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On free d.g. seminear-rings (**)

1 - Introduction

The idea of a seminear-ring was introduced in [5], as an algebraic system that can be constructed from a set S with two binary operations addition $+$ and multiplication $\cdot$ such that

i. $(S, +)$ and $(S, \cdot)$ are semigroups

ii. one distributive law is satisfied, namely, $a(b+c) = ab + ac$ for all $a, b, c \in S$.

S as defined above is called a *left seminear-ring* (which we are considering in the present work) and it is called a *right seminear-ring* if it satisfies **i** and

ii'. $(a+b)c = ac + bc$, for all $a, b, c \in S$.

A natural example of a left seminear-ring is the set $M(S)$ of all mappings on a semigroup $(S, +)$ with the operations of pointwise addition and multiplication given by composition of maps.

A seminear-ring (s.n.r. for short) S is called *distributively generated* (d. g.) if S contains a multiplicative subsemigroup $(T, .)$ of distributive elements which generates $(S, +)$. It should be noticed that if a semigroup $(S, +)$ is generated by $(T, .)$, then S is a seminear-ring. We notice that T need not be the semigroup of all distributive elements, and such a d.g. seminear-ring (d.g.s.n.r.) is denoted by (S, T). A. Fröhlich [1], [2] and J. D. P. Meldrum [3] have given some results concerning free d.g. near-rings in a variety $\mathfrak{V}$.

In this paper we generalize some of these results to free d.g.s.n.rs. and we can

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prove the existence of free (S, T)-semigroups on a set X in a variety $\mathcal{V}$. In this direction we construct and define the free d.g.s.n.r. in a variety $\mathcal{V}$, then we proceed towards our results.

Although our scheme relies on the work of A. Fröhlich and J. D. P. Meldrum, as d.g.s.n.rs. lack the group structure, we have to proceed along a slightly different path dealing with semigroups in order to generalize the main results.

Before going through these results, we give some definitions and basic concepts which will be used subsequently. It should be noted that some fundamental ideas are defined in a way analogous to the case of d.g.n.rs. However, some concepts are not because semigroups are involved rather than groups.

2. - Preliminaries

The following definitions and results are needed.

Definition 1. Let T be a multiplicative semigroup. A semigroup H is called a *T-semigroup* if there exists a homomorphism $\theta\colon T \to \mathrm{End}\,(H)$. We write ht for $h(t\theta)$. If S is a seminear-ring, then a semigroup H is called an *S-module* if there is a seminear-ring homomorphism $\theta\colon S \to M(H)$.

Definition 2. Let S be a s.n.r. and let T be a multiplicative subsemigroup of S. Let H and K be two S-modules. Then a homomorphism $\theta\colon H \to K$ is called a *T-homomorphism* if $(ht)\,\theta = (h)\,\theta t$ for all $h \in H$, $t \in T$.

The following three lemmas are the d.g.s.n.r. versions of results given, for the d.g.n.r. case, by J. D. P. Meldrum [4].

Lemma 1. *Let (S, T) be a d.g.s.n.r. and let H and K be two S-modules. Then a homomorphism $\theta\colon H \to K$ is an S-homomorphism if and only if it is a T-homomorphism.*

Definition 3. Let (N, T) and (M, U) be two d.g.s.n.rs. A seminear-ring homomorphism $\theta\colon N \to M$ such that $T\theta \subseteq U$ is called a d.g. homomorphism.

Lemma 2. *Let (N, T) and (M, U) be two d.g.s.n.rs. If θ is a map from N to M which is both a semigroup homomorphism from $(N, +)$ to $(M, +)$ and a semigroup homomorphism from $(T, \cdot)$ to $(U, \cdot)$, then it is a d.g.s.n.r. homomorphism from (N, T) to (M, U).*

Definition 4. Let $\mathfrak{V}$ be a variety of semigroups. If the additive semigroup of the seminear-ring S lies in $\mathfrak{V}$, then we say that the seminear-ring S *belongs to* $\mathfrak{V}$.

