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Comments on pseudo-differential operators (**)

Introduction

In this work we make some comments on various elementary properties of pseudo differential operators as they appear in past work:

1. The $L^2(\boldsymbol{R}^n)$ estimate in [3] is here replaced with $H^s(\boldsymbol{R}^n)$ and $B^{1,s}(\boldsymbol{R}^n)$ estimates (see [4] for discussion on these spaces of distributions).

2. The H^s-estimate which is established in [2] is here replaced by a corresponding $B^{1,s}$-estimate.

3. The order and the true order of some pseudo-differential operators which appear in $H^s(\boldsymbol{R}^n)$ spaces are discussed, in a similar way, in $B^{1,s}(\boldsymbol{R}^n)$ spaces (see our monographs [4], [5]).

4. We terminate by a $B^{1,s}(\boldsymbol{R}^n)$-form of a $H^s(\boldsymbol{R}^n)$ inequality for the reversor operator: $A(x, D) - \mathcal{A}(x, D)$ corresponding to Kohn-Nirenberg regular symbols (see [5]).

1 - H^s- and $B^{1,s}$-estimates

Consider a continuous function $\psi(\xi)$, $\boldsymbol{R}^n \to \boldsymbol{C}$, $\xi = (\xi_1, \xi_2 \ldots \xi_n)$. Let $\operatorname{supp} \psi = \overline{\{\xi;\, \psi(\xi) \neq 0\}}$. We assume that

$$\xi, \eta \in \operatorname{supp} \psi \qquad \text{implies} \qquad |\xi - \eta| \leq c\sqrt{1 + |\xi|} \tag{1.1}$$

where $|\xi| = (\xi_1^2 + \ldots + \xi_n^2)^{\frac{1}{2}}$, as usual.

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Remark. From assumption (1.1) it follows that supp ψ is compact in $\boldsymbol{R}^n$. In fact, let us fix $\xi_0 \in \operatorname{supp} \psi$. Then, $\forall \eta \in \operatorname{supp} \psi$ we obtain

$$(1.2) \qquad |\eta - \xi_0| \leqslant c\sqrt{1 + |\xi_0|} \qquad \text{hence} \qquad |\eta| \leqslant |\xi_0| + c\sqrt{1+|\xi_0|}.$$

Accordingly, $\psi(\cdot) \in C_0(\boldsymbol{R}^n)$ (continuous functions with compact support). It is immediate that, for the *Friedrichs operator* $\psi(D) = \mathcal{F}^{-1}\mathcal{M}_\psi \mathcal{F}$ (where $\mathcal{F}$, $\mathcal{F}^{-1}$ are the *direct* and *inverse Fourier transform* while $\mathcal{M}_\psi$ is the *multiplication operator* by ψ), the estimates:

$$(1.3) \qquad \|\psi(D)\, u\|_{H^s} \leqslant \sup_{\boldsymbol{R}^n} |\psi(\xi)|\, \|u\|_{H^s} \qquad \forall u \in H^s, \quad \forall s \in \boldsymbol{R}$$

and

$$(1.4) \qquad \|\psi(D)\, u\|_{B^{1,s}} \leqslant \sup_{\boldsymbol{R}^n} |\psi(\xi)|\, \|u\|_{B^{1,s}} \qquad \forall u \in B^{1,s}, \quad \forall s \in \boldsymbol{R}$$

hold true.

We shall prove

Theorem 1. *Let $\zeta \in \operatorname{supp} \psi$. Then the following inequalities are valid:*

$$(1.5) \qquad \|\psi(D)\, u\|_{H^s} \leqslant C(1+|\zeta|)^{\frac{1}{2}} \|\psi(D)\, u\|_{H^{s-\frac{1}{2}}} \qquad \forall u \in H^s, \quad \forall s \in \boldsymbol{R}$$

$$(1.6) \qquad \|\psi(D)\, u\|_{B^{1,s}} \leqslant C_1(1+|\zeta|) \|\psi(D)\, u\|_{B^{1,s-1}}, \qquad \forall u \in B^{1,s}, \quad \forall s \in \boldsymbol{R}.$$

In fact, the Lemma in [3] implies that $\forall \eta, \zeta \in \operatorname{supp} \psi$, the estimate

$$(1.7) \qquad 1 + |\eta| \leqslant C(1+|\zeta|)$$

holds true.

