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A monotone iterative method for semilinear parabolic systems (**)

1 - Introduction

In this work we propose a finite element method for reaction-diffusion equation systems. This method is based on a monotone iterative scheme proposed by C. V. Pao (see [6] and [4]) which treats reaction-diffusion equations by the method of upper and lower solutions [5] and its associate monotone iterations.

The method we propose is shown first for a single equation and then is extended to a reaction-diffusion system of equations. We proceed working from a weak formulation of the problem to obtain a semidiscrete Galerkin system (see [7]). This system is approximated by means of a θ-method and then we apply an upper and lower solution iterative scheme. Results about uniqueness and existence of the solution, stability and convergence of the numerical scheme are shown.

2 - The case of one equation

Consider $\Omega = [a, b] \subset \boldsymbol{R}$ and, for each time $T > 0$, define $D_T = (0, T] \times \Omega$ and $S_T = (0, T] \times \partial\Omega$. We want to study the semilinear reaction-diffusion parabolic equation

$$(1) \qquad \begin{aligned} & u_t - \frac{\partial}{\partial x}(s(x)\, u_x) = f(u) \quad \text{in } D_T \\ & Bu = g \quad \text{on } S_T, \qquad u(x, 0) = u_0(x) \quad \text{in } \Omega \end{aligned}$$

where $x \in [a, b]$, $t \geqslant 0$, $s(x)$ is a positive known function, B is a Robin boundary

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operator, f, g, u_0 are assumed to be Hölder continuous in their respective domains and the function f is such that exists $\frac{\partial f}{\partial u}$ in D_T (see [5]).

We apply the Galerkin semidiscretization method making a partition of the domain $[a, b]$ in p subintervals with knots $x_i = a + ih$ $\left(h = \frac{b-a}{p}\right)$. Using a finite element method, we make the approximation $u = \sum_{i=0}^{p} \alpha_i(t)\varphi_i(x)$ where the φ_i are B-splines (and they are used as test functions too). Then we use the trapezoidal method for integration and we obtain a semidiscrete scheme

$$D\alpha' + A\alpha = D\cdot \underline{f}(t, \alpha) \tag{2}$$

where D is the diagonal matrix $D = hI$, $\alpha = (\alpha_0, \dots, \alpha_p)$ and

$$\underline{f}(\alpha) = (f(x_0, t, \alpha_0(t)), \dots, f(x_p, t, \alpha_p(t)))\,.$$

A is a tridiagonal, diagonal dominant and symmetric matrix of the form

$$A = \frac{1}{h}\begin{pmatrix} a_0 & -b_0 & \cdots & 0 \\ -b_0 & a_1 & -b_1 & \vdots \\ \vdots & & \ddots & -b_{p-1} \\ 0 & \cdots & -b_{p-1} & a_p \end{pmatrix} = \frac{1}{h}\tilde{A}\,. \tag{3}$$

Now we can apply a θ-method to the equations (2). We define $\Delta t_{n+1} = t_{n+1} - t_n$, $r_n = \frac{\Delta t_n}{h^2}$ and we define $\alpha_n = \alpha(t_n)$, $\alpha_{i,n} = (\alpha_n)_i$. From (2) we obtain

$$(I + \theta r_n \tilde{A})\alpha_n = (I - (1-\theta)r_n\tilde{A})\alpha_{n-1} + \Delta t_n(\theta \underline{f}(\alpha_n) + (1-\theta)\underline{f}(\alpha_{n-1})) \tag{4}$$

where $0 \leqslant \theta \leqslant 1$.

Definition. Given two vectors $v, w \in \boldsymbol{R}^q$, we write $v \leqslant w$ (or $v \geqslant w$) iff $v(i) \leqslant w(i)$ (or $v(i) \geqslant w(i)$) $\forall i = 1, \dots, q$.

Definition. A vector $\tilde{\alpha}_n$ is called an *upper solution* of (4) if $\tilde{\alpha}_0 \geqslant \mathbf{u}_0$ (where $\mathbf{u}_0 = (u_0(z_0), \dots, u_0(z_p))$) and

$$(I + \theta r_n\tilde{A})\tilde{\alpha}_n \geqslant (I - (1-\theta)r_n\tilde{A})\tilde{\alpha}_{n-1} + \Delta t_n(\theta\underline{f}(\tilde{\alpha}_n) + (1-\theta)\underline{f}(\tilde{\alpha}_{n-1}))$$

where $n = 1, 2, \dots, N$.

