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On the matrix representation of Lie algebras for quantized Hamiltonians and their central extensions (**)

1 - Introduction

The method, introduced by S. Steinberg [6], to solve certain types of linear partial differential equations exploits the Lie algebraic decomposition formulae of Baker, Campbell, Hausdorff and Zassanhaus (cf. [7]) as well as their matrix representations. The required matrix representation should be faithful and of low-dimension [6]. This method is applied to solve the Schrödinger wave equations for some Hamiltonian systems in quantum mechanics. The faithful matrix representations of least degrees of three Lie algebras, namely L^+, L^- and L^c, are discussed and treated as special cases of L_r^s and its central extension ${}^cL_r^s$.

L^+, L^- and L_r^s are generated by K_0, K_+ and $K_- = K_+^\dagger$ († is used for hermitian conjugation) and are submitted to the physical condition that $K_+ + K_-$ is real, to satisfy the hermiticity of the Hamiltonians they correspond to. Also, K_0 is real and diagonal.

L^+ corresponds to the model of two-level optical atom of Hamiltonian $K_0 + \lambda(K_+ + K_-)$, $\lambda \in \boldsymbol{R}^*$ (the non-zero real numbers) [5]. L^+ is defined by the commutation relations:

$$(1) \qquad [K_+, K_-] = 2K_0 \qquad \text{and} \qquad [K_0, K_\pm] = \pm K_\pm .$$

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L^- corresponds to the model of light amplifier of Hamiltonian $K_0 + \lambda(K_+ + K_-)$, $\lambda \in \boldsymbol{R}^*$ (cf. [1], and references). L^- is defined by the commutation relations:

$$(2) \qquad [K_+, K_-] = -2K_0 \qquad \text{and} \qquad [K_0, K_\pm] = \pm K_\pm .$$

L_r^s is the Lie algebra defined by the commutation relations:

$$[K_+, K_-] = sK_0 \qquad \text{and} \qquad [K_0, K_\pm] = \pm rK_\pm$$

$r, s \in \boldsymbol{R}^*$. This is a generalization of L^+ and L^-. We prove that L_r^s has no non-trivial representation only if r and s have the same sign. Hence L^- has no non-trivial matrix representation satisfying the required mentioned physical conditions.

L^c is generated by K_1, $K_2 = K_1^\dagger$, K_3 and K_4 which satisfy:

$$[K_1, K_2] = K_3 + 2K_4 \quad [K_1, K_3] = 0 \quad [K_1, K_4] = K_1$$

$$[K_2, K_3] = 0 \quad [K_2, K_4] = -K_2 \quad [K_3, K_4] = 0 .$$

K_3 and K_4 are real diagonal.

L^c has the Hamiltonian model $\omega K_3 + 2\omega K_4 + \lambda(K_1 + K_2)$, $\lambda, \omega \in \boldsymbol{R}^*$, which is an alternative representation for the light amplifier model.

L^c is a special case of ${}^cL_r^s$. While ${}^cL_r^s$ is a generated by K_0, K_+, $K_- = K_+^\dagger$ and the central real diagonal elements $K_1, K_2, \dots, K_k$ satisfy

$$[K_+, K_-] = sK_0 + a_1K_1 + a_2K_2 + \dots + a_kK_k \qquad [K_0, K_\pm] = \pm rK_\pm$$

where $r, s, a_i \in \boldsymbol{R}^*$; $i = 1, 2, \dots, k$. It is also subjected to the physical condition that $(K_+ + K_-)$ is real. It is shown that ${}^cL_r^s$ has no non-trivial representation unless r and s have the same sign and so L^c has no non-trivial representation satisfying the physical conditions.

2 - Faithful non-trivial matrix representations of least degrees for L_r^s

Consider a representation of degree n for L_r^s. Let $X = [x_{ij}]$, $Y = [y_{ij}] = X^\dagger$, and Z be the representation matrices of K_+, K_- and K_0, respectively. To satisfy the physical properties, let $Z = [\delta_{ij} z_{ij}]$ (δ_{ij} is the Kronecker delta), $z_{ij} \in \boldsymbol{R}$, $y_{ij} = x_{ij}^\dagger = \bar{x}_{ji}$ (the bar is for complex conjugation), $1 \leqslant i, j \leqslant n$ and $X + Y = R = [r_{ij}]$ is a real matrix.

Lemma 1. *The defining relations of L_r^s can be either*

(3) $$[X, Y] = sZ \quad \text{and} \quad [Z, X] = rX \quad \text{or}$$

(4) $$[X, Y] = sZ \quad \text{and} \quad [Z, Y] = -rY.$$

Proof. $[Z, Y] = [Z^\dagger, X^\dagger] = [X, Z]^\dagger = (-rX)^\dagger = -rX^\dagger = -rY$. Also $[Z, X] = [Z^\dagger, Y^\dagger] = [Y, Z]^\dagger = -[Z, Y]^\dagger = -(-rY)^\dagger = rY^\dagger = rX$.