Next, for $u \in H^s(\boldsymbol{R}^n)$ we have

$$\|\psi(D)\, u\|_{H^s} = \Big[\int_{\boldsymbol{R}^n} (1+|\eta|^2)^s |\psi(\eta)|^2 |\hat{u}(\eta)|^2 d\eta\Big]^{\frac{1}{2}}$$

$$= \Big[\int_{\operatorname{supp}\psi} |\psi(\eta)|^2 |\hat{u}(\eta)|^2 (1+|\eta|^2)^{\frac{1}{2}} (1+|\eta|^2)^{-\frac{1}{2}+s} d\eta\Big]^{\frac{1}{2}}$$

$$\leqslant \Big[\int_{\operatorname{supp}\psi} |\psi(\eta)|^2 |\hat{u}(\eta)|^2 (1+|\eta|)(1+|\eta|^2)^{s-\frac{1}{2}} d\eta\Big]^{\frac{1}{2}}$$

$$\leqslant C_1\sqrt{1+|\zeta|}\Big[\int_{\boldsymbol{R}^n} |\psi(\eta)|^2 |\hat{u}(\eta)|^2 (1+|\eta|^2)^{s-\frac{1}{2}} d\eta\Big]^{\frac{1}{2}}$$

(we used here inequalities: $\sqrt{1+|\eta|^2} \leqslant 1+|\eta|$ and $1+|\eta| \leqslant C(1+|\zeta|)$ from (1.7)). This is in fact (1.5) and for $s=0$ we obtain the Theorem in [3].

Next, consider (1.6). We have, $\forall u \in B^{1,s}(\boldsymbol{R}^n)$

$$\|\psi(D)\, u\|_{B^{1,s}} = \int_{\boldsymbol{R}^n} (1+|\xi|^2)^{\frac{s}{2}}\, |\psi(\xi)|\, |\widehat{u}(\xi)|\, \mathrm{d}\xi$$

$$= \int_{\operatorname{supp}\psi} (1+|\eta|^2)^{\frac{s}{2}-\frac{1}{2}}\, |\psi(\eta)|\, (1+|\eta|^2)^{\frac{1}{2}}\, |\widehat{u}(\eta)|\, \mathrm{d}\eta$$

$$\leqslant \int_{\operatorname{supp}\psi} (1+|\eta|^2)^{\frac{s-1}{2}}\, |\psi(\eta)|\, (1+|\eta|)\, |\widehat{u}(\eta)|\, \mathrm{d}\eta$$

$$\leqslant C(1+|\zeta|) \int_{\boldsymbol{R}^n} (1+|\eta|^2)^{\frac{s-1}{2}}\, |(\psi(D)\, u)^\wedge(\eta)|\, \mathrm{d}\eta = C(1+|\zeta|)\, \|\psi(D)\, u\|_{B^{1,s-1}}$$

which is (1.6).

2 - Commutator inequality in $B^{1,s}$-space

In this section we present a $B^{1,s}$-version of a result in [2], concerning commutators of some pseudo-differential operators, which is in H^s-space. We refer to [5], p. 34-37 (for the special case $r=0$). Thus, let us consider measurable functions $g(x, \xi)$, $\boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$, such that

$$g(x, \xi) = (2\pi)^{-\frac{n}{2}} \int_{\boldsymbol{R}^n} e^{i\langle x, \lambda\rangle}\, \gamma(\lambda, \xi)\, \mathrm{d}\lambda \qquad \forall x \in \boldsymbol{R}^n,\ \xi \in \boldsymbol{R}^n \tag{2.1}$$

where $\langle x, \lambda\rangle = x_1\lambda_1 + \dots x_n\lambda_n$, the function $\gamma(\lambda, \xi)$ is measurable, $\boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{C}$, as well as the partial function $\lambda \mapsto \gamma(\lambda, \xi)$ $\forall \xi \in \boldsymbol{R}^n$, and the estimate

$$|\gamma(\lambda, \xi)| \leqslant k(\lambda) \qquad \forall \xi \in \boldsymbol{R}^n,\ \ \forall \lambda \in \boldsymbol{R}^n \tag{2.2}$$

is verified, where

$$(1+|\lambda|)^{|s|}\, k(\lambda) \in L^1(\boldsymbol{R}^n) \tag{2.3}$$

for some real number s.