Similarly $\hat{\alpha}_n$ is called a *lower solution* if $\hat{\alpha}_0 \leqslant \mathbf{u}_0$ and satisfies the reversed inequality.

Definition. The pair $\tilde{\alpha}_n$, $\hat{\alpha}_n$ are said to be *ordered* if $\tilde{\alpha}_n \geqslant \hat{\alpha}_n$ for every n.

Definition. For any pair of ordered upper and lower solutions $\tilde{\alpha}_n = (\tilde{\alpha}_{0,n}, \tilde{\alpha}_{1,n}, \dots, \tilde{\alpha}_{p,n})^T$, $\hat{\alpha}_n = (\hat{\alpha}_{0,n}, \hat{\alpha}_{1,n}, \dots, \hat{\alpha}_{p,n})^T$, we define a *sector* in $\boldsymbol{R}^{p+1}$ by

$$\langle \hat{\alpha}_n, \tilde{\alpha}_n \rangle = \{ \alpha_n \in \boldsymbol{R}^{p+1} \mid \hat{\alpha}_n \leqslant \alpha_n \leqslant \tilde{\alpha}_n \}$$

and put $$c_{i,n} = \max\{ -\frac{\partial f}{\partial u}(\alpha_{i,n}) \mid \hat{\alpha}_{i,n} \leqslant \alpha_{i,n} \leqslant \tilde{\alpha}_{i,n} \} \qquad i = 0, 1, \dots, p$$

$$C_n = \operatorname{diag}(\bar{c}_{0,n}, \bar{c}_{1,n}, \dots, \bar{c}_{p,n})$$

where $\bar{c}_{i,n}$ are functions satisfying $\bar{c}_{i,n} \geqslant \max\{0, c_{i,n}\}$.

Hence we can write equation (4) in the following form

$$(5) \qquad (I + \theta \boldsymbol{A}_n)\alpha_n = (I - (1-\theta)\boldsymbol{A}_{n-1})\alpha_{n-1} + \theta F_n(\alpha_n) + (1-\theta)F_n(\alpha_{n-1})$$

where $F_n(\alpha_n) = \Delta t_n(\underline{f}(\alpha_n) + C_n \alpha_n)$ and $\boldsymbol{A}_n = r_n \tilde{A} + \Delta t_n C_n$.

It's easy to show that, in the sector $\langle \hat{\alpha}, \tilde{\alpha}_n \rangle$, $F_n(\alpha)$ is monotone nondecreasing component by component. Hence we can apply the following iterative scheme

$$(6) \qquad (I + \theta \boldsymbol{A}_n)\alpha_n^{(m)} = (I - (1-\theta)\boldsymbol{A}_{n-1})\alpha_{n-1}^* + \theta F_n(\alpha_n^{(m-1)}) + (1-\theta)F_n(\alpha_{n-1}^*)$$

$n, m = 1, 2, \dots$; where $\alpha_0^{(m)} = \mathbf{u}_0$ for all $m = 0, 1, \dots$ and α_n^* is the limit of the sequence at the time step n.

In fact, if we denote the sequences beginning with $\bar{\alpha}_n^{(0)} = \tilde{\alpha}_n$ and $\underline{\alpha}_n^{(0)} = \hat{\alpha}_n$ with $\{\bar{\alpha}_n^{(m)}\}$ and $\{\underline{\alpha}_n^{(m)}\}$ respectively, then it's possible to show that for every n they converge to a unique limit, that we denote with α_n^* (this result is immediate for $n = 0$). This is the subject of the following theorems.

Now we define $\bar{\sigma}_n = \max\{ \frac{\partial f}{\partial u}(\alpha_{i,n}) \mid \hat{\alpha}_{i,n} \leqslant \alpha_{i,n} \leqslant \tilde{\alpha}_{i,n},\ i = 0, \dots, p \}$.

Theorem 1 (uniqueness of the limit). *Let $\hat{\alpha}_n$, $\tilde{\alpha}_n$ be a pair of ordered upper and lower solutions of* (4) *and let the following conditions hold:*

1a. $1 - \Delta t_n \left(\frac{a_i}{h^2} - \bar{\sigma}_n \right)(1-\theta) > 0 \qquad \forall i = 0, \dots, p$

1b. $1 - \theta \Delta t_n \bar{\sigma}_n > 0$.

