) = \varphi(x) \\ & \langle -\frac{\partial u}{\partial t} + Au,\ v - u\rangle \geq \langle f,\ v - u\rangle\ \ \forall v \in X, v(t,x) \geq \psi(t,x) \text{ a.e. } x \in \boldsymbol{R}^N, \forall t \in [0, T] \end{aligned}$$

where:

$X = L^2(0, T; H^1_{\mu, w}(\boldsymbol{R}^N))$ is the space of the square integrable functions on

$[0, T]$ with values into the space $H^1_{\mu, w}(\boldsymbol{R}^N)$, the completion of $C_0^\infty(\boldsymbol{R}^N)$ with respect to the norm

$$\|v\| = \Big(\int_{\boldsymbol{R}^N} |v(x)|^2 e^{-\mu|x|}\, dx + \int_{\boldsymbol{R}^N} |\nabla v(x)|^2 w(x) e^{-\mu|x|}\, dx\Big)^{\frac{1}{2}}$$

where μ is a fixed positive number and $w(x)$ is a measurable function on $\boldsymbol{R}^N$ with $w(x) > 0$ a.e. $x \in \boldsymbol{R}^N$

$\dfrac{\partial}{\partial t}$ is the distributional derivative with respect to the t-variable on $[0, T]$

X' is the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X'

ψ belongs to X, f belongs to X'

$\varphi, \dfrac{\partial \varphi}{\partial x_i}$ $(1 = 1, \dots, N)$ belong to $H^1_{\mu, w}(\boldsymbol{R}^N)$, $\varphi(x) \geq \psi(t, x)$ a.e. $x \in \boldsymbol{R}^N, \forall t \in [0, T]$

$A: X \to X'$ is the second order partial differential operator associated with the bilinear form a on $X \times X$ defined as

$$\begin{aligned} a(v_1, v_2) = & \sum_{i,j=1}^{N} \int_0^T \int_{\boldsymbol{R}^N} a_{ij}(t, x) \frac{\partial v_1}{\partial x_i} \frac{\partial v_2}{\partial x_j} e^{-\mu|x|}\, dx\, dt \\ & + \sum_{j=1}^{N} \int_0^T \int_{\boldsymbol{R}^N} \Big(a_j(t, x) - \mu \sum_{i=1}^{N} a_{ij}(t, x) \frac{x_i}{|x|}\Big) \frac{\partial v_1}{\partial x_j} v_2 e^{-\mu|x|}\, dx\, dt \\ & + \int_0^T \int_{\boldsymbol{R}^N} a_0(t, x) v_1 v_2 e^{-\mu|x|}\, dx\, dt \end{aligned}$$

with a_{ij}, a_j, a_0 measurable functions on $[0, T] \times \boldsymbol{R}^N$ such that:

(2) $$a_{ij} w^{-1} \in L^\infty([0, T]; \boldsymbol{R}^N) \qquad \forall i, j = 1, \dots, N$$

(3) $$a_{ij} w^{-\frac{1}{2}} \in L^\infty([0, T]; \boldsymbol{R}^N) \qquad \forall i, j = 1, \dots, N$$

(4) $$a_i w^{-\frac{1}{2}} \in L^\infty([0, T]; \boldsymbol{R}^N) \qquad \forall i = 1, \dots, N$$

(5) $$a_0 \in L^\infty([0, T]; \boldsymbol{R}^N).$$

Note that assumptions (2), ..., (5) easily imply that a is continuous on $X \times X$, thus the position $\langle A v_1, v_2 \rangle = a(v_1, v_2)$, $\forall v_1, v_2 \in X$ actually defines A as a continuous linear operator from X into X'.

Some weaker assumptions than (2), ..., (5) could be taken as suitable consequences of the continuous embedding properties of the Sobolev spaces $H^1_{\mu, w}$ into the L^p-spaces (see [18], which deals with the stationary case – i.e. independence from the time variable t – and the case where $\boldsymbol{R}^N$ is replaced by a bounded open domain of $\boldsymbol{R}^N$, so $\mu \equiv 0$).