Define the *pseudo-differential operator* $\mathcal{G}(x, D)$ on $B^{1,s}$ by

$$\mathcal{G}(x, D)\, u = \mathcal{F}^{-1}\Big[(2\pi)^{-\frac{n}{2}} \int_{\boldsymbol{R}^n} \gamma(\xi-\eta, \eta)\, \widehat{u}(\eta)\, \mathrm{d}\eta\Big] \qquad \forall u \in B^{1,s}. \tag{2.4}$$

As seen in [5], p. 37, this is a linear continuous operator, $B^{1,s} \to B^{1,s}$, where

$$\|\mathcal{G}(x, D)\, u\|_{B^{1,s}} \leq (2\pi)^{-\frac{n}{2}} \Big(\int_{R^n} (1 + |\lambda|)^{|s|} k(\lambda)\, d\lambda\Big) \|u\|_{B^{1,s}} \qquad \forall u \in B^{1,s}.$$

Next, consider a bounded measurable function $\psi(\xi)$, $\boldsymbol{R}^n \to \boldsymbol{C}$, such that

$$|\psi(\xi) - \psi(\eta)| \leq C|\xi - \eta|\, |\eta|^{-\frac{1}{2}} \qquad \text{for } |\xi - \eta| \leq \frac{1}{2}|\eta|$$

holds true, as well as a continuous function $\zeta(\xi)$, $\boldsymbol{R}^n \to [0, 1]$, where $\zeta(\xi) = 0$ for $|\xi| < \dfrac{1}{2}$ while $\zeta(\xi) = 1$ for $|\xi| \geq 1$.

The associated operators $\psi(D)$, $\zeta(D)$ belong to $\mathcal{L}(B^{1,s})$ (see (1.4)).

Consider also the *commutator operator*

$$L = [\psi(D), \mathcal{G}(x, D)] = \psi(D)\, \mathcal{G}(x, D) - \mathcal{G}(x, D)\, \psi(D). \tag{2.5}$$

We prove the following

Theorem 2. *Let us assume the supplementary assumption (replacing* (2.3))

$$(1 + |\lambda|)^{|s|+1} k(\lambda) \in L^1(\boldsymbol{R}^n). \tag{2.6}$$

Then the following estimate holds true

$$\|\zeta(D)\, Lu\|_{B^{1,s}} \leq C\|(1 + |\lambda|)^{|s|+1} k(\lambda)\|_{L^1} \|u\|_{B^{1,s-\frac{1}{2}}} \qquad \forall u \in B^{1,s}. \tag{2.7}$$

Proof. The *Fourier transform* of $\zeta(D)\, Lu$ is easily seen to be given by the formula

$$(\zeta(D)\, Lu)^\wedge(\xi) = \zeta(\xi)(2\pi)^{-\frac{n}{2}} \int_{R^n} \gamma(\xi - \eta, \eta)[\psi(\xi) - \psi(\eta)]\, \hat{u}(\eta)\, d\eta\,. \tag{2.8}$$

Thus, we have to estimate the $L^1(\boldsymbol{R}^n)$-norm of the expression

$$W_s(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} (\zeta(D)\, Lu)^\wedge(\xi) \tag{2.9}$$

$$= (2\pi)^{-\frac{n}{2}} \int_{R^n} (1 + |\xi|^2)^{\frac{s}{2}} (1 + |\eta|^2)^{-\frac{s}{2}} \zeta(\xi)\, \gamma(\xi - \eta, \eta)[\psi(\xi) - \psi(\eta)](1 + |\eta|^2)^{\frac{s}{2}}\, \hat{u}(\eta)\, d\eta\,.$$