Lemma 3. *Let $w(x_1, \ldots, x_n)$ be a semigroup word in n variables $x_1, \ldots, x_n$. Then $hw(s_1, \ldots, s_n) = w(hs_1, \ldots, hs_n)$, whenever $s_1, \ldots, s_n$ lie in a seminear-ring S and h lies in an S-module H.*

Using Lemma 3 we can prove

Theorem 1. *Let S be a s.n.r. with a faithful representation on the S-module H. If H belongs to variety $\mathfrak{V}$, then so does S.*

Proof. Let $w(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)$ be a law in the variety $\mathfrak{V}$ for which $H \in \mathfrak{V}$. By Lemma 3, for $s_1, \ldots, s_n \in S$ and $h \in H$, we have

$$\begin{aligned} hw(s_1, \ldots, s_n) &= w(hs_1, \ldots, hs_n) \\ hu(s_1, \ldots, s_n) &= u(hs_1, \ldots, hs_n) \quad \text{for all } h \in H . \end{aligned} \tag{1}$$

Since the law $w = u$ holds in $H \in \mathfrak{V}$ and $hs_i \in H$, it follows that

$$w(hs_1, \ldots, hs_n) = u(hs_1, \ldots, hs_n) .$$

Substituting from (1) yields

$$hw(s_1, \ldots, s_n) = hu(s_1, \ldots, s_n) \quad \text{for all } h \in H$$

so $w(s_1, \ldots, s_n)$ and $u(s_1, \ldots, s_n)$ induce the same map on H. As H is faithful, we conclude that $w(s_1, \ldots, s_n) = u(s_1, \ldots, s_n)$, which means that the law $w = u$ holds in $(S, +)$. But this is true for all the laws of $\mathfrak{V}$. Thus $(S, +) \in \mathfrak{V}$.

Definition 5. Let (S, T) be a d.g.s.n.r. A representation θ of S is a *d.g. representation* if θ: $S \to M(H)$ satisfies $T\theta \subseteq \mathrm{End}(H)$, where H is the S-module associated with the representation θ.

We note that a d.g. representation of (S, T) on the S-module H is a d.g. homomorphism from (S, T) to $(E(H), \mathrm{End}(H))$.

The above definition leads to

Definition 6. Let (S, T) be a d.g.s.n.r. A semigroup H is called an (S, T)-*semigroup* if (S, T) has a d.g. representation on H.

3. – Free d.g. seminear-rings

Let $\mathcal{V}$ be a variety of semigroups. Given a set X, $F_{\mathcal{V}}(X)$ denotes the free additive semigroup in $\mathcal{V}$ on X. Let T be a multiplicative semigroup and define the semigroup $\mathrm{Frs}_{\mathcal{V}}(X, T)$ as the free semigroup in the variety $\mathcal{V}$ on the set of symbols: $\{x, t_x : x \in X, t \in T\} = T_x$. Thus we may write

$$\mathrm{Frs}_{\mathcal{V}}(X, T) = F_{\mathcal{V}}\langle T_x \rangle .$$

For each $t \in T$, define a map t^* from T_x into $\mathrm{Frs}_{\mathcal{V}}(X, T)$ by

$$(2) \qquad x \cdot t^* = t_x \qquad (m_x)\, t^* = (mt)_x \qquad \text{for all } x \in X, \quad m \in T$$

which we extend to be an endomorphism of $\mathrm{Frs}_{\mathcal{V}}(X, T)$. Thus the elements of T are mapped to $\mathrm{End}(\mathrm{Frs}_{\mathcal{V}}(X, T))$, the semigroup of all endomorphisms of $\mathrm{Frs}_{\mathcal{V}}(X, T)$. Indeed this map, $t \to t^*$, is a monomorphism of semigroups. Note that we may replace t^* by t itself where no confusion would result; then we can write (2) as

$$(2)' \qquad x \cdot t = t_x \qquad (m_x)\, t = (mt)_x \qquad \text{for all } x \in X, \quad m \in T .$$

Now, assume that T is a semigroup of endomorphisms of $\mathrm{Frs}_{\mathcal{V}}(X, T)$, then T generates a d.g.s.n.r. which we will denote $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ and call *the free d.g.s.n.r. on T in $\mathcal{V}$*. One of our main purposes is to clarify the reason for the name given to $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ as defined above. Before doing so, we give a lemma, which is a consequence of Theorem 1.