Lemma 2. *The representation matrices of the elements of L_r^s are all real trace-less matrices.*

Proof. Since $[Z, X] = ZX - XZ = rX$, then $\sum_{l=1}^{n} (\delta_{il} z_{il} x_{lj} - x_{il} \delta_{lj} z_{lj}) = r x_{ij}$. Thus:

(5) $$x_{ij}(z_{ii} - z_{jj} - r) = 0$$

(6) $$x_{ji}(z_{jj} - z_{ii} - r) = 0$$

for $i, j = 1, 2, \ldots, n$. In particular we have

(7) $$x_{ii}(z_{ii} - z_{ii} - r) = 0.$$

which implies $x_{ii} = 0$.

Since, $R = X + Y$, we have for $1 \leqslant i, j \leqslant n$ that $r_{ij} = x_{ij} + x_{ij}^\dagger = x_{ij} + \overline{x}_{ji}$ is real. But from (5), (6) we derive that x_{ij} and x_{ji} cannot be both different from zero for $i \neq j$. Thus both X and Y are real matrices with diagonal entries zeros and Y is just the transpose of X. From (3) we have

$$s\delta_{ij} z_{ij} = \sum_{l=1}^{n} (x_{il} x_{lj}^\dagger - x_{il}^\dagger x_{lj})$$

and consequently

(8) $$z_{ii} = \frac{1}{s} \sum_{l=1}^{n} (x_{il}^2 - x_{li}^2).$$

Therefore $$\text{trace } Z = \sum_{i=1}^{n} z_{ii} = \frac{1}{s} \sum_{i,l} (x_{il}^2 - x_{li}^2) = 0.$$

Hence the lemma.

From (8) and (5), if $x_{ij} \neq 0$ then

$$r = z_{ii} - z_{jj} = \frac{1}{s} \sum_{l=1}^{n} (x_{il}^2 - x_{li}^2 - x_{jl}^2 + x_{lj}^2). \tag{9}$$

Theorem. *L_r^s has non-trivial matrix representation only when $rs > 0$, i.e. only when r and s have the same sign.*

Proof. Inducing on n, the degree of the representation, we get for $n = 1$, that $X = Y = Z$ is the zero 1×1 matrix, since $x_{11} = 0$ is the only element of X. The representation is trivial. Consider the following cases:

Case 1. Let $n = 2$. For non-trivial representation let, $x_{ij} \neq 0$, $i \neq j$, $1 \leqslant i, j \leqslant 2$. From (6) $x_{ji} = 0$ and from (9) and (7) we have $r = \frac{2}{s} x_{ij}^2$ which is satisfied only when $rs > 0$.

Now, assume that the theorem is true for $n = m - 1$. Consider a matrix representation for L_r^s of degree m. Let X be partitioned as follows

$$X = \begin{bmatrix} X_{m-1} & C_{m-1} \\ R_{m-1} & 0 \end{bmatrix}$$

where X_{m-1} is an $(m-1) \times (m-1)$ matrix. C_{m-1} and R_{m-1} are $(m-1)$-component column and row, respectively. From (7), $x_{mm} = 0$.

Consider now the following cases and sub-cases:

Case 2. If $X_{m-1} = O_{m-1}$, the zero $(m-1) \times (m-1)$ matrix. For a non-trivial representation, there must be a non-zero element in C_{m-1} or R_{m-1}.

Case 2.1. If all the elements of C_{m-1} are zeros and $x_{mj} \neq 0$, $j \neq m$. Then from (9)

$$r = z_{mm} - z_{jj} = \frac{1}{s} (\sum_{l=1}^{m} x_{ml}^2 + x_{mj}^2) \tag{10}$$

which is only satisfied when $rs > 0$.

Case 2.2. If all the elements of R_{m-1} are zeros and $x_{im} \neq 0,\ i \neq m$. Then from (9)

$$r = z_{ii} - z_{mm} = \frac{1}{s}\left(x_{im}^2 + \sum_{l=1}^{m} x_{lm}^2\right) \tag{11}$$

which is satisfied only when $rs > 0$.

Case 2.3. If $x_{im} \neq 0,\ i \neq m$, and R_{m-1} is a non-zero row. For each element $x_{mj} \neq 0,\ j \neq m$, in the m^{th} row, we have from a previous remark, $x_{jm} = 0 = x_{mi}$. From (5) we have $r = z_{ii} - z_{mm} = z_{mm} - z_{jj}$. Thus, $2r = z_{ii} - z_{jj}$. And from (8) we get

$$2r = \frac{1}{s}\left(x_{im}^2 + x_{mj}^2\right) \tag{12}$$

which is satisfied only when $rs > 0$.

Therefore the theorem is true for **case 2** and its **sub-cases 2.1-2.3.**

Case 3. If $X_{m-1} \neq O_{m-1}$. Let $P = [p_{ij}]$ be an $(m-1) \times (m-1)$ matrix. In each of the **sub-cases 3.1-3.2** P will be acquired to be a non-zero matrix representing K_+, which contradicts our assumption unless $rs > 0$, as required.