We shall make use of the inequality (2.7) at p. 34 in [5] and obtain accordingly the inequality

$$|W_s(\xi)| \leqslant c \int_{R^n} (1+|\xi-\eta|)^{|s|} \zeta(\xi)\, k(\xi-\eta)\, |\psi(\xi)-\psi(\eta)|\, (1+|\eta|^2)^{\frac{s}{2}}\, |\widehat{u}(\eta)|\, d\eta$$

$$\leqslant W_{s,1}(\xi) + W_{s,2}(\xi)$$

where

$$W_{s,1}(\xi) = C_1 \int_{\eta;\ |\xi-\eta| \leqslant \frac{1}{2}|\eta|} (1+|\xi-\eta|)^{|s|} \zeta(\xi)\, k(\xi-\eta) \frac{|\xi-\eta|}{\sqrt{|\eta|}} (1+|\eta|^2)^{\frac{s}{2}}\, |\widehat{u}(\eta)|\, d\eta$$

and

$$W_{s,2}(\xi) = C_2 \int_{\eta;\ |\xi-\eta| \geqslant \frac{1}{2}|\eta|} (1+|\xi-\eta|)^{|s|} k(\xi-\eta)(1+|\eta|^2)^{\frac{s}{2}}\, |\widehat{u}(\eta)|\, d\eta\,.$$

We see that $W_{s,1}(\xi) = 0$ for $|\xi| \leqslant \frac{1}{2}$. For $|\xi| \geqslant \frac{1}{2}$ and $\frac{|\eta|}{2} > |\xi-\eta|$ we get $|\eta| \geqslant \frac{1}{3}$ and for $|\eta| \geqslant \frac{1}{3}$ we get $|\xi-\eta|\,|\eta|^{-\frac{1}{2}} \leqslant C(1+|\xi-\eta|)(1+|\eta|^2)^{-\frac{1}{4}}$. This entails estimate for $W_{s,1}(\xi)$:

$$W_{s,1}(\xi) \leqslant C \int_{R^n} (1+|\xi-\eta|)^{|s|} k(\xi-\eta)(1+|\xi-\eta|)(1+|\eta|^2)^{-\frac{1}{4}} (1+|\eta|^2)^{\frac{s}{2}}\, |\widehat{u}(\eta)|\, d\eta$$

$$= C \int_{R^n} (1+|\xi-\eta|)^{|s|+1} k(\xi-\eta)(1+|\eta|^2)^{\frac{s}{2}-\frac{1}{4}}\, |\widehat{u}(\eta)|\, d\eta$$

and accordingly

(2.10) $$\|W_{s,1}(\cdot)\|_{L^1(R^n)} \leqslant C\|(1+|\lambda|)^{|s|+1} k(\lambda)\|_{L^1} \|u\|_{B^{1,s-\frac{1}{2}}}\,.$$

As for $W_{s,2}(\xi)$, from $|\xi-\eta| \geqslant \frac{1}{2}|\eta|$ we derive:

$$1 \leqslant 2(1+|\xi-\eta|)\, c(1+|\eta|^2)^{-\frac{1}{2}}$$

hence $$W_{s,2}(\xi) \leqslant C \int_{R^n} (1+|\xi-\eta|)^{|s|+1} k(\xi-\eta)(1+|\eta|^2)^{\frac{s-1}{2}}\, |\widehat{u}(\eta)|\, d\eta$$

and accordingly

(2.11) $$\|W_{s,2}(\cdot)\|_{L^1(R^n)} \leqslant C\|(1+|\lambda|)^{|s|+1} k(\lambda)\|_{L^1} \|u\|_{B^{1,s-1}}.$$

We are able now to conclude with (2.7).

3 – Order and true order in the scale $B^{1,s}$

We first refer to [5], p. 4. Similar facts are true for the scale of Banach spaces $\{B^{1,s}(\boldsymbol{R}^n)\}$. For instance, if $u \in B^{1,\infty}(\boldsymbol{R}^n) = \cap B^{1,s}(\boldsymbol{R}^n)$ and if $\varphi(\xi)$, $\boldsymbol{R}^n \to \boldsymbol{C}$ is a measurable function such that

$$|\varphi(\xi)| \leq C(1+|\xi|^2)^\sigma \quad \text{a.e in } \boldsymbol{R}^n, \quad \text{for some real number } \sigma \tag{3.1}$$

then one obtains

$$|(1+|\xi|^2)^{\frac{s}{2}}\varphi(\xi)\,\widehat{u}(\xi)| \leq C(1+|\xi|^2)^{\frac{s}{2}+\sigma}|\widehat{u}(\xi)| \tag{3.2}$$

almost everywhere, so that

$$\int_{\boldsymbol{R}^n}(1+|\xi|^2)^{\frac{s}{2}}|\varphi(\xi)\,\widehat{u}(\xi)|\,\mathrm{d}\xi = \|\varphi(D)\,u\|_{B^{1,s}} \leq C\|u\|_{B^{1,s+2\sigma}} \qquad \forall s \in \boldsymbol{R} \tag{3.3}$$

which implies that $[2\sigma, +\infty) \subset \mathcal{O}(\varphi(D))$.