Lemma 4. *The free d.g.s.n.r. on T in $\mathcal{V}$ lies in $\mathcal{V}$.*

Now we can prove an important result concerning $(\mathrm{Frs}_{\mathcal{V}}(T), T)$.

Theorem 2. *Let $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ be the free d.g.s.n.r. on T in the variety $\mathcal{V}$. Then the following hold :*

i. *$(\mathrm{Frs}_{\mathcal{V}}(T), +)$ is the free semigroup in $\mathcal{V}$ on the set T*

ii. *Every T-semigroup H in $\mathcal{V}$ is a $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-semigroup*

iii. Let *(S, U) be a d.g.s.n.r. in $\mathcal{V}$ generated by the multiplicative semigroup U. Then every semigroup homomorhism $\theta: T \to U$ can be extended to a d.g.s.n.r. homomorphism from $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ to (S, U).*

Proof.

i. Evidently $(\mathrm{Frs}_{\mathfrak{V}}(T), +)$ lies in $\mathfrak{V}$, by Lemma 4. Let $t_1 + \ldots + t_n$ be an element of $(\mathrm{Frs}_{\mathfrak{V}}(T), +)$, the subsemigroup of $\mathrm{End}(\mathrm{Frs}_{\mathfrak{V}}(X, T))$, where $t_i \in T$, $1 \leqslant i \leqslant n$. By (2)′ we have

$$x(t_1 + \ldots + t_n) = t_{1x} + \ldots + t_{nx}\,.$$

Consider $\mathrm{sg}\langle t_x;\, t \in T\rangle$, the semigroup which is generated by the set $\{t_x;\, t \in T\}$. For each $x \in X$, define a map θ_x where

$$\theta_x\colon (\mathrm{Frs}_{\mathfrak{V}}(T), +) \to \mathrm{sg}\langle t_x;\, t \in T\rangle \qquad \text{is given by}$$

(3)

$$(t_1 + \ldots + t_n)\,\theta_x = x(t_1 + \ldots + t_n) = t_{1x} + \ldots + t_{nx}\,.$$

The map θ_x is well defined because, if we assume that $a = b$ in $(\mathrm{Frs}_{\mathfrak{V}}(T), +)$ where $a = t_1 + \ldots + t_n$, $b = k_1 + \ldots + k_m$, $t_i, k_j \in T$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, then

$$x(t_1 + \ldots + t_n) = x(k_1 + \ldots + k_m)$$

$$(t_1 + \ldots + t_n)\,\theta_x = (k_1 + \ldots + k_m)\,\theta_x\,.$$

It follows that $(a)\theta_x = (b)\,\theta_x$. That is θ_x is well defined.

Now we show that $(\mathrm{Frs}_{\mathfrak{V}}(T), +) \cong \mathrm{sg}\langle t_x;\, t \in T\rangle$. Let $a, b \in (\mathrm{Frs}_{\mathfrak{V}}(T), +)$. Then $a = t_1 + \ldots + t_n$, $b = k_1 + \ldots + k_m$, where $t_i, k_j \in T$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Thus

$$\begin{aligned}(a + b)\theta_x &= (t_1 + \ldots + t_n + k_1 + \ldots + k_m)\,\theta_x \\ &= x(t_1 + \ldots + t_n + k_1 + \ldots + k_m) \\ &= t_{1x} + \ldots + t_{nx} + k_{1x} + \ldots k_{mx} \\ &= (t_1 + \ldots + t_n)\,\theta_x + (k_1 + \ldots + k_m)\,\theta_x \\ &= (a)\,\theta_x + (b)\,\theta_x\,.\end{aligned}$$

Hence θ_x is a homomorphism.