We are in the position to state two different existence results, either in case that $\varphi \equiv 0$, or in case that $\varphi \not\equiv 0$, but the coefficients a_{ij}, a_j, a_0 must satisfy some further regularity assumptions.

Theorem 1. *Let all the previous positions and assumptions be taken and let the further hypothesis hold:*

(6) $$\sum_{i,j=1}^{N} a_{ij}(t,x)\,\xi_i\xi_j \geq w(x)\,|\xi|^2 \qquad \text{a.e. } (t,x) \in [0,T] \times \boldsymbol{R}^N\, \forall \xi \in \boldsymbol{R}^N$$

(7) $$\exists a_0 > 0 : a_0(t,x) \geq a_0 \qquad \text{a.e. } (t,x) \in [0,T] \times \boldsymbol{R}^N$$

(8) $$\frac{\partial \psi}{\partial t} \in L^2(0,T; H^1_{\mu,w}(\boldsymbol{R}^N))$$

(9) $$\frac{\partial^2 \psi}{\partial t^2} \in L^2(0,T; L^2_\mu(\boldsymbol{R}^N))$$

($L^2_\mu(\boldsymbol{R}^N)$ is the usual space $L^2(\boldsymbol{R}^N)$ with the Lebesgue measure dx *replaced by* $e^{-\mu|x|}$ dx).

(10) $$-\frac{\partial \psi}{\partial t} + A\psi - f \text{ belongs to the order dual space } X^* \text{ of } X \text{ where}$$
$$X^* = \{v' \in X' : v' = v'_+ - v'_-, \text{ where } v'_+, v'_- \text{ are nonnegative elements in } X'\}$$

(11) $$\psi(T,x) \leq \varphi(x) = 0 \qquad \text{a.e. } x \in \boldsymbol{R}_N.$$

Then there exists one and only one solution u to (1). *Moreover, u satisfies the following pair of dual estimates with respect to the dual ordering of X'*

(12) $$f \leq -\frac{\partial u}{\partial t} + Au \leq f + \left(-\frac{\partial \psi}{\partial t} + A\psi - f\right)^+$$

where $(v')^+$ denotes the positive part of an element $v' \in X^*$.

Theorem 2. *Let all the assumptions of Theorem* 1 *hold with possibly $\varphi \not\equiv 0$ and let the further conditions be verified:*

(13) $a_{ij} \in C^1([0,T)] \times \boldsymbol{R}^N) \cap L^\infty([0,T] \times \boldsymbol{R}^N)$, $\dfrac{\partial}{\partial x_k} a_{ij}\, w^{-1} \in L^\infty(0,T; \boldsymbol{R}^N)$ *for any* $i, j, k = 1, \dots, N$

(14) $a_i(t,x)$, $\dfrac{\partial a_i}{\partial t}(t,x)$ *are continuous with respect to* $t \in [0,T]$ *a.e.* $x \in \boldsymbol{R}^N$ *and* $\left(\dfrac{\partial}{\partial x_k} a_i\right) w^{-1} \in L^\infty([0,T] \times \boldsymbol{R}^N)$, $\quad \forall i,\ K = 1, \dots N$

(15) $a_0(t, x)$, $\dfrac{\partial a_0}{\partial t}(t, x)$ *are continuous with respect to* $t \in [0, T]$ *a.e.* $x \in \boldsymbol{R}^N$

(16) $a'_{ij} = \dfrac{\partial a_{ij}}{\partial t}$, $a'_i = \dfrac{\partial a_i}{\partial t}$, $a'_0 = \dfrac{\partial a_0}{\partial t}$ *verify all the assumptions* (2), (3), (4), (5) *with* a_{ij}, a_i, a_0 *replaced by* a'_{ij}, a'_i, a'_0 *respectively.*

Then there exists one and only one solution to (1). *Moreover* u *satisfies the following pair of dual estimates with respect to the dual ordering of* X':

$$f \leqslant -\frac{\partial u}{\partial t} + Au \leqslant f + \Big(-\frac{\partial \psi}{\partial t} + A\psi - f\Big)^+ . \tag{17}$$

3. - Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. From now on, we denote by $\|\cdot\|$ the norm of the space X and divide the proof into steps.