If the sequences $\{\bar{\alpha}_n^{(m)}\}$ and $\{\underline{\alpha}_n^{(m)}\}$ given by (6) *with $\bar{\alpha}_n^{(0)} = \tilde{\alpha}_n$ and $\underline{\alpha}_n^{(0)} = \hat{\alpha}_n$ converge to $\bar{\alpha}_n$ and $\underline{\alpha}_n$ respectively, then $\underline{\alpha}_n = \bar{\alpha}_n$ and this is the unique solution in the sector $\langle \hat{\alpha}_n, \tilde{\alpha}_n \rangle$.*

Proof. Let's consider $w_n = \overline{\alpha}_n - \underline{\alpha}_n$, then $w_n \geqslant 0$, $w_0 = 0$, and

$$(7) \quad \begin{aligned}&(I + \theta A_n) w_n = (I - (1-\theta) A_{n-1}) w_{n-1} \\ &+ \{\theta F_n(\overline{\alpha}_n) + (1-\theta) F_n(\overline{\alpha}_{n-1}) - \theta F_n(\underline{\alpha}_n) - (1-\theta) F_n(\underline{\alpha}_{n-1})\}.\end{aligned}$$

Since F_n is monotone nondecreasing, we have

$$F_n(\overline{\alpha}_n) - F_n(\underline{\alpha}_n) \leqslant B_n(\overline{\alpha}_n - \underline{\alpha}_n) = B_n w_n$$

where $B_n = \Delta t_n \cdot \overline{\sigma}_n I + \Delta t_n C_n$ is a diagonal matrix. Hence we deduce that

$$(8) \quad (I + \theta \boldsymbol{P}_n) w_n \leqslant (I - (1-\theta) \boldsymbol{P}_{n-1}) w_{n-1}$$

where $\boldsymbol{P}_n = \boldsymbol{A}_n - B_n$.

From **1b** we have that $(I + \theta \boldsymbol{P}_n)$ is diagonal dominant with positive diagonal entries, thus $(I + \theta \boldsymbol{P}_n)^{-1}$ exists and is positive (see [8], p. 85).

So we obtain

$$(9) \quad w_n \leqslant (I + \theta \boldsymbol{P}_n)^{-1} (I - (1-\theta) \boldsymbol{P}_{n-1}) w_{n-1}.$$

From **1a** we can verify that $(I - (1-\theta)\boldsymbol{P}_{n-1})$ is positive too. It follows from $w_0 = 0$ that $w_n \leqslant 0$ for every n. Since $w_n \geqslant 0$, we must have $w_n = 0$. This proves the relation $\underline{\alpha}_n = \overline{\alpha}_n$. The uniqueness follows from the maximal and minimal property of $\underline{\alpha}_n$ and $\overline{\alpha}_n$.

As $(I + \theta \boldsymbol{A}_n)$ is diagonal dominant and, for $1 \leqslant i \leqslant p$, $(I + \theta \boldsymbol{A}_n)_{ii} > 0$, then $(I + \theta \boldsymbol{A}_n)^{-1}$ exists and is positive. From (6) we can write ($x^m = \alpha_n^{(m)}$):

$$(10) \quad \begin{aligned}&x^m = (I + \theta \boldsymbol{A}_n)^{-1} (I - (1-\theta) \boldsymbol{A}_{n-1}) \alpha_{n-1}^* \\ &+ (I + \theta \boldsymbol{A}_n)^{-1} [\theta F_n(x^{m-1}) + (1-\theta) F_n(\alpha_{n-1}^*)] = \boldsymbol{G}_n(x^{m-1}) \quad (m, n = 1, 2, \ldots).\end{aligned}$$

Lemma 1. *Let $G: \boldsymbol{R}^m \to \boldsymbol{R}^m$ be a contraction on a closed set $D \subset \boldsymbol{R}^m$ (i.e. there exists $\overline{K} < 1$ such that $\|G(x) - G(y)\| \leqslant \overline{K} \|x - y\|$ $\forall x, y \in D$) and such that $G(D) \subset D$. Then G has a unique fixed point $x^* \in D$ such that $\forall x^0 \in D$ the sequence defined by $x^{k+1} = G(x^k)$ converges to x^*. Moreover*

$$\|x^k - x^*\| \leqslant \frac{\overline{K}}{1 - \overline{K}} \|x^k - x^{k-1}\|, \qquad k = 1, 2, \ldots,$$

where $\overline{K}$ is the contraction constant.

Let $J(G(x))$ be the Jacobian matrix of $G(x)$, Then we have

Lemma 2. *If $G: \boldsymbol{R}^m \to \boldsymbol{R}^m$ belongs to $C^1(D)$ and $\|J(G(x))\| \leqslant \overline{K} < 1$ $\forall x \in D$, then G is a contraction on D with constant $\overline{K}$.*

For the proofs of Lemmas 1 and 2 see [2] and [3].

