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Factoring a certain type of 2-group by subsets (**)

1 - Introduction

Throughout this paper we will use *multiplicative notation* for abelian groups. Let G be a *finite abelian group*. We denote the identity element of G by e.

If $B, A_1, \ldots, A_n$ are subsets of G such that each b in B is uniquely expressible in the form

$$b = a_1 \ldots a_n \qquad a_1 \in A_1, \ldots, a_n \in A_n$$

and each product $a_1 \ldots a_n$ belongs to B, that is, if the product $A_1 \ldots A_n$ is direct and is equal to B, then we say that B is factored by subsets $A_1, \ldots, A_n$. The equation $B = A_1 \ldots A_n$ is also said to be a *factorization* of B.

We say that a subset A of a finite abelian group G is *periodic* if there is an element $a \in G$ such that $a \neq e$ and $aA = A$. The element a is called a *period* of A. The periods of A together with the identity element form a subgroup H of G. Moreover there is a subset B of G such that $A = BH$ is a factorization of A.

If the group G is a direct product of cyclic groups of orders $t_1, \ldots, t_s$ respectively, then we express this fact shortly saying that G is of *type* $(t_1, \ldots, t_s)$.

A. D. Sands ([1], Theorem 7) proved that if a finite abelian group G is of type $(2^\lambda, 2)$ and G is a direct product of two of its subsets, then at least one of the factors must be periodic. As the main result of this paper we will show that Sands theorem holds in the more general case when G is factored into any number of subsets.

In his proof Sands used cyclotomic polynomials. Besides cyclotomic polynomials we will rely on characters too. If A is a subset and χ is a character of G, then

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we will use the notation $\chi(A)$ to denote the sum

$$\sum_{a \in A} \chi(a).$$

If $G = A_1 \dots A_n$ is a factorization of G, then $\chi(G) = \chi(A_1)\dots\chi(A_n)$. When χ is a nonprincipal character of G, then $\chi(G) = 0$ and hence $\chi(A_i) = 0$ for some i, $1 \leqslant i \leqslant n$.

2 - The result

In the remaining part of the paper let G be a group of type $(2^\lambda, 2)$. This means that $G = \langle a, b \rangle$, where $|a| = 2^\lambda$, $|b| = 2$. Each element g of G can be written in the form $g = a^\alpha b^\beta$, where $0 \leqslant \alpha \leqslant 2^\lambda - 1$ and $0 \leqslant \beta \leqslant 1$. Let A be a subset of G. If there are no elements a^α and $a^\alpha b$ together in A we will say that A is a *type* 1 subset of G. Otherwise we say that A is a *type* 2 subset of G. Also in what follows we will use the characters χ and χ' of G defined by

$$\chi(a) = \varrho \quad \chi(b) = -1 \qquad \text{and} \qquad \chi'(a) = \varrho \quad \chi'(b) = 1$$

respectively, where ϱ is a primitive 2^λth root of unity. We need two lemmas.

Lemma 1. Let A *be a subset of* G. *If* $\chi(A) = 0$, *then there are subsets* U, V *of* G *such that* $A = U\langle a^{2^{\lambda-1}} \rangle \cup V\langle b \rangle$, *where the union is disjoint and the products* $U\langle a^{2^{\lambda-1}} \rangle$ *and* $V\langle b \rangle$ *are direct.*

Proof. We can partition A in the form $A = B \cup C$, where B is a type 1 subset of G and C is periodic with period b. Simply put a^α to B if $a^\alpha \in A$ and $a^\alpha b \notin A$ or put $a^\alpha b$ to B if $a^\alpha \notin A$ and $a^\alpha b \in A$; and put a^α and $a^\alpha b$ to C if $a^\alpha \in A$ and $a^\alpha b \in A$. Now there is a subset V of G such that $C = V\langle b \rangle$, where the product is direct. Clearly B can be written in the form

$$B = \{a^{\alpha_1}, \dots, a^{\alpha_r}, a^{\beta_1} b, \dots, a^{\beta_s} b\},$$

where $0 \leqslant \alpha_i, \beta_j \leqslant 2^\lambda - 1$ and $\alpha_i \neq \beta_j$. Now

$$0 = \chi(A) = \chi(B) + \chi(V)\,\chi(\langle b \rangle) = \chi(B)$$

since $\chi(\langle b \rangle) = 1 + \chi(b) = 1 - 1 = 0$. Hence

$$0 = \chi(B) = \varrho^{\alpha_1} + \dots + \varrho^{\alpha_r} - \varrho^{\beta_1} - \dots - \varrho^{\beta_s}.$$