Step 1. *The form* $a(\cdot, \cdot)$ *can be supposed coercive on* $X \times X$, *i.e.*

$$a(v, v) \geqslant const. \ \|v\|^2 \qquad \forall v \in X \tag{18}$$

without loss of generality.

Proof. Actually the assumptions (6), (7) easily imply the existence of a number $k_0 > 0$ such that $a_k(v, v) = a(v, v) + k(v, v)_{L^2_\mu}$ (where $(\cdot, \cdot)_{L^2_\mu}$ denotes the inner product of $L^2([0, T]; L^2_\mu(\boldsymbol{R}^N))$ is coercive on $X \times X$ for any $k \geqslant k_0$. On the other side, the parabolic nature of the variational inequality (1) enables to avoid the assumption of further conditions on the first order coefficients in order to guarantee the coerciveness of $a(\cdot, \cdot)$, since it is standard to check that (1) is equivalent, for any $k > 0$, to the following variational inequality (see e.g. [**1**] for cases where $w(x) \equiv 1$)

$$(1_k) \qquad \begin{aligned} & u_k \in X, \ \frac{\partial u_k}{\partial t} \in X', \ u_k \geqslant \psi_k, \ u_k(T, x) = 0 \\ & \Big\langle -\frac{\partial u_k}{\partial t}, v - u_k \Big\rangle + a_k(u_k, v - u_k) \geqslant \langle f_k, v - u_k \rangle \qquad \forall v \in X, \quad v \geqslant \psi_k \end{aligned}$$

where $\varphi_k = e^{-k(T-t)}\psi$, $f_k = e^{-k(T-t)} f$. The equivalence is to be intended in the sense that u solves (1) iff $u_k = e^{k(T-t)} u$ solves (1_k). This remark and the coerciveness of the form a_k, for $k \geqslant k_0$, allow to reduce oneself always to the case where the original form $a(\cdot, \cdot)$ is coercive on $X \times X$, i.e. verifies (18).

Step 2. *The operator A is strictly T-monotone in the sense that*

$$\langle A(v_1 - v_2), (v_1 - v_2)^+ \rangle \geq 0 \qquad \forall v_1, v_2 \in X$$
$$\text{with } \langle A(v_1 - v_2), (v_1 - v_2)^+ \rangle = 0 \qquad \text{iff } (v_1 - v_2)^+ = 0 .$$

Proof. It is an immediate consequence of Step 1 and the fact that $\langle Av^+, v^- \rangle = 0$, $\forall v \in X$.

Step 3. *Let us consider the following spaces*

$$V = \{v \in X: \frac{\partial v}{\partial t} \in L^2(0, T; L^2_\mu(\boldsymbol{R}^N))$$
$$V_0 = \{v \in X: \frac{\partial v}{\partial t} \in L^2(0, T; L^2_\mu(\boldsymbol{R}^N)), v(T, x) = 0, \text{ a.e. } x \in \boldsymbol{R}^N\}$$

equipped with the $\frac{\partial}{\partial t}$*-graph norm* $|||v||| = \|v\|_X + \|\frac{\partial v}{\partial t}\|_{L^2_\mu}$. *Let, for any* $n \in \boldsymbol{N}$:

$$\pi_n, \eta_n \in L^2(0, T; L^2_\mu(\boldsymbol{R}^N)) \qquad \pi_n \geq 0 \qquad \eta_n \geq 0$$
$$\pi_n \to (-\frac{\partial \psi}{\partial t} + A\psi - f)^+ \qquad \eta_n \to (-\frac{\partial \psi}{\partial t} + A\psi - f)^- \text{ in } X'$$
$$f_n = (-\frac{\partial \psi}{\partial t} + A\psi - (\psi_n - \eta_n))$$