In particular, if $\varphi(\cdot) \in C_0(\boldsymbol{R}^n)$ then (3.1) holds $\forall \sigma \in \boldsymbol{R}$ and accordingly $\mathcal{O}(\varphi(D))$, evaluated in $B^{1,\infty}(\boldsymbol{R}^n)$, is $\boldsymbol{R}$, as it is in the scale $\{H^s(\boldsymbol{R}^n)\}$.

Next we refer to the class of $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$-symbols denoted in [4] p. 94 with $\tilde{S}^r$; if $p(x,\xi)$ is such a symbol and $\mathcal{P}(x,D)$ its associated operator (see (10.3) in [4], p. 95) we have

Theorem 3. *The following estimate holds true*

$$\|\mathcal{P}(x,D)\,u\|_{B^{1,s}} \leq C_s\|u\|_{B^{1,s+r}} \qquad \forall s \in \boldsymbol{R}, \quad \forall u \in \mathcal{S}(\boldsymbol{R}^n). \tag{3.4}$$

Proof. As seen in [4], if $\widehat{p}(\lambda,\xi) = (2\pi)^{-\frac{n}{2}}\int_{\boldsymbol{R}^n} e^{-i\langle x,\lambda\rangle}p(x,\xi)\,\mathrm{d}x$, then the *Fourier transform* of $\mathcal{P}(x,D)\,u$ is given by relation

$$(\mathcal{P}(x,D)\,u)^\wedge(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\boldsymbol{R}^n}\widehat{p}(\xi-\eta,\eta)\,\widehat{u}(\eta)\,\mathrm{d}\eta \qquad \forall u \in \mathcal{S}(\boldsymbol{R}^n).$$

Therefore $(\mathcal{P}(x,D)\,u)^\wedge \in L^1$ and $\mathcal{P}(x,D)\,u \in B^{1,0}(\boldsymbol{R}^n)$.

Take now any $s \in \boldsymbol{R}$ and establish that the function $U_s(\xi)$ given by

$$U_s(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\boldsymbol{R}^n}(1+|\xi|^2)^{\frac{s}{2}}\,\widehat{p}(\xi-\eta,\eta)\,\widehat{u}(\eta)\,\mathrm{d}\eta \tag{3.5}$$

belongs to $L^1(\boldsymbol{R}^n)$.

We write the equality

$$U_s(\xi) = (2\pi)^{-\frac{n}{2}} \int_{R^n} (1+|\xi|^2)^{\frac{s}{2}} (1+|\eta|^2)^{-\frac{s}{2}} \hat{p}(\xi-\eta, \eta)(1+|\eta|^2)^{\frac{s}{2}} \hat{u}(\eta)\, d\eta$$

whence the estimate

$$|U_s(\xi)| \leq (2\pi)^{-\frac{n}{2}} 2^{\frac{|s|}{2}} \int_{R^n} (1+|\xi-\eta|^2)^{\frac{|s|}{2}} |\hat{p}(\xi-\eta, \eta)| (1+|\eta|^2)^{\frac{s}{2}} |\hat{u}(\eta)|\, d\eta .$$

Furthermore one has: $|\hat{p}(\xi-\eta, \eta)| \leq C_l (1+|\eta|^2)^{\frac{r}{2}} (1+|\xi-\eta|^2)^{-l}$ for any $l = 1, 2, \ldots$ so that

$$(3.6) \qquad |U_s(\xi)| \leq C_s \int_{R^n} (1+|\xi-\eta|^2)^{\frac{|s|}{2}-l} (1+|\eta|^2)^{\frac{s+r}{2}} |\hat{u}(\eta)|\, d\eta .$$

This obviously entails (3.4) (when l is sufficiently large).