Now if $(a)\,\theta_x = (b)\,\theta_x$ then

$$t_{1x} + \ldots + t_{nx} = k_{1x} + \ldots + k_{mx}$$

$$t_1 + \ldots + t_n = k_1 + \ldots + k_m\,.$$

Thus $a = b$ and θ_x is a monomorphism.

Finally if $t_{1x} + \ldots + t_{nx}$ is an element of $\text{sg}\langle t_x;\, t \in T\rangle$, then as $xt_i = t_{ix}$, $1 \leqslant i \leqslant n$, we have

$$t_{1x} + \ldots + t_{nx} = xt_1 + \ldots + xt_n = x(t_1 + \ldots + t_n)$$

where $t_1 + \ldots + t_n \in (\text{Frs}_{\mathfrak{V}}(T),\ +)$. So that $(t_1 + \ldots + t_n)\,\theta_x = t_{1x} + \ldots + t_{nx}$ which proves that θ_x is an epimorphism and we have shown that

$$(\text{Frs}_{\mathfrak{V}}(T),\ +) \cong \text{sg}\langle t_x;\, t \in T\rangle.$$

The right hand side of the above isomorphism is the free semigroup on the set $\{t_x;\, t \in T\}$ in which for each t_x there corresponds an element $t \in T$. This implies that $(\text{Frs}_{\mathfrak{V}}(T),\ +)$ is the free semigroup in $\mathfrak{V}$ on the set T.

ii. Let H be a T-semigroup in $\mathfrak{V}$; then there exists a homomorphism θ of T into End(H). As $H \in \mathfrak{V}$, $E(H)$ also lies in $\mathfrak{V}$ by Theorem 1. Since $(\text{Frs}_{\mathfrak{V}}(T),\ +)$ is the free semigroup in $\mathfrak{V}$ on T and $E(H) \in \mathfrak{V}$, we can extend θ to a semigroup homomorphism ψ such that $\psi\colon (\text{Frs}_{\mathfrak{V}}(T),\ +) \to (E(H),\ +)$. By Lemma 2, ψ is a d.g. homomorphism which means that ψ is a d.g. representation of $(\text{Frs}_{\mathfrak{V}}(T),\ T)$ on H or equivalently H is a $(\text{Frs}_{\mathfrak{V}}(T),\ T)$-semigroup.

iii. Since $(\text{Frs}_{\mathfrak{V}}(T),\ +)$ is the free semigroup in $\mathfrak{V}$ on the set T and $(S,\ +)$ lies in $\mathfrak{V}$, then θ can be extended uniquely to a semigroup homomorphism

$$\eta\colon (\text{Frs}_{\mathfrak{V}}(T),\ +) \to (S,\ +).$$

By Lemma 2, η is a d.g.s.n.r. homomorphism from $(\text{Frs}_{\mathfrak{V}}(T),\ T)$ to $(S,\ U)$.

Remark. It should be noticed that the structure of $(\text{Frs}_{\mathfrak{V}}(T),\ T)$ is independent of the set X.

Corollary 1. *Let $(S,\ T)$ be a d.g.s.n.r. in $\mathfrak{V}$. Then there is a d.g. epimorphism $\theta : (\text{Frs}(T),\ T) \to (S,\ T)$ extending the identity map on T.*

In order to give the next result, we need to define the free $(S,\ T)$-semigroup in a given variety $\mathfrak{V}$.

Definition 7. Let $(S,\ T)$ be a d.g.s.n.r. in a variety $\mathfrak{V}$. An $(S,\ T)$-semigroup H is a *free $(S,\ T)$-semigroup in $\mathfrak{V}$ on a set X* if $H \in \mathfrak{V}$, $X \subseteq H$ and if K is an $(S,\ T)$-semigroup in $\mathfrak{V}$ then any map $\theta\colon X \to K$ can be extended uniquely to an $(S,\ T)$ homomorphism from H to K.