and the operator $A_n : V \to V_0'$ *defined as*

$$\langle A_n v_1, v_2 \rangle = -\frac{1}{n}\langle \frac{\partial v_1}{\partial t}, \frac{\partial v_2}{\partial t} \rangle + \langle \frac{\partial v_1}{\partial t}, v_2 \rangle + \langle Av_1, v_2 \rangle \qquad \forall v_1 \in V \qquad \forall v_2 \in V_0$$

where $\langle \cdot, \cdot \rangle$ *denotes the «pairing» between* X *and* X' *as well as the «pairing» between* V_0 *and* V_0'. *Then, for any* $n \in \boldsymbol{N}$, *there exists one and only one solution* u_n *of the variational inequality*

$$(1)_n \qquad \begin{aligned} &u_n \in V_0 \quad u_n \geq \psi \\ &\langle A_n u_n, v - u_n \rangle \geq \langle f_n, v - u_n \rangle \quad \forall v \in V_0, \ v \geq \psi . \end{aligned}$$

Moreover u_n *satisfies the pair of dual estimates*

$$(19) \qquad f_n \leq A_n u_n \leq f_n + \frac{1}{n}(\frac{\partial^2 \psi}{\partial t^2})^+ + \pi_n$$

in the sense of the dual ordering of V_0'.

Proof. First of all let us note that the convex set $K = \{v \in V_0 : v \geq \psi\}$ is actually not empty, as (8), (11) guarantee that the element $\sup(\psi, v_0)$ belongs to K for any $v_0 \in V_0$.

Moreover, due to Step 1 and the positivity of the operator $-\frac{\partial}{\partial t}$ on the space V_0 (as a consequence of an integration by parts with respect to the variable t and condition $v(T)=0$), the operator A_n is coercive on the space V_0 with respect to its $\frac{\partial}{\partial t}$-graph norm.

Obviously A_n is continuous on V and it is strictly T-monotone for V into V_0' in the sense that

$$\langle A_n(v_1-v_2),(v_1-v_2)^+\rangle \geqslant 0 \qquad \forall v_1, v_2 \in V \quad \text{such that } (v_1-v_2)^+ \in V_0$$
$$\text{with } \langle A_n(v_1-v_2),(v_1-v_2)^+\rangle = 0 \Leftrightarrow (v_1-v_2)^+ = 0$$

(as an easy consequence of Step 2). Therefore one can use a general result due to Mosco [17] (Corollary of Th. 4.1 p. 133) for abstract variational inequalities in a framework of lattice structure: it ensures the existence and uniqueness of the solution u_n of $(1)_n$ and that (19) holds.

Step 4. *For any $n \in \boldsymbol{N}$, $\frac{\partial^2 u_n}{\partial t^2}$ belongs to X'. Moreover there exists a subsequence $\{u_\nu\}$ of $\{u_n\}$ and an $\bar{u} \in W_0 = \{v \in X,\ \frac{\partial v}{\partial t} \in X',\ v(T,x)=0$ a.e. $x \in \boldsymbol{R}^N\}$ such that*

$$u_\nu \to \bar{u} \quad \textit{weakly in } X \tag{20}$$

$$\frac{\partial u_\nu}{\partial t} \to \frac{\partial \bar{u}}{\partial t} \quad \textit{weakly in } X' \tag{21}$$

$$\frac{1}{n}\,\frac{\partial^2 u_\nu}{\partial t^2} \to 0 \quad \textit{weakly in } X'. \tag{22}$$

Proof. Indeed the estimates (19), the definition of f_n and the assumption (10) yield

$$\|A_n u_n\|_{X'} \leqslant \text{const.} \qquad \forall n \in \boldsymbol{N}. \tag{23}$$

Taking the coerciveness of A into account, one gets

$$\frac{1}{n}\left\|\frac{\partial u_n}{\partial t}\right\|_{L^2_\mu} + \text{const.}\|u_n\|_X^2 \leqslant \|A_n u_n\|_{X'}\,\|u_n\|_X \leqslant \text{const.}\|u_n\|_X \qquad \forall n \in \boldsymbol{N}. \tag{24}$$

