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Linear connections on manifolds admitting an almost s-tangent structure (**)

1 - Introduction

Let V be a differentiable manifold (C^∞, connected, second-countable) of dimension $2n+1$ and $\mathfrak{X}(V)$ the module of the vector fields on V.

An *almost s-tangent structure* on V (J. A. Oubiña [5]) is a triple (I, ω, ξ), where I is a tensor field of type (1, 1), ω is a differential 1-form and ξ is a vector field on V such that

$$\omega(\xi)=1 \qquad I^2=\omega\otimes\xi \qquad \text{rank}\,(I)=n+1\,. \tag{1.1}$$

Then $I\xi=\lambda\omega$ and $\omega I=\lambda\omega$, where $\lambda=\omega(I\xi)$, $\lambda^2=1$. A differentiable manifold with an almost s-tangent structure is called an almost s-tangent manifold.

J. A. Oubiña proved in [6] that a $(2n+1)$-dimensional differentiable manifold V admits an almost s-tangent structure if and only if it admits a vector field ξ, a differential 1-form ω and tensor fields φ and ψ of type (1, 1) such that:

a. (φ, ξ, ω) is an almost contact structure
b. (ψ, ξ, ω) is an almost paracontact structure
c. $\varphi\psi=-\psi\varphi$
d. $\text{rank}\,(k)=\text{rank}\,(s)=n$, being

$$k=\frac{1}{2}(1+\varphi\psi-p) \qquad s=\frac{1}{2}(1-\varphi\psi-p) \qquad p=\omega\otimes\xi. \tag{1.2}$$

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Then we have [**6**]:

$$k^2 = k \qquad s^2 = s \qquad p^2 = p$$

$$sp = ps = 0 \qquad kp = pk = 0 \qquad ks = sk = 0 \tag{1.3}$$

$$k + s + p = 1$$

and hence k, s and p define complementary distributions K, S and P. If D is a distribution on V, then let $\mathcal{S}(D)$ be the vector fields module in the distribution D.

2 - Linear connections on almost s-tangent manifold

Let $V(I, \omega, \xi)$ be an almost s-tangent manifold of dimension $2n+1$, $\mathcal{C}(V)$ the affine module of the linear connections on V. In the following paragraphs $\nabla \in \mathcal{C}(V)$ will be a fixed linear connection on V.

We consider two operators $\overline{\nabla}$ and $\tilde{\nabla}$ on V defined by

$$\overline{\nabla}_X Y = k\nabla_X(kY) + s\nabla_X(sY) + p\nabla_X(pY) \tag{2.1}$$

$$\begin{aligned} \tilde{\nabla}_X Y &= k\nabla_{kX}(kY) + s\nabla_{sX}(sY) + p\nabla_{pX}(pY) \\ &+ k[(1-k)X, kY] + s[(1-s)X, sY] + p[(1-p)X, pY] \end{aligned} \tag{2.2}$$

for any vector fields X, Y on V, where $[X, Y]$ is the Lie product of X and Y.

One can easily verify that $\overline{\nabla}$ and $\tilde{\nabla}$ are linear connections.

A distribution D on V is said to be ∇-*parallel* if for every $Y \in \mathcal{S}(D)$ we have $\nabla_X Y \in \mathcal{S}(D)$ for every $X \in \mathfrak{X}(V)$.

Theorem 1. *In an almost s-tangent manifold $V(I, \omega, \xi)$ the distributions K, S, and P are $\overline{\nabla}$-parallel as well as $\tilde{\nabla}$-parallel.*

Proof. Since $sk = 0$ and $pk = 0$, we have by (2.1), (2.2)

$$s\overline{\nabla}_X Y = 0 \qquad p\overline{\nabla}_X Y = 0 \qquad s\tilde{\nabla}_X Y = 0 \qquad p\tilde{\nabla}_X Y = 0$$

for any $X \in \mathfrak{X}(V)$. Consequently, K is $\overline{\nabla}$-parallel as well as $\tilde{\nabla}$-parallel. The same can be shown for the distributions S and P.

Theorem 2. *In an almost s-tangent manifold $V(I, \omega, \xi)$ the distributions K, S and P are ∇-parallel if and only if the connections $\overline{\nabla}$ and ∇ are equal.*

Proof. If K, S and P are ∇-parallel, then

$$\nabla_X(kY) = k\nabla_X(kY) \qquad \nabla_X(sY) = s\nabla_X(sY) \qquad \nabla_X(pY) = p\nabla_X(pY)$$

for arbitrary vector fields $X, Y \in \mathfrak{X}(V)$.

Therefore, since $k + s + p = 1$, we obtain by (2.1) that ∇ and $\overline{\nabla}$ are equal. In the same way, it can be shown the converse.

3 - Distribution's involutivity

A distribution D on V is called involutive if for every two vector fields X, Y of $\mathcal{J}(D)$ we have $[X, Y] \in \mathcal{J}(D)$.