The following theorem tells us about the free (S, T)-semigroup that we have already met.

Theorem 3. *The free semigroup* $\mathrm{Frs}_{\mathcal{V}}(X, T)$ *in a variety* $\mathcal{V}$ *on the set* T_x, *is the free* $(\mathrm{Frs}_{\mathcal{V}}(T), T)$*-semigroup in* $\mathcal{V}$ *on the set* X.

Proof. Let H be a $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-semigroup in $\mathcal{V}$. Let θ be a map such that $\theta\colon X \to H$. Then we can extend θ in a unique way to be a T-homomorphism from $\mathrm{Frs}_{\mathcal{V}}(X, T)$ to H as the following shows.

Since we already have $x \cdot t = t_x$, $x \in X$, $t \in T$, it follows that

$$(t_x)\,\theta = (x \cdot t)\,\theta = (x\theta)\,t\,.$$

Hence θ is extended (as a map) to T_x.

Since $\mathrm{Frs}_{\mathcal{V}}(X, T) = \mathrm{sg}\langle T_x \rangle$ and $\mathrm{Frs}_{\mathcal{V}}(X, T)$ is a free semigroup on the set T_x, therefore θ is indeed extended uniquely to be a homomorphism from $\mathrm{Frs}_{\mathcal{V}}(X, T)$ to H. Now it only remains to show that θ is an (S, T)-homomorphism. Let $y \in (\mathrm{Frs}_{\mathcal{V}}(X), T)$, then $y = t_{1x_1} + \ldots + t_{nx_n}$, where $t_i \in T \cup \{1\}$, $x_i \in X$, $1 \leqslant i \leqslant n$, and 1_{x_i} represents x_i. For $t \in T$, we have

$$\begin{aligned}
(yt)\,\theta &= ((t_{1x_1} + \ldots + t_{nx_n})\,t)\,\theta = (t_{1x_1}t + \ldots + t_{nx_n}t)\,\theta \\
&= ((t_1 t)_{x_1} + \ldots + (t_n t)_{x_n})\,\theta = (t_1 t)_{x_1}\theta + \ldots + (t_n t)_{x_n}\theta \\
&= (x_1\theta)\,t_1 t + \ldots + (x_n\theta)\,t_n t = ((x_1\theta)\,t_1 + \ldots + (x_n\theta)\,t_n)\,t \\
&= ((x_1 t_1)\,\theta + \ldots + (x_n t_n)\,\theta)\,t = ((t_{1x_1})\,\theta + \ldots + (t_{nx_n}\theta))\,t \\
&= (t_{1x_1} + \ldots + t_{nx_n})\,\theta t = (y\theta)t\,.
\end{aligned}$$

That is θ is a T-homomorphism and the result is proved.

Now we are able to prove the existence of the free (S, T)-semigroup in $\mathcal{V}$ on a set X.

Theorem 4. *The free* (S, T)*-semigroup in* $\mathcal{V}$ *on a set* X *always exists.*

Proof. Let (S, T) be a d.g.s.n.r.; then by Corollary 1, there is a d.g. epimorphism θ which extends the identity map on T such that

$$\theta\colon (\mathrm{Frs}_{\mathcal{V}}(T), T) \to (S, T)\,,$$

where $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ is the free d.g.s.n.r. on T in $\mathcal{V}$.