Hence

$$\frac{1}{n}\left\|\frac{\partial u_n}{\partial t}\right\|^2_{L^2_\mu} \leqslant \text{const.} \qquad \forall n \in \boldsymbol{N} \tag{25}$$

and

$$\|u_n\|_X \leqslant \text{const.} \qquad \forall n \in \boldsymbol{N}. \tag{26}$$

At this point one can quite easily adapt to the present case an argument given, in an abstract framework (when the time derivative is replaced by an infinitesimal generator of a suitable semigroup) in [12] (Chap. 3, Prop. 7.1, p. 262) (details on this matter would go beyond the spirit of the present paper), in order to get that (25), (26) yield

$$\left\| \frac{\partial u_n}{\partial t} \right\|_{X'} \leqslant \text{const.} \qquad \forall n \in N \tag{27}$$

$$\frac{\partial^2 u_n}{\partial t^2} \quad \text{belongs to } X' \qquad \forall n \in N. \tag{28}$$

Therefore there exists a subsequence $\{u_\nu\}$ of $\{u_n\}$ and an element $\bar{u} \in X$, with $\bar{u} \in X'$ and $\bar{u}(T) = 0$ (i.e. $\bar{u} \in W_0$), such that (20), (21) hold as consequences of (26) and (27) respectively.

As for (22), one first notices the boundedness in X', of the sequence $\{A_n u_n - \frac{\partial u_n}{\partial t} - Au_n\}$ and that, thanks to an integration by parts with respect to the time variable t, one has

$$A_n u_n + \frac{\partial u_n}{\partial t} - Au_n = \frac{1}{n} \frac{\partial^2 u_n}{\partial t^2}. \tag{29}$$

On the other side, always by integrating by parts, one gets

$$\frac{1}{n} \langle \frac{\partial^2 u_n}{\partial t^2}, v \rangle = -\frac{1}{n} \langle \frac{\partial u_n}{\partial t}, \frac{\partial v}{\partial t} \rangle \qquad \forall v \in X, \ \frac{\partial v}{\partial t} \in X \tag{30}$$

so that (27) and (30) yield

$$\frac{1}{n} \langle \frac{\partial^2 u_n}{\partial t^2}, v \rangle \to 0 \qquad \forall v \in X, \ \frac{\partial v}{\partial t} \in X. \tag{31}$$

At this point, (31), the density of the space $\{v \in X: \frac{\partial v}{\partial t} \in X'\}$ into the space X and the boundedness of $\frac{1}{n} \frac{\partial^2 u_n}{\partial t^2}$ given by (29) yield (22).

Step 5. *Let* $\{u_\nu\}, \bar{u}$ *be given by* Step 4. *Then one has*

$$\overline{\lim} \langle Au_\nu, u_\nu - \bar{u} \rangle \leqslant 0 \tag{32}$$

$$\underline{\lim} \langle Au_\nu, u_\nu - v \rangle \geqslant \langle A\bar{u}, \bar{u} - v \rangle \qquad \forall v \in X \tag{33}$$

$$\underline{\lim} \langle -\frac{\partial u_\nu}{\partial t}, u_\nu - v \rangle \geqslant \langle -\frac{\partial \bar{u}}{\partial t}, \bar{u} - v \rangle \qquad \forall v \in X \tag{34}$$

the limits being taken as $\nu \to +\infty$.

Proof. First, let $\{\bar{u}_\nu\}$ be a sequence converging to $\bar{u}$ in X with $\frac{\partial \bar{u}_\nu}{\partial t} \in X$ $\forall \nu \in N$ and $\bar{u}_\nu \geqslant \psi$ (the existence of such a sequence is easily deduced by the density of the space $\{v \in V: \frac{\partial v}{\partial t} \in X\}$ in X and the lattice properties of X).