Consider
$$L_n = \max_{x \in D_n} \frac{\|J(F_n(x))\|_2}{\Delta t_n}.$$

Lemma 3. *Suppose that $\boldsymbol{G}_n \in C^1(D_n)$, where $D_n = \{x \in \boldsymbol{R}^m \mid \hat{\alpha}_n \leq x \leq \tilde{\alpha}_n\}$, that $\underline{f}$ is almost differentiable in the domain Ω and that the following condition holds*

$$\exists \overline{K}_n < 1 \quad \text{such that} \quad \Delta t_n \theta \|(I + \theta \boldsymbol{A}_n)^{-1} L_n\|_2 \leq \overline{K}_n. \tag{11}$$

Then $\boldsymbol{G}_n$ is a contraction on D_n with constant $\overline{K}_n$.

Proof. Notice that $J(\boldsymbol{G}_n(x)) = \theta(I + \theta \boldsymbol{A}_n)^{-1} J(F_n(x))$. Our aim is to prove that there exists $\overline{K}_n$ such that $\|J(\boldsymbol{G}_n)\|_2 \leq \overline{K}_n < 1$. We have that

$$\|J(F_n(x))\|_2 \leq \Delta t_n \cdot L_n$$

and, from (11)

$$\|J(\boldsymbol{G}_n(x))\|_2 \leq \Delta t_n \theta \|(I + \theta \boldsymbol{A}_n)^{-1} L_n\|_2 \leq \overline{K}_n.$$

Lemma. 4. *Let $\hat{\alpha}_n$, $\tilde{\alpha}_n$ be a couple of ordered lower and upper solutions of* (5) *and $D_n = \langle \hat{\alpha}_n, \tilde{\alpha}_n \rangle$. Suppose that the following condition holds $\forall n = 0, \dots, N$*

$$1 - \Delta t_n \left(\frac{a_{i-1}}{h^2} + \bar{c}_{i,n} \right)(1 - \theta) > 0 \qquad \forall i = 1, \dots, p. \tag{12}$$

Consider the iterative scheme (10) *and suppose that $\boldsymbol{G}_n$ is a contraction on D_n, with constant $\overline{K}_n$. Then, $\forall n = 0, 1, \dots, \boldsymbol{G}_n(D_n) \subset D_n$ and the two sequences $\{\overline{\alpha}_n^{(m)}\}$ and $\{\underline{\alpha}_n^{(m)}\}$ converge to a unique limit α_n^*. Moreover the following error estimates hold:*

$$\begin{aligned} \|\underline{\alpha}_n^{(k)} - \alpha_n^*\| &\leq \left(\frac{\overline{K}_n}{1 - \overline{K}_n} \right) \cdot \|\underline{\alpha}_n^{(k)} - \underline{\alpha}_n^{(k-1)}\| \\ \|\overline{\alpha}_n^{(k)} - \alpha_n^*\| &\leq \left(\frac{\overline{K}_n}{1 - \overline{K}_n} \right) \cdot \|\overline{\alpha}_n^{(k)} - \overline{\alpha}_n^{(k-1)}\| \end{aligned} \tag{13}$$

for $k = 1, 2, \dots$.

Proof. By induction we prove that, for every $n = 0, \dots, N$

$$\boldsymbol{G}_n(\hat{\alpha}_n) \geq \hat{\alpha}_n \qquad \boldsymbol{G}_n(\tilde{\alpha}_n) \leq \tilde{\alpha}_n. \tag{14}$$

Define $w = \overline{\alpha}_0^{(0)} - \overline{\alpha}_0^{(1)}$. So we have $w = \tilde{\alpha}_0 - \overline{\alpha}_0^{(1)} = u_0 - u_0 = 0$; hence $\boldsymbol{G}_0(\tilde{\alpha}_0) \leq \tilde{\alpha}_0$. Similarly we prove that $\boldsymbol{G}_0(\hat{\alpha}_0) \geq \hat{\alpha}_0$. We deduce $\boldsymbol{G}_0(D_0) \subset D_0$.

Supposing that (14) is true for $l-1$ (hence there exists the unique limit α^*_{l-1} and $\boldsymbol{G}_{l-1}(D_{l-1}) \subset D_{l-1}$).