Consider the polynomial

$$B(x) = x^{\alpha_1} + \dots + x^{\alpha_r} - x^{\beta_1} - \dots - x^{\beta_s}$$

associated with $\chi(B)$. The 2^λth cyclotomic polynomial $F_{2^\lambda}(x) = 1 + x^{2^{\lambda-1}}$ is irreducible over the field of rational numbers. From this and from $B(\varrho) = 0$ it follows that $F_{2^\lambda}(x)$ divides $B(x)$ over the rationals, that is, there is a polynomial $D(x)$ with rational coefficients such that $B(x) = F_{2^\lambda}(x)\, D(x)$. As $\deg\, B(x) \leqslant 2^\lambda - 1$ and $\deg\, F_{2^\lambda}(x) = 2^{\lambda-1}$ from

$$\deg\, B(x) = \deg\, F_{2^\lambda}(x) + \deg\, D(x)$$

it follows that $\deg\, D(x) \leqslant 2^{\lambda-1} - 1$. So if x^{α_i} and $-x^{\beta_j}$ are terms of $B(x)$, then so are $x^{\alpha_i + 2^{\lambda-1}}$ and $-x^{\beta_j + 2^{\lambda-1}}$. Therefore $a^{2^{\lambda-1}}$ is a period of B. Hence there is a subset U of G such that $B = U\langle a^{2^{\lambda-1}}\rangle$, where the product is direct. This completes the proof.

Corollary. 1. *Let A be a type 1 subset of G. If $\chi(A) = 0$, then A is periodic with period a^{2^λ}.*

Proof. By Lemma 1 for each subset A of G there are subsets U, V of G such that $A = U\langle a^{2^{\lambda-1}}\rangle \cup V\langle b\rangle$, where the union is disjoint and the products are direct. But now A is a type 1 subset and so $V = \emptyset$. This completes the proof.

Lemma 2. *Let A be a type 1 subset of G. If $\chi'(A) = 0$, then there are subsets U, V of G such that $A = U\langle a^{2^{\lambda-1}}\rangle \cup V\langle a^{2^{\lambda-1}}b\rangle$, where the union is disjoint and the products $U\langle a^{2^{\lambda-1}}\rangle$ and $V\langle a^{2^{\lambda-1}}b\rangle$ are direct.*

Proof. We can partition A in the form $A = B \cup C$, where C is periodic with period $a^{2^{\lambda-1}}b$. Simply put a^α to B if $a^\alpha \in A$ and $a^{\alpha+2^{\lambda-1}}b \notin A$ or put $a^{\alpha+2^{\lambda-1}}b$ to B if $a^\alpha \notin A$ and $a^{\alpha+2^{\lambda-1}}b \in A$; and put a^α and $a^{\alpha+2^{\lambda-1}}b$ to C if $a^\alpha \in A$ and $a^{\alpha+2^{\lambda-1}}b \in A$. There is a subset V of G such that $C = V\langle a^{2^{\lambda-1}}b\rangle$ and the product is direct. It is clear that B can be written in the form

$$B = \{a^{\alpha_1}, \dots, a^{\alpha_r}, a^{\beta_1}b, \dots, a^{\beta_s}b\}$$

where $0 \leqslant \alpha_i, \beta_j \leqslant 2^\lambda - 1$ and $\alpha_i \neq \beta_j$. Note that

$$0 = \chi'(A) = \chi'(B) + \chi'(V)\chi'(\langle a^{2^{\lambda-1}}b\rangle) = \chi'(B)$$

since $\chi'(\langle a^{2^{\lambda-1}}b\rangle) = 1 + \chi'(a^{2^{\lambda-1}})\,\chi'(b) = 1 - 1 = 0$. Hence

$$0 = \chi'(B) = \varrho^{\alpha_1} + \dots + \varrho^{\alpha_r} + \varrho^{\beta_1} + \dots + \varrho^{\beta_s}.$$

Consider the polynomial

$$B(x) = x^{\alpha_1} + \ldots + x^{\alpha_r} + x^{\beta_1} + \ldots + x^{\beta_s}$$

associated with $\chi'(B)$. In a similar way as we have seen in the proof of Lemma 1 we can conclude that B is periodic with period $a^{2^{\lambda-1}}$. Hence there is a subset U of G such that $B = U\langle a^{2^{\lambda-1}}\rangle$. This completes the proof.