Theorem 3. *If the linear connection ∇ is symmetric and the distributions K, S and P are involutive, then the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ is given by*

$$\tilde{T}(X, Y) = -\sum k[sX, pY] \tag{3.1}$$

where the sum is obtained of all the permutations (k, s, p).

Proof. Since the connection ∇ is symmetric, the corresponding torsion tensor

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0\,. \tag{3.2}$$

On the other hand, the corresponding torsion tensor of the connection $\tilde{\nabla}$ is

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]\,.$$

By simple calculation and taking account of (3.2) $k[sX, sY] = 0$, $k[pX, pY] = 0$ and the others, we have (3.1).

Theorem 4. *If the connection $\overline{\nabla}$ is symmetric, then the distributions K, S and P are involutive.*

Proof. The distribution K is involutive if

$$s[kX, kY] = 0 \qquad p[kX, kY] = 0 \qquad \forall X, Y \in \mathfrak{X}(V). \tag{3.3}$$

Since $\overline{\nabla}$ is symmetric, the first relation (3.3) becomes

$$s\overline{\nabla}_{kX}kY - s\overline{\nabla}_{kY}kX = 0.$$

This condition is verified by Theorem 1. Likewise, it is shown that the second condition (3.3) is verified.

Remark 1. Let $\overline{p}$: $\mathcal{C}(V) \to \mathcal{C}(V)$, $\tilde{p}$: $\mathcal{C}(V) \to \mathcal{C}(V)$ be the applications defined by $\overline{p}(\nabla) = \overline{\nabla}$, $\tilde{p}(\nabla) = \tilde{\nabla}$ where ∇ is an arbitrary linear connection on V and $\overline{\nabla}$ and $\tilde{\nabla}$ are the connections given by (2.1) and (2.2).

One can easily verify that $\overline{p}$ and $\tilde{p}$ are the affine projections on the affine module $\mathcal{C}(V)$:

$$\overline{p}^2 = \overline{p} \qquad \tilde{p}^2 = \tilde{p}.$$

Moreover, we have the properties

$$\overline{p} \circ \tilde{p} = \tilde{p} \circ \overline{p} = \tilde{p}.$$

Introducing the notations $\overline{\mathcal{C}}(V) = \overline{p}\,\mathcal{C}(V)$ and $\tilde{\mathcal{C}}(V) = \tilde{p}\,\mathcal{C}(V)$ we find that $\overline{\mathcal{C}}(V)$ and $\tilde{\mathcal{C}}(V)$ are affine submodules and that $\tilde{\mathcal{C}}(V) \subset \overline{\mathcal{C}}(V)$.

Remark 2. Let us consider now the operators:

$$\overline{\nabla}^k: \mathfrak{X}(V) \times \mathcal{S}(K) \to \mathcal{S}(K) \qquad \overline{\nabla}^k_X Y = k\nabla_X Y$$

$$\overline{\nabla}^s: \mathfrak{X}(V) \times \mathcal{S}(S) \to \mathcal{S}(S) \qquad \overline{\nabla}^s_X Y = s\nabla_X Y$$

$$\overline{\nabla}^p: \mathfrak{X}(V) \times \mathcal{S}(P) \to \mathcal{S}(P) \qquad \overline{\nabla}^p_X Y = p\nabla_X Y$$

where $\nabla \in \mathcal{C}(V)$.

It is easy to see that $\overline{\nabla}^k$, $\overline{\nabla}^s$, $\overline{\nabla}^p$, are linear connections on the distributions (vectorial bundles) K, S and P, respectively. We observe that the connection $\overline{\nabla}$ can be regarded as the sum of the three connections $\overline{\nabla}^k$, $\overline{\nabla}^s$ and $\overline{\nabla}^p$ induced on the vectorial subbundles K, S and P:

$$\nabla_X Y = \overline{\nabla}^k_X(kY) + \overline{\nabla}^s_X(sY) + \overline{\nabla}^p_X(pY) \qquad X, Y \in \mathfrak{X}(V).$$

On the other hand the connection $\tilde{\nabla}$ can be written as

$$\tilde{\nabla}_X Y = \nabla'_{kX} kY + \nabla''_{sX} sY + \nabla'''_{pX} pY$$

$$+ k[(l-k)X, kY] + s[(1-s)X, sY] + p[(1-p)X, pY]$$

where ∇', ∇'', ∇''' are the restriction of the connections $\overline{\nabla}^k$, $\overline{\nabla}^s$ and $\overline{\nabla}^p$ on $\mathcal{I}(K)$, $\mathcal{I}(S)$ and $\mathcal{I}(P)$ respectively.

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Sommario

Il lavoro introduce due connessioni, denotate con $\overline{\nabla}$ e $\tilde{\nabla}$, sulle varietà quasi s-tangenti e ne indica alcune proprietà.

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