Then, by definition of A_n, the fact that u_ν solves $(1)_\nu$, and the lower semicontinuity of the operator $F_1(v) = \langle -\frac{\partial v}{\partial t}, v\rangle$ $\forall v \in V_0$, one gets

$$\begin{aligned}\langle Au_\nu, u_\nu - \overline{u}\rangle &= \langle A_\nu u_\nu, u_\nu - \overline{u}_\nu\rangle + \frac{1}{n}\langle \frac{\partial u_\nu}{\partial t}, \frac{\partial}{\partial t}(u_\nu - \overline{u}_\nu)\rangle \\ &\quad + \langle \frac{\partial u_\nu}{\partial t}, u_\nu - \overline{u}_\nu\rangle + \langle Au_\nu, \overline{u}_\nu - u\rangle \\ &\leq \langle f_n, u_\nu - \overline{u}_\nu\rangle - \frac{1}{n}\langle \frac{\partial^2 u_\nu}{\partial t^2}, \overline{u}_\nu\rangle + \langle Au_\nu, \overline{u}_\nu - \overline{u}\rangle .\end{aligned} \tag{35}$$

Hence, from the convergence properties of $\{u_\nu\}$ and the boundedness of A, (32) is deduced by (35). As for (33) and (34), they follow from the fact that the functional $F_2(v) = \langle -\frac{\partial v}{\partial t}, v\rangle$, $\forall v \in W_0$ as well is weakly lower semicontinuous on W_0.

Step 6. *The variational inequality* (1) *admits a solution.*

Proof. Let $v \in X$, $v \geq \psi$ and let $\{v_\nu\}$ be a sequence converging to v in X with $v_\nu \in X$, $\frac{\partial v_\nu}{\partial t} \in X$ and $v_\nu \geq \psi$. Then by (20), (21), (22), (32), (33), (34) one gets

$$\langle A\overline{u}, \overline{u} - v\rangle + \langle -\frac{\partial \overline{u}}{\partial t}, \overline{u} - v\rangle$$

$$\leq \underline{\lim}\,\{\langle Au_\nu, u_\nu - v_\nu\rangle + \langle Au_\nu, v_\nu - v\rangle + \langle -\frac{\partial u_\nu}{\partial t}, v_\nu - v\rangle + \langle -\frac{\partial u_\nu}{\partial t}, u_\nu - v_\nu\rangle\}$$

$$\leq \underline{\lim}\,\{\langle A_\nu u_\nu, u_\nu - v_\nu\rangle - \frac{1}{n}\langle \frac{\partial^2 u_\nu}{\partial t^2}, v_\nu\rangle\} \leq \lim \langle f_\nu, u_n - u_\nu\rangle = \langle f, \overline{u} - v\rangle .$$

Therefore, as $\overline{u} \geq \psi$ (as u_ν has this property $\forall \nu$ and (20) holds) and $\overline{u}(T) = 0$, one concludes that $\overline{u}$ is a solution of (1).

Step 7. *The solution of the variational inequality* (1) *is unique.*

Proof. The uniqueness is due to a standard argument based on the *strict* monotonicity of A as an operator from X into X', and of $-\frac{\partial}{\partial t}$ as an operator from W_0 into X'.

Proof of Theorem 2.

Step 1. *Let us consider the following problem*

$$u_0 \in X, \ \frac{\partial u_0}{\partial t} \in X' : \ -\frac{\partial u_0}{\partial t} + Au_0 = 0 \qquad u_0(T, x) = \varphi(x) \tag{36}$$

Then there exists a unique solution u_0 *of* (36) *which satisfies properties* (8), (9) *as well as obstacle* ψ *in Theorem* 1, *that is*

$$\frac{\partial u_0}{\partial t} \in L^2(0, T; H^1_{\mu, w}(\boldsymbol{R}^N)) \tag{37}$$

$$\frac{\partial^2 u_0}{\partial t^2} \in L^2(0, T; L^2_{\mu, w}(\boldsymbol{R}^N)). \tag{38}$$

Proof. The existence and uniqueness of the solution u_0 of (36) as well as its further regularity expressed by (37) can be obtained as a consequence of a very general result, i.e. the theorem of Hille and Yosida (see e.g. [3], p. 105), whose conditions are all satisfied, due to the continuity and coerciveness property of A, the further regularity assumptions on the coefficients expressed by (13), ..., (16) and the conditions given on φ.