We next complete reasonings in [5], p. 10, taking the scale $\{B^{1,s}(\boldsymbol{R})\}$ instead of $\{H^s(\boldsymbol{R})\}$, and $V = B^{1,\infty}(\boldsymbol{R})$.

Let $\psi(\xi) = 1$ for $\xi \geq 0$, $\psi(\xi) = 0$ for $\xi < 0$ and $\psi(D) = \mathcal{F}^{-1} \mathcal{M}_{\psi(\cdot)} \mathcal{F}$. Then obviously

$$(3.7) \qquad \|\psi(D)\, u\|_{B^{1,s}} \leq \|u\|_{B^{1,s}} \qquad \forall s \in \boldsymbol{R}, \quad \forall u \in V \text{ and } \quad [0, \infty) \subset \mathcal{O}(\psi(D))$$

(the order is computed with respect to the scale $\{B^{1,s}(\boldsymbol{R})\}$).

Actually, the order of $\psi(D)$ equals $[0, \infty)$. This is a consequence of

Proposition 1. *There exists no positive number ε_0, such that $-\varepsilon_0 \in \mathcal{O}(\psi(D))$.*

Proof. Let us assume that for some $\varepsilon_0 > 0$, $-\varepsilon_0$ belongs to $\mathcal{O}(\psi(D))$. Then the estimates

$$\|\psi(D)\, u\|_{B^{1,s}} \leq C_s \|u\|_{B^{1,s-\varepsilon_0}} \qquad \forall u \in \mathcal{S}(\boldsymbol{R}), \quad \forall s \in \boldsymbol{R}$$

hold true. In particular, if $s = 0$, we obtain that

$$(3.8) \qquad \int_{\boldsymbol{R}} |\psi(\xi)\, \hat{u}(\xi)|\, d\xi \leq C \int_{\boldsymbol{R}} (1+|\xi|^2)^{-\frac{\varepsilon_0}{2}} |\hat{u}(\xi)|\, d\xi \qquad \forall u \in \mathcal{S}(\boldsymbol{R}).$$

Take now (as in [5], p. 11), a sequence $(g_p(\xi))$ where $g_p(\xi) \in C_0^\infty(\boldsymbol{R})$, $0 \leq g_p(\cdot) \leq 1$, $g_p(\xi) = 0$ for $\xi \leq p-1$ and $\xi \geq 2p+1$, $g_p(\xi) = 1$ for $p \leq \xi \leq 2p$

and then $u_p(x) = \mathcal{F}^{-1}(g_p(\cdot))$. Introducing in (3.8) we get

$$\int_R |\psi(\xi)\, g_p(\xi)|\,d\xi \leqslant C \int_R (1+|\xi|^2)^{-\frac{\varepsilon_0}{2}} |g_p(\xi)|\,d\xi \qquad \forall p = 1, 2, \ldots$$

and consequently

$$\int_0^\infty |g_p(\xi)|\,d\xi \leqslant C \int_{p-1}^{2p+1} (1+|\xi|^2)^{-\frac{\varepsilon_0}{2}}\,d\xi \leqslant C(p+2)(1+(p-1)^2)^{-\frac{\varepsilon_0}{2}}$$

for any $p \in N$.

Also, $\int_0^\infty |g_p(\xi)|\,d\xi \geqslant p$ and one gets

$$p \leqslant C(p+2)(1+(p-1)^2)^{-\frac{\varepsilon_0}{2}} \qquad \forall p \in N$$

which of course is impossible.

A similar result (cf. Prop. 5.1, p. 14 in [5]) is given below as

Theorem 4. *Let* $\psi(\cdot)$ *be a continuous, real-valued function on* $\boldsymbol{R}^n$, $0 \leqslant \psi(\cdot) \leqslant 1$, $\psi(\xi) = 0$ *for* $|\xi| \leqslant \frac{1}{2}$, $\psi(\xi) = 1$ *for* $|\xi| \geqslant 1$. *The operator* $\psi(D)$ *is continuous,* $B^{1,\infty} \to B^{1,\infty}$ *and its true order equals zero.*

Proof. The only non-trivial thing to demonstrate: there is no negative number r belonging to $\mathcal{O}(\psi(D))$, in $\{B^{1,s}\}_{s\in \boldsymbol{R}}$.