Let $\varrho = \operatorname{Ker}\theta$. Consider $\mathrm{Frs}_{\mathfrak{V}}(X, T)$ and let N be the least congruence in $\mathrm{Frs}_{\mathfrak{V}}(X, T)$ containing $\mathrm{Frs}_{\mathfrak{V}}(X, T)\varrho$, where

$$\mathrm{Frs}_{\mathfrak{V}}(X, T)\,\varrho = \{(ky_1, ky_2)\colon k \in \mathrm{Frs}_{\mathfrak{V}}(X, T),\ (y_1, y_2) \in \varrho\}.$$

Let $H = \mathrm{Frs}_{\mathfrak{V}}(X, T)/N$. We show that H is the free (S, T)-semigroup in $\mathfrak{V}$ on a set X. Certainly H is an $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$-semigroup. From above, the kernel of the representation of $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$ contains ϱ. Thus as $(S, T) \cong (\mathrm{Frs}_{\mathfrak{V}}(T), T)/\varrho$, we can define H canonically as an (S, T)-semigroup. Let K be an (S, T)-semigroup. Then it is a T-semigroup and, by Theorem 2 **ii**, it is an $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$-semigroup. Since K is an (S, T)-semigroup, the representation of $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$ on K contains ϱ in its kernel.

Let $\phi\colon X \to K$. Then by Theorem 3, ϕ can be extended to an $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$-homomorphism $\phi\colon \mathrm{Frs}_{\mathfrak{V}}(X, T) \to K$. Let $k \in \mathrm{Frs}_{\mathfrak{V}}(X, T)$, $(y_1, y_2) \in \varrho$, then $k(y_1, y_2)$ equals $(ky_1, ky_2) \in \ker\phi$, since the representation of $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$ on K contains ϱ in its kernel. Thus $N \subseteq \ker\phi$, N being the least congruence in $\mathrm{Frs}_{\mathfrak{V}}(X, T)$ containing $\mathrm{Frs}_{\mathfrak{V}}(X, T)\varrho$. This means that ϕ can be factored as a product $\psi\phi'$ where ψ is the natural $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$-homomorphism such that $\psi\colon \mathrm{Frs}(X, T) \to H$. We deduce that ϕ' is a T-homomorphism from the (S, T)-semigroup H to K extending ϕ. Since ϕ' agrees with ϕ on the T-generating set X of H, it is uniquely defined, forcing H to be the free (S, T)-semigroup in $\mathfrak{V}$ on the set X.

Our next result shows that given a d.g.s.n.r. (S, T) then it is not possible in general to find a semigroup H such that (S, T) is embedded in $(E(H), \mathrm{End}(H))$ with T embedded in $\mathrm{End}(H)$, in other words, *not every d.g.s.n.r. has a faithful d.g. representation*, as the following theorem says:

Theorem 5. *There exist d.g.s.n.rs. in a variety $\mathfrak{V}$ which do not have a faithful d.g. representation.*

To prove this result we need to state a definition and a lemma regarding this definition.

Definition 8. A semigroup $(H, +)$ is said to be *subcommutative* if it satisfies

$$a + b + c + d = a + c + b + d \qquad \text{for all } a, b, c, d \in H.$$

Lemma 5. *Let (S, T) be a d.g.s.n.r. Suppose that T contains three elements a, b, c such that $a + b = c$. If (S, T) has a faithful d.g. representation, then a and* *b are relatively subcommutative in the sense of the relation*

$$a+b+a+b=a+a+b+b\,.$$

Proof. Let (S, T) be a d.g.s.n.r. having a faithful d.g. representation on a semigroup H. Then for all $h \in H$, we have

$$(h+h)\,c = hc + hc, \quad \text{as } c \in T \text{ is mapped to an endomorphism of } H$$

$$= h(c+c) = h(a+b+a+b)\,.$$

On the other hand, we have

$$(h+h)\,c = (h+h)(a+b) = (h+h)\,a + (h+h)\,b$$

$$= ha + ha + hb + hb, \quad \text{as } a, b \in T \text{ are mapped to } \mathrm{End}(H)$$

$$= h(a+a+b+b)\,.$$

Thus $h(a+b+a+b) = h(a+a+b+b)$. Since H is a faithful S-module, we conclude

$$a+b+a+b=a+a+b+b\,.$$

Now we are able to prove Theorem 5.

