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Remark on the semiconductor equations (**)

1 - Introduction

We consider the Van Roosbroeck model for the carrier transport in a semiconductor device [19]; such model describes the transport of carriers in absence of ionized impurities by means of the following system of non-linear partial differential equations:

$$\begin{aligned} & p_t - \nabla \cdot (D_1 \nabla p + \mu_1 p \nabla \Phi) = R(p, n) \\ & n_t - \nabla \cdot (D_2 \nabla n - \mu_2 n \nabla \Phi) = R(p, n) \qquad \text{on } Q_T \\ & -\nabla \cdot (\varepsilon \nabla \Phi) = p - n \end{aligned} \tag{1.1}$$

where p and n are the densities of mobile holes and electrons, while Φ is the electric potential. The positive constants D_i, μ_i and ε are respectively the diffusion coefficients, the mobilities of holes and electrons, and the dielectric permittivity. The term $R(p, n)$ represents the law of recombination which, according to the Shockley-Read-Hall model, can be expressed by

$$R(p, n) = \frac{1 - pn}{r_0 + r_1 p + r_2 n}$$

where r_i, $i = 0, 1, 2$, are positive constants. The domain Q_T is the product $\Omega \times (0, T)$, where Ω is an open bounded subset of $\boldsymbol{R}^N$ with regular boundary.

Due to their physical and technological interest (see e.g. [11] and [15] for more detailed information on the physical background), the above equations have recently received a great deal of attention from the mathematical point of view. The

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problems of existence and uniqueness, regularity, and asymptotic behaviour of the solutions have been widely studied by different authors, as M. S. Mock [12], [13], [14], T. I. Seidman [16], [17], H. Beirão da Veiga [2], [3], [4], H. Gajewski [7], [8], and K. Gröger [7], [8], [9], [10].

In this paper we show the existence of an invariant region for the mobile carriers p and n. That immediately yields the existence of a solution global in time to (1.1), as well as an a priori sup-norm bound for p and n. The main aim of this paper is then to notice that such results, which have been already proved in the literature by using essentially a more complicated version of the maximum principle [17], [9], can be obtained in a quite simplified way.

In Section 2 we briefly recall the definition of invariant region, and a basic related criterion, while in Section 3 we state and prove the existence of an invariant region for (1.1).

2 – Invariant regions for parabolic systems

Let us introduce the notion of invariant region and some related results [6]. For a complete survey about invariant regions, we refer to the book by J. Smoller [18].

Let us consider the weakly coupled parabolic system:

$$
(2.1) \quad \begin{aligned}
&(u_k)_t = D\Delta u_k + \sum_{j=1}^{N} a_j(x, t)(u_k)_{x_j} + \psi_k(u_1, \ldots, u_M) && \text{on } Q_T \\
&u_k(x, 0) = (u_k)_0(x) && \text{on } \Omega \\
&u_k(x, t) = (u_k)_b(x, t) && \text{on } \partial\Omega \times (0, T)
\end{aligned}
$$

where $k = 1, \ldots, M$. Here D denotes a positive constant, while the coefficients a_j, the vector field $\boldsymbol{V} = (\psi_1, \ldots, \psi_M)$, as well as the initial and boundary data $(u_k)_0$ and $(u_k)_b$ are assumed to be continuous. In the following $\boldsymbol{u}$ will be a vector of components $(u_1, \ldots, u_M)$.

Definition 1. A subset S of $\boldsymbol{R}^M$ is said to be an *invariant region* for the parabolic system (2.1) when the following condition holds: if the initial and boundary data of (2.1) lie in S, i.e. if

$$\boldsymbol{u}_0(x) \in S \quad \forall x \in \Omega \qquad \text{and} \qquad \boldsymbol{u}_b(x, t) \in S \quad \forall (x, t) \in \partial\Omega \times (0, T)$$

then every solution $\boldsymbol{u}(x, t)$ of (2.1) belonging to $C^2(Q_T) \cap C^0(\overline{Q}_T)$ remains in S, i.e. $\boldsymbol{u}(x, t) \in S$, $\forall (x, t) \in Q_T$.

We recall the following result [20], [1], which is the main tool used in the proof of Theorem 2.

Theorem 1. *Let S be a closed and convex subset of $\boldsymbol{R}^M$ such that the vector field V does not point out of S whenever $\boldsymbol{u}$ is on the boundary of S, that is*

$$V(\boldsymbol{u})\cdot \boldsymbol{\nu} \leqslant 0 \tag{2.2}$$

for every point $\boldsymbol{u} \in \partial S$ and for every $\boldsymbol{\nu}$ outward normal vector at $\boldsymbol{u}$. Then S is an invariant region for the parabolic system (2.1).