If such $r < 0$ would exist, then, in particular, it would follow that

$$\|\psi(D)u\|_{B^{1,0}} \leqslant C\|u\|_{B^{1,r}} \qquad \forall u \in \mathcal{S}(\boldsymbol{R}^n)$$

that is $\displaystyle\int_{\boldsymbol{R}^n} |\psi(\xi)\,\hat{u}(\xi)|\,d\xi \leqslant C \int_{\boldsymbol{R}^n} (1+|\xi|^2)^{\frac{r}{2}} |\hat{u}(\xi)|\,d\xi, \quad \forall u \in \mathcal{S}(\boldsymbol{R}^n)$ hence

$$\int_{|\xi|\geqslant 1} |\hat{u}(\xi)|\,d\xi \leqslant c \int_{\boldsymbol{R}^n} (1+|\xi|^2)^{\frac{r}{2}} |\hat{u}(\xi)|\,d\xi \qquad \forall u \in \mathcal{S}(\boldsymbol{R}^n). \tag{3.9}$$

Take now the sequence $(u_p(\cdot))$ in $\mathcal{S}(\boldsymbol{R}^n)$ where $\hat{u}_p(\xi) = (1+|\xi|^2)^{-\frac{n}{2}}$, $|\xi| \leqslant p$ and $\hat{u}_p(\xi) = 0$ for $|\xi| \geqslant 2p$, $\hat{u}_p(\xi) \in C^\infty(\boldsymbol{R}^n)$, $0 \leqslant |\hat{u}_p(\xi)| \leqslant (1+|\xi|^2)^{-\frac{n}{2}}$, $\forall \xi \in \boldsymbol{R}^n$ (see [5], p. 15).

Thus, (3.9) entails

$$\int_{1 \leq |\xi| \leq p} (1+|\xi|^2)^{-\frac{n}{2}} d\xi \leq C \int_{|\xi| \leq 2p} (1+|\xi|^2)^{\frac{r}{2}} (1+|\xi|^2)^{-\frac{n}{2}} d\xi \tag{3.10}$$

for any $p = 1, 2, \ldots$. In spherical coordinates, the inequality (3.10) becomes

$$\int_1^p (1+\varrho^2)^{-\frac{n}{2}} \varrho^{n-1} d\varrho \leq C \int_0^{2p} (1+\varrho^2)^{\frac{r-n}{2}} \varrho^{n-1} d\varrho \qquad \forall p = 1, 2, \ldots$$

which is impossible.

As in [5], p. 15, 16, Theorem 4 has the following

Corollary 1. *Let* $\psi(\cdot)$ *as in Theorem* 4 *and, for some* $\sigma \in \boldsymbol{R}$, *define* $\psi_\sigma(\xi) = |\xi|^\sigma \psi(\xi)$ *for* $\xi \neq 0$ *and* $\psi_\sigma(\xi) = 0$ *for* $\xi = 0$. *Then, the true order of the operator* $\psi_\sigma(D)$ *relatively to* $\{B^{1,s}(\boldsymbol{R}^n)\}$ *equals* σ.

Proof. Note the estimate: $|\psi_\sigma(\xi)| \leq C(1+|\xi|^2)^{\frac{\sigma}{2}}$, $\forall \xi \in \boldsymbol{R}^n$. Use of (3.3) shows that $t \cdot o(\psi_\sigma(D)) \leq \sigma$.

If this inequality is strict and $t \cdot o(\psi_\sigma(D)) = \sigma_1 < \sigma$, consider the operators $\psi_\sigma(D)$, $\psi_{-\sigma}(D)$, $B^{1,\infty} \to B^{1,\infty}$, and then apply Proposition 4.2, p. 12 in [5], with $V = B^{1,\infty}$, $L_1 = \psi_\sigma(D)$, $L_2 = \psi_{-\sigma}(D)$. We obtain

$$t \cdot o(\psi_\sigma(D) \cdot \psi_{-\sigma}(D)) \leq \sigma_1 - \sigma < 0\,. \tag{3.11}$$

On the other hand, for any $u \in B^{1,\infty}$ we have

$$\psi_\sigma(D)\, \psi_{-\sigma}(D)\, u = \psi_\sigma(D)(\psi_{-\sigma}(D)\, u) = \mathcal{F}^{-1}(\psi_\sigma(\xi)(\psi_{-\sigma}(D)\, u)^\wedge(\xi))$$

$$= \mathcal{F}^{-1}(\psi_\sigma(\xi)(\psi_{-\sigma}(\xi)\, \widehat{u}(\xi))) = \mathcal{F}^{-1}(\psi^2(\xi)\, \widehat{u}(\xi)) = \psi^2(D)\, u\,.$$

Note that $\psi^2(\xi)$ has same properties as $\psi(\xi)$, so that, from Theorem 4, $t \cdot o \psi^2(D)$ (in $B^{1,\infty}$) equals 0.