Proof of Theorem 5. Let $\mathfrak{V}$ be a variety of semigroups which are not subcommutative. Let $T = \{x, y, z, 0\}$ be a semigroup with all products equal to zero. Let $(\mathrm{Frs}_{\mathfrak{V}}(T), T)$ be the free d.g.s.n.r. on T in $\mathfrak{V}$. Then all products in $\mathrm{Frs}_{\mathfrak{V}}(T)$ are zero, and hence a semigroup congruence on $(\mathrm{Frs}(T), +)$ is a seminear-ring congruence on $(\mathrm{Frs}_{\mathfrak{V}}(T), +, \cdot)$. Let ϱ be the least congruence on $\mathrm{Frs}_{\mathfrak{V}}(T)$ containing $(x+y, z)$. Define (S, T) to be $(\mathrm{Frs}_{\mathfrak{V}}(T), T)/\varrho$, the canonical homomorphic image of $\mathrm{Frs}_{\mathfrak{V}}(T)$ by ϱ.

We show that S is the free semigroup in $\mathfrak{V}$ on two generators, namely $x+\varrho$ and $y+\varrho$. Let Q be a free semigroup in $\mathfrak{V}$ on two generators x' and y', say. Consider the map $\phi\colon T \to \{x', y'\}$ given by

$$(x)\,\phi = x' \qquad (y)\phi = y' \qquad (z)\,\phi = x' + y'\,.$$

Then ϕ extends to be a homomorhism ϕ^* from $(\mathrm{Frs}_{\mathfrak{V}}(T), +)$ to $(Q, +)$ with $\ker \phi^* \geqslant \{(x+y, z)\}$. Hence $\ker \phi^* \geqslant \varrho$, and ϕ^* can be factored through $(S, +)$. In other words there exists a homomorphism ψ from $(S, +)$ onto $(Q, +)$ such that $(x+\varrho)\,\psi = x'$ and $(y+\varrho)\,\psi = y'$. This shows that $(S, +)$ is free in $\mathfrak{V}$ on

$\{x+\varrho, y+\varrho\}$. Letting $a=x+\varrho$, $b=y+\varrho$, we have

$$a+b=(x+\varrho)+(y+\varrho)=z+\varrho=c\,.$$

By Lemma 5, if (S, T) has a faithful d.g. representation, then a and b are relatively subcommutative. But by the hypothesis, two free generators of a free semigroup in $\mathcal{V}$ can not be subcommutative. This implies that (S, T) can not have a faithful d.g. representation.

References

[1] A. FRÖHLICH, *Distributively generated near-rings I: ideal theory*, Proc. London Math. Soc. 8 (1958), 74-94.

[2] A. FRÖHLICH, *Distributively generated near-rings II: representation theory*, Proc. London Math. Soc. 8 (1958), 95-108.

[3] J. D. P. MELDRUM, *The representation of d.g. near-rings*. J. Austral. Math. Soc. **16** (1973), 467-480.

[4] J. D. P. MELDRUM, *Near-rings and their links with groups*, Research Notes in Math. **134**, Pitman, London 1985.

[5] B. VAN ROOTSELAAR, *Algebraische Kennzeichung freier Wortarithmetiken*, Compositio Math. **15** (1963), 156-168.

Sommario

In questo lavoro si considerano semiquasianelli (seminear-rings). Essi costituiscono una generalizzazione dei quasi anelli che sono semigruppi sia rispetto alla struttura additiva che rispetto alla struttura moltiplicativa. Questo tipo di struttura si presenta in modo naturale considerando l'insieme delle applicazioni di un semigruppo in sè con l'addizione definita puntualmente e la composizione. Gli endomorfismi forniscono esempi di elementi distributivi e conducono all'idea di semiquasianelli distributivamente generati (brevemente d.g.s.n.r.). In questo lavoro, che estende risultati sui quasi anelli distributivamente generati, si studiano l'esistenza e la costruzione di d.g.s.n.r. liberi e di d.g.s.n.r. semigruppi liberi. Si mostra inoltre che non tutti i d.g.s.n.r. sono fedeli.

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