Now observe that from $\boldsymbol{G}_{l-1}(D_{l-1}) \subset D_{l-1}$ we deduce that the two sequences $\overline{\alpha}^m_{l-1}$ and $\underline{\alpha}^m_{l-1}$ are bounded in D_{l-1} and $\hat{\alpha}_{l-1} \leqslant \alpha^*_{l-1} \leqslant \tilde{\alpha}_{l-1}$. Remember that F_n is monotone nondecreasing (component by component) $\forall n = 0, \ldots, N$; so we have that

$$F_n(\hat{\alpha}_{l-1}) \leqslant F_n(\alpha^*_{l-1}) \leqslant F_n(\tilde{\alpha}_{l-1}) \qquad \forall n = 0, \ldots, N. \tag{15}$$

From (12) $(I-(1-\theta)\boldsymbol{A}_{l-1})$ is a nonnegative matrix. In fact it is tridiagonal and its elements over and under the diagonal are nonnegative.

Then, if we consider the diagonal elements:

$$(I-(1-\theta)\boldsymbol{A}_{l-1})_{ii} = 1-(1-\theta)(r_{l-1}a_{i-1} - \Delta t_{l-1}\bar{c}_{i,l-1})$$

$$= 1-(1-\theta)\Delta t_{l-1}\left(\frac{a_{i-1}}{h^2} + \bar{c}_{i,l-1}\right) > 0\,.$$

Let's show that (14) holds for l too. Remember that $\tilde{\alpha}_l = \overline{\alpha}^{(0)}_l$ and $\boldsymbol{G}_l(\tilde{\alpha}_l) = \tilde{\alpha}^{(1)}_l$ and put $\overline{w} = \overline{\alpha}^{(0)}_l - \overline{\alpha}^{(1)}_l$. Using the fact that $(I-(1-\theta)\boldsymbol{A}_{l-1})$ is nonnegative, the definition of upper solution and the (15) we have:

$$(I+\theta\boldsymbol{A}_l)\overline{w} = (I+\theta\boldsymbol{A}_l)\tilde{\alpha}_l - \theta F_l(\tilde{\alpha}_l) - (I-(1-\theta)\boldsymbol{A}_{l-1})\alpha^*_{l-1} - (1-\theta)F_l(\alpha^*_{l-1})$$

$$\geqslant (I+\theta\boldsymbol{A}_l)\tilde{\alpha}_l - \theta F_l(\tilde{\alpha}_l) - (1-\theta)F_l(\tilde{\alpha}_{l-1}) - (I-(1-\theta)\boldsymbol{A}_{l-1})\tilde{\alpha}_{l-1} \geqslant 0\,.$$

$(I+\theta\boldsymbol{A}_l)$ is an M-matrix and its inverse is positive. It follows that $\overline{w} \geqslant 0$, which implies $\overline{\alpha}^{(0)}_l \geqslant \overline{\alpha}^{(1)}_l$ and $\boldsymbol{G}_l(\tilde{\alpha}_l) \leqslant \tilde{\alpha}_l$.

Now we prove that $\boldsymbol{G}_l(\hat{\alpha}_l) \geqslant \hat{\alpha}_l$. Define $\underline{w} = \underline{\alpha}^{(0)}_l - \underline{\alpha}^{(1)}_l$. In the same way as shown before we obtain:

$$(I+\theta\boldsymbol{A}_l)\underline{w} = (I+\theta\boldsymbol{A}_l)\hat{\alpha}_l - \theta F_l(\hat{\alpha}_l) - (I-(1-\theta)\boldsymbol{A}_{l-1})\alpha^*_{l-1} - (1-\theta)F_l(\alpha^*_{l-1})$$

$$\leqslant (I+\theta\boldsymbol{A}_l)\hat{\alpha}_l - \theta F_l(\hat{\alpha}_l) - (1-\theta)F_l(\hat{\alpha}_{l-1}) - (I-(1-\theta)\boldsymbol{A}_{l-1})\hat{\alpha}_{l-1} \leqslant 0$$

It follows that $(I+\theta\boldsymbol{A}_l)\underline{w} \leqslant 0$ and $\underline{w} \leqslant 0$, so $\boldsymbol{G}_l(\hat{\alpha}_l) \geqslant \hat{\alpha}_l$.

Remember that $\hat{\alpha}_n$ and $\tilde{\alpha}_n$ are ordered and that $\boldsymbol{G}_n$ is monotone nondecreasing. This implies that $G_n(\hat{\alpha}_n) \leqslant \boldsymbol{G}_n(\tilde{\alpha}_n)$, so $\boldsymbol{G}_n(\hat{\alpha}_n) \in D_n$ and $\boldsymbol{G}_n(\tilde{\alpha}_n) \in D_n$ for all $n = 1, 2, \ldots$. From the monotone property of $\boldsymbol{G}_n$ we have that $\boldsymbol{G}_n(D_n) \subset D_n$, $\forall n = 1, 2, \ldots$.