Corollary 2. *Let A be a subset of G. If $\chi(A) = \chi'(A) = 0$, then A is periodic with period $a^{2^{\lambda-1}}$.*

Proof. As $\chi(A) = 0$ by Lemma 1 there are subsets U, V of G such that $A = U\langle a^{2^{\lambda-1}}\rangle \cup V\langle b\rangle$, where the union is disjoint and the products are direct. Now

$$0 = \chi'(A) = \chi'(U)\,\chi'(\langle a^{2^{\lambda-1}}\rangle) + \chi'(V)\,\chi'(\langle b\rangle) = \chi'(V)$$

since $\chi'(\langle a^{2^{\lambda-1}}\rangle) = 1 + \chi'(a^{2^{\lambda-1}}) = 1 - 1 = 0$ and $\chi'(\langle b\rangle) = 1 + \chi'(b) = 1 + 1 = 2$. We know that V is a type 1 subset of G. So by Corollary 1 V is periodic with period $a^{2^{\lambda-1}}$. Therefore A is periodic with period $a^{2^{\lambda-1}}$. This completes the proof.

Now we are ready to prove the main result.

Theorem 1. *If $G = A_1 \ldots A_n$ is a factorization of G, then at least one of the factors $A_1, \ldots, A_n$ is periodic.*

Proof. As a first step we show that at most one of the factors $A_1, \ldots, A_n$ can be a type 2 subset of G. Each A_i can be partitioned in the form $A_i = B_i \cup V_i\langle b\rangle$, where B_i is a type 1 subset of G and the product $V_i\langle b\rangle$ is direct. Clearly A_i is a type 1 subset of G if and only if $V_i = \emptyset$. Now assume the contrary that, say, A_1 and A_2 are type 2 subsets of G. So there are elements $v_1 \in V_1$ and $v_2 \in V_2$. Multiplying the factorization $G = A_1 \ldots A_n$ by $g = v_1^{-1}v_2^{-1}$ yields the factorization

$$G = Gg = (v_1^{-1}A_1)(v_2^{-1}A_2)\,A_3 \ldots A_n.$$

Here $b \in v_1^{-1}A_1$ and $b \in v_2^{-1}A_2$ contradict the definition of the factorization.

In the remaining part of the proof we distinguish two cases:

Case 1. Each A_i is a type 1 subset of G

Case 2. A_1 is a type 2 and $A_2, \ldots, A_n$ are type 1 subsets of G

Turn to **Case 1** first. As χ is a nonprincipal character of G there is an i, $1 \leqslant i \leqslant n$ such that $\chi(A_i) = 0$. By Corollary 1 A_i is a periodic subset of G.

Turn to **Case 2.** As χ is a nonprincipal character of G, there is an i, $1 \leqslant i \leqslant n$, such that $\chi(A_i) = 0$. If $i \geqslant 2$, then by Corollary 1 A_i is periodic. So we assume that $i = 1$, that is, $\chi(A_1) = 0$. By Lemma 1 there are subsets U_1, V_1 of G such that $A_1 = U_1\langle a^{2^{\lambda-1}}\rangle \cup V_1\langle b\rangle$, where the union is disjoint and the products are direct. If $U_1 = \emptyset$ or $V_1 = \emptyset$, then A_1 is periodic. So we assume that $U_1 \neq \emptyset$ and $V_1 \neq \emptyset$.

As χ' is a nonprincipal character of G, there is an i, $1 \leqslant i \leqslant n$, such that $\chi'(A_i) = 0$. If $i \geqslant 2$, say $i = 2$, then by Lemma 2 there are subsets U_2, V_2 of G such that $A_2 = U_2\langle a^{2^{\lambda-1}}\rangle \cup V_2\langle a^{2^{\lambda-1}}b\rangle$, where the union is disjoint and the products are direct.

If $U_2 = \emptyset$ or $V_2 = \emptyset$, then A_2 is periodic. So we assume that $U_2 \neq \emptyset$ and $V_2 \neq \emptyset$. Since $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$, there are elements $u_1 \in U_1$ and $u_2 \in U_2$. Multiplying the factorization $G = A_1 \dots A_n$ by $g = u_1^{-1}u_2^{-1}$ we obtain the factorization

$$G = Gg = (u_1^{-1}A_1)(u_2^{-1}A_2)A_3 \dots A_n.$$

Here $a^{2^{\lambda-1}} \in u_1^{-1}A_1$ and $a^{2^{\lambda-1}} \in u_2^{-1}A_2$ contradict the definition of the factorization.

If $i = 1$, then by Corollary 2 A_1 is periodic. This completes the proof.

References

[1] A. D. Sands, *The factorization of abelian groups*, Quart. J. Math. Oxford **10** (1959), 81-91.

Sommario

Sia G il prodotto diretto di due gruppi ciclici di ordini 2^λ *e* 2, *rispettivamente. A. D. Sands ha dimostrato in* [1] *che, se G è prodotto diretto di due suoi sottinsiemi, almeno uno di essi deve essere periodico. Si mostra che il risultato di Sands sussiste anche quando G è fattorizzato in più di due sottoinsiemi.*

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