Moreover one can state $\dfrac{\partial u_0}{\partial t}$ belongs to $C^0([0, T]; L^2_\mu(\boldsymbol{R}^N))$, so that, in particular, the *final Cauchy datum* $u_0(T, x)$ belongs to $L^2_\mu(\boldsymbol{R}^N)$, then $v_0 = \dfrac{\partial u_0}{\partial t}$ is the unique solution of the problem

$$v_0 \in X, \ \frac{\partial v_0}{\partial t} \in X', \ -\frac{\partial v_0}{\partial t} + Av_0 = g, \qquad v_0(T, x) = \frac{\partial u_0}{\partial t}(T, x) \in L^2_\mu(\boldsymbol{R}^N) \tag{39}$$

where g is a suitable function, which belongs to $L^2(0, T; L^2_\mu(\boldsymbol{R}^N))$ as a consequence of (13), ..., (16). Applying another classical result for parabolic equations (see e.g. [1]) to problem (39), one can conclude that u_0 actually satisfies condition (38) as well.

Step 2. *Let* u_0 *be given by Step* 1. *Then there exists one and only one solution* $\tilde{u}$ *to the variational inequality*

$$\begin{aligned} &\tilde{u} \in X, \ \frac{\partial \tilde{u}}{\partial t} \in X', \ \tilde{u} \geq \tilde{\psi} = \psi - u_0, \ \tilde{u}(T, x) = 0 \ \text{a.e.} \ x \in \boldsymbol{R}^N \\ &\langle -\frac{\partial \tilde{u}}{\partial t} + A\tilde{u}, v - \tilde{u} \rangle \geq \langle f, v - \tilde{u} \rangle \qquad \forall v \in X, \ v \geq \tilde{\psi}. \end{aligned} \tag{40}$$

Moreover $\tilde{u}$ *satisfies the following pair of dual estimates in* X'

$$f \leq -\frac{\partial \tilde{u}}{\partial t} + A\tilde{u} \leq f + \left(-\frac{\partial \psi}{\partial t} + A\psi - f\right)^+. \tag{41}$$

Proof. It is sufficient to apply the thesis of Theorem 1. Indeed the obstacle $\tilde{\psi} = \psi - u_0$ verifies assumptions (8), (9) as well as ψ since u_0 satisfies (37), (38). Furthermore, (10) holds by the properties of ψ and the fact that u_0 solves the differential equation in problem (36). Moreover, $\varphi(x) = 0$ in this case, as in the statement of Theorem 1. Finally, (41) follows from (12), since $-\frac{\partial\tilde{\psi}}{\partial t} + A\tilde{\psi} = -\frac{\partial\psi}{\partial t} + A\psi$ in this case.

Step 3. *The conclusion: the function* $u = u_0 + \tilde{u}$ *is the unique solution of* (1). *Moreover* u *satisfies* (15).

Proof. The existence is an obvious consequence of the linearity of the operators A and $-\frac{\partial}{\partial t}$ and the definitions themselves of u_0 and $\tilde{u}$. Similarly (15) is a trivial consequence of (41) and the differential relation in (36). Finally the uniqueness of the solution u follows from standard arguments due to the strict monotonicity of A from X into X' and of $-\frac{\partial}{\partial t}$ from the space $W_0 = \{v \in X, \frac{\partial v}{\partial t} \in X, v(T, x) = 0 \text{ a.e. } x \in \boldsymbol{R}^N\}$ into its dual space W_0'.

Final remark. We decided to present our results in case that x varies into the whole space $\boldsymbol{R}^N$. Actually it is easy to verify that all the results hold even if $\boldsymbol{R}^N$ is replaced by an open bounded subset Ω of $\boldsymbol{R}^N$ with a sufficiently smooth boundary $\partial\Omega$, if one considers the homogeneous Dirichlet problem. Indeed, in this case one chooses $\mu = 0$ in the definition of the exponential weight and makes some obvious changes in the definition of the spaces of the x-variable functions, which take into account the homogeneity condition on $\partial\Omega$.