Using the previous lemmas, we conclude that, for all $n = 0, 1, 2, \ldots$ and for every initial value in D_n, the two upper and lower sequences $\underline{\alpha}^{(m)}_n$ and $\overline{\alpha}^{(m)}_n$ defined in (6) converge to a unique limit $\alpha^*_n \in D_n$. Moreover we obtain the following error estimates:

$$\|\underline{\alpha}_n^{(k)} - \alpha_n^*\| \leq \left(\frac{\overline{K}_n}{1-\overline{K}_n}\right)\cdot\|\underline{\alpha}_n^{(k)} - \underline{\alpha}_n^{(k-1)}\|$$

$$\|\overline{\alpha}_n^{(k)} - \alpha_n^*\| \leq \left(\frac{\overline{K}_n}{1-\overline{K}_n}\right)\cdot\|\overline{\alpha}_n^{(k)} - \overline{\alpha}_n^{(k-1)}\|$$

where $\overline{K}_n$ is the contraction constant of G_n.

Lemma. 5. *The iterative schemes of the upper and lower sequences defined in* (6) *are stable for* $\overline{K}_n < \frac{1}{3}$ *(where* $\overline{K}_n$ *is the contraction constant of* $\boldsymbol{G}_n$*).*

Consider $\underline{e}_n^k = \|\underline{\alpha}_n^{(k)} - \alpha_n^*\|$. From (13) we can deduce that

$$\|\underline{\alpha}_n^{(k)} - \alpha_n^*\| \leq \left(\frac{\overline{K}_n}{1-\overline{K}_n}\right)\cdot\|\underline{\alpha}_n^{(k)} - \underline{\alpha}_n^{(k-1)}\| \leq \left(\frac{\overline{K}_n}{1-\overline{K}_n}\right)\cdot(\|\underline{\alpha}_n^{(k)} - \alpha_n^*\| + \|\alpha_n^* - \underline{\alpha}_n^{(k-1)}\|)$$

that is
$$\underline{e}_n^k \leq \left(\frac{\overline{K}_n}{1-\overline{K}_n}\right)\cdot(\underline{e}_n^k + \underline{e}_n^{k-1}).$$

For $\overline{K}_n < \frac{1}{2}$ we have $\underline{e}_n^k \leq \left(\frac{\overline{K}_n}{1-2\overline{K}_n}\right)\cdot\underline{e}_n^{k-1}$. If $\overline{K}_n < \frac{1}{3}$, then $\frac{\overline{K}_n}{1-2\overline{K}_n} < 1$.

For the upper sequence we follow the same way.

We summarize of the previous results in

Theorem 2 (existence, uniqueness and stability). *Let* $\tilde{\alpha}_n$ *and* $\hat{\alpha}_n$ *be a couple of ordered upper and lower solutions of* (5). *Suppose that the conditions* **1a**, **1b** *and* (11), (12) *hold. Then* $\forall n = 0, \dots, N$ *the upper and lower sequences* $\{\overline{\alpha}_n^{(m)}\}$ *and* $\{\underline{\alpha}_n^{(m)}\}$ *converge to a unique limit* $\alpha_n^* \in D_n = \{x \in \boldsymbol{R}^m \mid \hat{\alpha}_n \leq x \leq \tilde{\alpha}_n\}$. *Moreover if in* (11) $\overline{K}_n < \frac{1}{3}$, *then the algorithm is stable.*

Proof. The proof follows immediately from the previous lemmas.

3 - The case of a system of reaction-diffusion semilinear parabolic equations

Consider the semilinear parabolic system

$$\frac{\partial U(x,t)}{\partial t} = \nabla\cdot(S(x)\nabla U(x,t)) + F(U(x,t)) \quad (x,t)\in\Omega\times\boldsymbol{R}^+,\ U(x,t)\in\boldsymbol{R}^m \tag{16}$$

with convenient Robin-type boundary conditions and initial condition $U(x,0) = U_0(x) = (U_{01}, \dots, U_{0m})$ and where S is a positive diagonal matrix. Moreover Ω (subset of $\boldsymbol{R}^q$) and $F(x,t,U) = (F_1(x,t,U), \dots, F_m(x,t,U))^T$ satisfy convenient regularity conditions (see [5], p. 382, 16, 21).