From (3.11) we then derive the contradiction $0 < 0$.

4 - Reversor inequality in $B^{1,s}$-space

In this final (and short) section of this paper we present the $B^{1,s}$-version of the inequality (2.19) at p. 63 of [5] concerning the *reversor operator* $A_a(x, D) - \mathfrak{a}_a(x, D)$ corresponding to a C^∞-zero homogeneous symbol $a(x, \xi)$.

We thus refer to [5], Ch. VI for the main definitions of symbols and associated operators and establish

Proposition 2. *If $a(x, \xi)$ is a $(K-N)$ symbol, the difference operator $A_a - \mathcal{A}_a$ is a linear continuous operator, $B^{1,s} \to B^{1,s+1}$, $\forall s \in \boldsymbol{R}$.*

Proof. We shall demonstrate inequality

$$\|(A_a - \mathcal{A}_a)\, u\|_{B^{1,s+1}} \leqslant C_s \|u\|_{B^{1,s}} \qquad \forall u \in B^{1,s}, \quad \forall s \in \boldsymbol{R}\,. \tag{4.1}$$

We have, as in [5] Ch. VI, the expression for the Fourier transform of $(A_a - \mathcal{A}_a)\, u$:

$$((A_a - \mathcal{A}_a)\, u)^{\wedge}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\boldsymbol{R}^n} [\tilde{a}'(\xi - \eta, \xi) - \tilde{a}'(\xi - \eta, \eta)]\, \hat{u}(\eta)\, \mathrm{d}\eta\,.$$

We consider next the expression

$$W_s(\xi) = (2\pi)^{-\frac{n}{2}} (1 + |\xi|^2)^{\frac{s+1}{2}} \int_{\boldsymbol{R}^n} [\tilde{a}'(\xi - \eta, \xi) - \tilde{a}'(\xi - \eta, \eta)]\, \hat{u}(\eta)\, \mathrm{d}\eta$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\boldsymbol{R}^n} (1 + |\xi|^2)^{\frac{s+1}{2}} (1 + |\eta|^2)^{-\frac{s+1}{2}} [\tilde{a}'(\xi - \eta, \xi) - \tilde{a}'(\xi - \eta, \eta)](1 + |\eta|^2)^{\frac{s+1}{2}}\, \hat{u}(\eta)\, \mathrm{d}\eta$$

whence, for any p of $\boldsymbol{N}$, the estimate

$$|W_s(\xi)| \leqslant C_{s,p} \int_{\boldsymbol{R}^n} (1 + |\xi - \eta|^2)^{\frac{|s+1|}{2} - p + \frac{1}{2}} (1 + |\eta|^2)^{\frac{s}{2}}\, |\hat{u}(\eta)|\, \mathrm{d}\eta$$

and then, for sufficiently large p

$$\int_{\boldsymbol{R}^n} |W_s(\xi)|\, \mathrm{d}\xi = \|(A_a - \mathcal{A}_a)\, u\|_{B^{1,s+1}} \leqslant C\Big(\int_{\boldsymbol{R}^n} (1 + |\lambda|^2)^{\frac{|s+1|}{2} + \frac{1}{2} - p}\, \mathrm{d}\lambda\Big) \|u\|_{B^{1,s}}$$

which is (4.1).

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Sommario

In questo lavoro vengono fatti alcuni commenti riguardo a varie proprietà elementari degli operatori pseudo-differenziali, incontrate in precedenti lavori dell'autore. Essenzialmente, si presenta una versione $B^{1,s}(\boldsymbol{R}^n)$ *di risultati precedentemente ottenuti in spazi* $H^s(\boldsymbol{R}^n)$.

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