References

[1] A. Bensoussan and J. L. Lions, *Applications of variational inequalities in stochastic control*, North-Holland, Amsterdam 1982.

[2] M. Biroli and U. Mosco, *Wiener criterion and potential estimates for obstacle problems relative to degenerate elliptic operators*, Ann. Mat. Pura Appl. **159** (1991), 255-281.

[3] H. Brezis, *Analyse fonctionelle. Théorie et applications*, Masson, Paris 1983.

[4] P. Charrier and G. M. Troianello, *On strong solutions of parabolic unilateral problems with obstacle dependent on time*, J. Math. Anal. Appl. **65** (1984), 110-125.

[5] F. M. Chiarenza and R. P. Serapioni, *A Harnack inequality for degenerate parabolic equations*, Comm. Partial Differential Equations **9** (1984), 719-749.

[6] F. Donati and M. Matzeu, *On the strong solutions of some nonlinear evolution problems in ordered Banach spaces*, Boll. Un. Mat. Ital. **16** (1979), 54-73.

[7] E. B. Fabes, D. S. Jerison and C. E. Kenig, *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier **32** (1982), 151-182.

[8] E. B. Fabes, C. E. Kenig and R. S. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), 77-116.

[9] J. C. FERNANDES, *Mean value and Harnack inequalities for a certain class of degenerate parabolic equations*, Rev. Mat. Iberoamericana 7 (1991), 247-286.

[10] P. JAILLET, D. LAMBERTON and B. LAPEYRE, *Variational inequalities and the pricing of American Options*, Acta Appl. Math. **21** (1990), 263-289.

[11] S. N. KRUSKOV, *Certain properties of solutions to elliptic equations*, Soviet Math. 4 (1963), 686-695.

[12] J. L. LIONS and E. MAGENES, *Problèmes aux limites non homogènes et applications*, **I**, Dunod, Paris 1968.

[13] L. MASTROENI and M. MATZEU, *An integro-differential parabolic variational inequality connected with the problem of the American option pricing*, Z. Anal. Anwendungen **14** (1995), 869-880.

[14] L. MASTROENI and M. MATZEU, *Stability for the integro-differential variational inequalities of the American option pricing problem*, Advances in Math. Sciences and Appl., to appear.

[15] L. MASTROENI and M. MATZEU, *Nonlinear variational inequalities for jump-diffusion processes with irregular obstacles and a financial application*, Nonlinear Anal. TMA, to appear.

[16] J. L. MENALDI, *On the optimal impulse control problem for degenerate diffusions*, SIAM J. Control Optim. 18 (1980), 722-739.

[17] U. MOSCO, *Implicit variational problems and quasi-variational inequalities*, Lecture Notes in Math. **543**, Springer, Berlin 1976.

[18] M. K. V. MURTHY and G. STAMPACCHIA, *Boundary value problems for some degenerate elliptic operators*, Ann. Mat. Pura Appl. **80** (1968), 1-122.

[19] N. TRUDINGER, *On the regularity of generalized solutions of linear nonuniformly elliptic equations*, Arch. Rational Mech. Anal. **42** (1971), 51-62.

[20] M. A. VIVALDI, *Existence and uniqueness results for degenerate elliptic integro-differential problems*, in Proceed. Second Europ. Conf. on Elliptic and Parabolic Problems, C. Bandle et alii eds., Pitman Res. Notes Math. Series **325**, Pitman, Harlow, UK 1995.

[21] X. ZHANG, *Options américaines et modèles des diffusion avec sauts*, C. R. Acad. Sci. Paris **317** (1993), 857-862.

Sommario

Si dimostrano due risultati di esistenza e unicità per soluzioni di disequazioni variazionali paraboliche con parte ellittica degenere, nel senso che la condizione di ellitticità non è uniforme, ma prevede la presenza di una «funzione peso». Le tecniche di dimostrazione si basano sull'uso di una regolarizzazione ellittica in opportuni spazi di Sobolev con peso e sull'uso di stime del tipo Lewy-Stampacchia per soluzioni di disequazioni variazionali ellittiche.

* * *