If we apply a finite element method to system (16), we obtain a new system of ordinary differential equations of the form:

$$\frac{d\mathbf{v}(t)}{dt} = -A\mathbf{v}(t) + f(\mathbf{v}(t)) \tag{17}$$

with an initial condition $\mathbf{v}(0) = \mathbf{v}_0$, and $\mathbf{v}(t) = (v_1(t)^T, v_2(t)^T, \dots, v_m(t)^T)^T \in \boldsymbol{R}^{lm}$ where $v_i(t)$ is a l-vector of elements $v_{ij}(t)$.

Here A denotes the $lm \times lm$ block-diagonal matrix $A = \frac{1}{h^2} D \otimes \overline{A} = \frac{1}{h} \overline{A}$ where $\otimes$ is the Kronecker product, D is a diagonal matrix with elements d_i, $i = 1, \dots, m$ and $\overline{A}$ is an $l \times l$ matrix. $\widetilde{A}$ is a matrix of the type:

$$\widetilde{A} = \begin{pmatrix} a_{11} & -b_{11} & \cdots & 0 \\ -b_{11} & a_{12} & -b_{12} & \vdots \\ \vdots & & \ddots & -b_{m(l-1)} \\ 0 & \cdots & -b_{m(l-1)} & a_{ml} \end{pmatrix}$$

Moreover it is $f(\mathbf{v}(t)) = (f_1(\mathbf{v}(t))^T, f_2(\mathbf{v}(t))^T, \dots, f_m(\mathbf{v}(t))^T))^T$ where $f_i(\mathbf{v}(t))$ is a l-vector and h is a positive constant (the spatial step).

Now, applying a θ-method to system (17) we obtain a scheme like (4), that is:

$$(I + \theta r_n \widetilde{A})\mathbf{v}_n = (I(1-\theta) r_n \widetilde{A})\mathbf{v}_{n-1} + \Delta t_n I(\theta f(\mathbf{v}_n) + (1-\theta) f(\mathbf{v}_{n-1})) \tag{18}$$

where $\mathbf{v}_n = \mathbf{v}(t_n)$. Defining $\mathbf{U}_0 = (\mathbf{U}_{01}, \dots, \mathbf{U}_{0m})$, where $\mathbf{U}_{0i} = (U_{0i}(x_1), \dots, U_{0i}(x_l))$, then the definition of upper and lower solutions, of sector and of ordered upper and lower solutions are analogous with that we used in the previous part.

For each time t_n, we can consider a couple of ordered upper and lower solutions of (18), $\tilde{\mathbf{v}}_n \hat{\mathbf{v}}_n$. Then we define:

$$\begin{aligned} c_{ijn} &= \max\left\{-\frac{\partial f_i}{\partial v_{ij}}(w_{ijn}) : \hat{v}_{ijn} \leqslant w_{ijn} \leqslant \tilde{v}_{ijn}\right\} \qquad \bar{c}_{ijn} = \max(c_{ijn}, 0) \\ C_n &= \operatorname{diag}(\bar{c}_{11n}, \dots, \bar{c}_{1ln}, \dots, \bar{c}_{m1n}, \dots, \bar{c}_{mln}) \\ \bar{\sigma}_n &= \max\left\{\frac{\partial f_i}{\partial v_{ij}}(w_{ijn}) : \hat{v}_{ijn} \leqslant w_{ijn} \leqslant \tilde{v}_{ijn},\ i = 1, \dots, m; j = 1, \dots, l\right\}. \end{aligned} \tag{19}$$

As we did in the first part of this paper, we consider

$$\widetilde{F}_n(\mathbf{v}) = [\Delta t_n (f(\mathbf{v}_n) + C_n \mathbf{v}_n)] \qquad \boldsymbol{A}_n = r_n \widetilde{A} + \Delta t_n C_n .$$

It's easy to see that $\widetilde{F}_n(\mathbf{v})$ is monotone nondecreasing component by component.

Hence we can rewrite (18) in the form of (5):

$$(20)\qquad (I+\theta A_n)\mathbf{v}_n=(I-(1-\theta)A_{n-1})\mathbf{v}_{n-1}+\theta\tilde{F}_n(\mathbf{v}_n)+(1-\theta)\tilde{F}_n(\mathbf{v}_{n-1}).$$

We can now apply the iterative scheme:

$$(21)\qquad (I+\theta A_n)\mathbf{v}_n^{(k)}=(I-(1-\theta)A_{n-1})\mathbf{v}^*_{n-1}+\theta\tilde{F}_n(\mathbf{v}_n^{(k-1)})+(1-\theta)\tilde{F}_n(\mathbf{v}^*_{n-1})$$

$k=1,2,\dots;\ n=1,\dots,N$; where $\mathbf{v}_0^{(k)}=\mathbf{U}_0$ for all $k=0,1,\dots$, and $\mathbf{v}^*_n$ is the limit of the sequence at the time step n.