Case 3.1. If $X_{m-1} \neq O_{m-1}$ and all the elements of R_{m-1} and C_{m-1} are zeros, then let $P = X_{m-1}$.

Case 3.2. If $X_{m-1} \neq O_{m-1}$and not all the elements of R_{m-1} and C_{m-1} are zeros, consider an $x_{ij} \neq 0,\ i \neq j;\ 1 \leqslant i, j \leqslant m-1$ (there must be such an element). Then let the only non-zero element of P be (see eq. (9))

$$p_{ij} = \sqrt{\frac{1}{2}\sum_{l=1}^{m}\left(x_{il}^2 - x_{li}^2 - x_{jl}^2 + x_{lj}^2\right)} = \sqrt{\frac{rs}{2}} \neq 0\,.$$

We have $\frac{1}{s}\sum_{l=1}^{m-1}\left(p_{il}^2 - p_{li}^2 - p_{jl}^2 + p_{lj}^2\right) = \frac{1}{s}\left(p_{ij}^2 + p_{ij}^2\right) = r$.

Hence the theorem.

We conclude this section by introducing all the non-trivial matrix representations of degree two, the lowest degree, for L_r^s.

From **Case 1**, $x_{ij}^2 = \dfrac{rs}{2}$ and we have the representations:

$$X = \begin{bmatrix} 0 & \pm\sqrt{\dfrac{rs}{2}} \\ 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 0 \\ \pm\sqrt{\dfrac{rs}{2}} & 0 \end{bmatrix} \quad \text{and} \quad Z = \frac{r}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 \\ \pm\sqrt{\dfrac{rs}{2}} & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & \pm\sqrt{\dfrac{rs}{2}} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Z = \frac{r}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly, L_r^s is $sl(2, \boldsymbol{R})$, the special linear Lie algebra of all 2×2 trace-less matrices.

The one-parameter subgroups of $SL(2, \boldsymbol{R})$ associated with X, Y, and Z in each representation are respectively:

$$\exp[\alpha(t)\, X] = \begin{bmatrix} 1 & \pm\alpha(t)\sqrt{\dfrac{rs}{2}} \\ 0 & 1 \end{bmatrix} \qquad \exp[\beta(t)\, Y] = \begin{bmatrix} 1 & 0 \\ \pm\beta(t)\sqrt{\dfrac{rs}{2}} & 1 \end{bmatrix}$$

and

$$\exp[\gamma(t)\, Z] = \begin{bmatrix} e^{\frac{r\gamma(t)}{2}} & 0 \\ 0 & e^{-\frac{r\gamma(t)}{2}} \end{bmatrix}$$

$$\exp[\alpha(t)\, X] = \begin{bmatrix} 1 & 0 \\ \pm\alpha(t)\sqrt{\dfrac{rs}{2}} & 1 \end{bmatrix} \qquad \exp[\beta(t)\, Y] = \begin{bmatrix} 1 & \pm\beta(t)\sqrt{\dfrac{rs}{2}} \\ 0 & 1 \end{bmatrix}$$

and

$$\exp[\gamma(t)\, Z] = \begin{bmatrix} e^{-\frac{r\gamma(t)}{2}} & 0 \\ 0 & e^{\frac{r\gamma(t)}{2}} \end{bmatrix}$$

with $t \in (-\infty, \infty)$.

3 - Faithful non-trivial matrix representation for ${}^cL_r^s$

Let $K = K_0 + b_1 K_1 + b_2 K_2 + \ldots + b_k K_k \in {}^cL_r^s$, where $b_i = \dfrac{a_i}{s}$ and $i = 1, \ldots, k$. From the above discussion there are no non-trivial matrices to represent

K_+, K_-, K unless $rs > 0$, because

$$[K_+, K_-] = sK \qquad [K, K_\pm] = \pm rK_\pm$$

as $[K_i, A] = 0$, $\forall A \in {}^cL_r^s$, $i = 1, \ldots, k$.

Therefore, L^c has no non-trivial matrix representations satisfying the mentioned physical requirements.

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Sommario

Si consideri l'algebra di Lie $L_r^s: [K_+, K_-] = sK_0$, $[K_0, K_\pm] = \pm rK_\pm$ *e la sua estensione centrale* ${}^cL_r^s: [K_+, K_-] = sK_0 + a_1K_1 + \ldots + a_kK_k$, $[K_0, K_\pm] = \pm rK_\pm$, *dove* $r, s, a_i \in \boldsymbol{R}^*$, $i = 1, \ldots, k$. K_0 *è l'operatore hermitiano,* $K_+ = K_-^\dagger$ *e* K_i *sono gli elementi centrali diagonali. Si dimostra che* L_r^s *e* ${}^cL_r^s$ *hanno rappresentazioni matriciali non banali soltanto quando è* $rs > 0$.

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