Theorem 3 (uniqueness of the limit). *Let $\tilde{\mathbf{v}}_n,\ \hat{\mathbf{v}}_n$ be a pair of ordered upper and lower solutions of* (20) *and let the following conditions hold for each time step n:*

2a. $\qquad 1-\Delta t_n\left(\dfrac{a_{ij}}{h^2}-\bar{\sigma}_n\right)(1-\theta)>0 \qquad \forall i=1,\dots,m;\ j=1,\dots,l$

2b. $\qquad 1-\Delta t_n\theta\bar{\sigma}_n>0\,.$

If the sequences $\{\bar{\mathbf{v}}_n^{(k)}\}$ and $\{\underline{\mathbf{v}}_n^{(k)}\}$, given by (21) *with $\bar{\mathbf{v}}_n^{(0)}=\tilde{\mathbf{v}}_n$ and $\underline{\mathbf{v}}_n^{(0)}=\hat{\mathbf{v}}_n$, converge to $\bar{\mathbf{v}}_n$ and $\underline{\mathbf{v}}_n$ respectively, then $\underline{\mathbf{v}}_n=\bar{\mathbf{v}}_n$ and this is the unique solution in the sector $\langle\hat{\mathbf{v}}_n,\tilde{\mathbf{v}}_n\rangle$.*

The proof is similar to that of Theorem 1.

Remark. With the same arguments used in the previous section we can say that there exists $(I+\theta A_n)^{-1}$ and is positive. From (21) we can write $(x^k=\mathbf{v}_n^{(k)})$

$$x^k=(I+\theta A_n)^{-1}(I-(1-\theta)A_{n-1})\mathbf{v}^*_{n-1}$$
$$+(I+\theta A_n)^{-1}[\theta\tilde{F}_n(x^{k-1})+(1-\theta)\tilde{F}_n(\mathbf{v}^*_{n-1})]=G_n(x^{k-1})$$

for $k=1,2\dots;\ n=1,\dots,N$.

As before we define $L_n=\dfrac{\max\limits_{x\in D_n}\|J(\tilde{F}_n(x))\|_2}{\Delta t_n}$.

It's easy to show that three lemmas, analogous with Lemma 3, 4 and 5, hold. Just use

$$(22)\qquad I-\Delta t_n\left(\frac{a_{ij}}{h^2}+\bar{c}_{ijn}\right)(1-\theta)>0 \qquad \forall i=1,\dots,m;\ j=1,\dots,l$$

instead of (12), and condition

$$(23)\qquad \exists\bar{K}_n<1 \quad\text{such that}\quad \theta\Delta t_n\|(I+\theta A_n)^{-1}L_n\|_2\leqslant\bar{K}_n\,.$$

Finally, the following theorem follows immediately

Theorem 4 (existence, uniqueness and stability). *Let* $\tilde{\mathbf{v}}_n$ *and* $\hat{\mathbf{v}}_n$ *be a couple of ordered upper and lower solutions of* (20). *Suppose that the conditions* **2a**, **2b**, (22), (23) *hold. Then* $\forall n = 1, \ldots, N$ *the upper and lower sequences* $\{\overline{\mathbf{v}}_n^{(k)}\}$ *and* $\{\underline{\mathbf{v}}_n^{(k)}\}$ *converge to a unique limit* $\mathbf{v}_n^* \in D_n = \{x \in \boldsymbol{R}^{lm} : \hat{\mathbf{v}}_n \leqslant x \leqslant \tilde{\mathbf{v}}_n\}$. *Moreover if in* (23) $\overline{K}_n < \frac{1}{3}$, *then the algorithm is stable.*

The proof follows immediately from the previous considerations.

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Sommario

A partire da una semidiscretizzatione di Galerkin per un sistema di equazioni paraboliche semilineari viene proposto un metodo di approssimazione numerica della soluzione che utilizza un θ-metodo e quindi uno schema iterativo monotono basato sulle soluzioni superiori ed inferiori. Vengono mostrate le proprietà di esistenza e unicità della soluzione, convergenza e stabilità dello schema numerico.

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