Remark 1.

i. Theorem 1 holds even if ∂S contains singular points. In this case we take as outward normal at a singular point $\boldsymbol{u}_s$ any vector $\boldsymbol{\nu}$ of $\boldsymbol{R}^M$ such that S is contained in the halfspace with exterior normal $\boldsymbol{\nu}$; the existence of such a vector $\boldsymbol{\nu}$ is guaranteed by the hypothesis on S of being a closed and convex set [5].

ii. Let α_k and β_k be real numbers, for $k = 1, \ldots, M$, and suppose that the set $S \subset \boldsymbol{R}^M$ is of the type

$$S = \{\boldsymbol{u} \in \boldsymbol{R}^M : \ \alpha_k \leqslant u_k \leqslant \beta_k, \quad k = 1, \ldots, M\}.$$

Then Theorem 1 still holds if we replace the coefficient D in equation $(2.1)_1$ by a positive constant D_k, depending on the index k.

3 - Existence of an invariant region for the semiconductor equations

We want to use Theorem 1 to prove the existence of an invariant region for the mobile carriers in semiconductor equations.

To this aim we observe that the model (1.1) can be reduced to a weakly coupled parabolic system of type (2.1). Indeed, if we substitute equation $(1.1)_3$ into equations $(1.1)_1$ and $(1.1)_2$, we get:

$$p_t = D_1 \Delta p + \mu_1 \nabla p \cdot \nabla \Phi - \frac{\mu_1}{\varepsilon} p(p-n) + R(p, n)$$

$$n_t = D_2 \Delta n - \mu_2 \nabla n \cdot \nabla \Phi + \frac{\mu_2}{\varepsilon} n(p-n) + R(p, n).$$

We consider the above equations coupled with initial data

$$p(x, 0) = p_0(x) \qquad n(x, 0) = n_0(x) \qquad \text{on } \Omega$$

and with Dirichlet boundary data

$$p = p_b(x, t) \qquad n = n_b(x, t) \qquad \text{on } \partial\Omega \times [0, T).$$

For simplicity we suppose that p_0, n_0, p_b, n_b are regular functions, satisfying the compatibility condition

$$p_b(x, 0) = p_0(x) \qquad n_b(x, 0) = n_0(x) \qquad \text{on } \Omega .$$

We are now in a position to state our main result.

Theorem 2. *For c sufficiently large, the set*

$$Q(c) = \{(p, n) \in \boldsymbol{R}^2\colon\ 0 \leqslant p \leqslant 1 + c,\ \ 0 \leqslant n \leqslant 1 + c\}$$

is an invariant region for the mobile carriers in the semiconductor equations.

Proof. It is sufficient to prove that, for c large enough, the vector field $\boldsymbol{V} = (f, g)$ of components

$$f = -\frac{\mu_1}{\varepsilon} p(p - n) + R(p, n) \qquad g = \frac{\mu_2}{\varepsilon} n(p - n) + R(p, n)$$

satisfies hypothesis (2.2) on the boundary of $Q(c)$. We set

$$\partial\big(Q(c)\big) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 ,$$

where:

$$\Gamma_1 = \{p = 0,\ 0 \leqslant n \leqslant 1 + c\} \qquad \Gamma_2 = \{n = 0,\ 0 \leqslant p \leqslant 1 + c\}$$

$$\Gamma_3 = \{p = 1 + c,\ 0 \leqslant n \leqslant 1 + c\} \quad \Gamma_4 = \{n = 1 + c,\ 0 \leqslant p \leqslant 1 + c\}.$$

On Γ_1 we have

$$(\boldsymbol{V} \cdot \boldsymbol{\nu})_{|\Gamma_1} = -f(0, n) = -\frac{1}{r_0 + r_2 n} < 0 \qquad \forall n \in [0, 1 + c].$$

Similarly we obtain $(V\cdot \nu)_{|\Gamma_2} < 0$. Let us consider Γ_3.

$$(V\cdot \nu)_{|\Gamma_3} = f(1+c, n) = -\frac{\mu_1}{\varepsilon}(1+c)[(1+c)-n] + \frac{1-(1+c)\,n}{r_0 + r_1(1+c) + r_2 n}$$

for any $n \in [0, 1+c]$.

Since
$$f(1+c, 0) = -\frac{\mu_1}{\varepsilon}(1+c)^2 + \frac{1}{r_0 + r_1(1+c)}$$

and
$$f(1+c, 1+c) = \frac{1-(1+c)^2}{r_0 + (r_1+r_2)(1+c)}$$

are both negative for c large enough, we have only to show that there exists $\bar{c} > 0$ such that the derivative of $f(1+c, n)$ with respect to n is non-negative when c is larger than $\bar{c}$. Actually, for $n \in [0, 1+c]$, we have the inequality

$$\frac{\partial f}{\partial n}(1+c, n) \geqslant \frac{\mu_1}{\varepsilon}(1+c) - \frac{(1+c)}{r_0 + r_1(1+c)} - \frac{r_2}{[r_0 + r_1(1+c)]^2}$$

so that
$$\lim_{c \to +\infty} \frac{\partial f}{\partial n}(1+c, n) = +\infty\,,$$

which gives the required condition on Γ_3. In an analogous way one can show that, for c large enough, $(V\cdot \nu)_{|\Gamma_4} < 0$, and the proof is achieved.

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Sommario

Si prova l'esistenza di una regione invariante *per il sistema parabolico che descrive il trasporto di lacune ed elettroni nei semiconduttori. Conseguenza immediata di questo risultato è che l'esistenza di una soluzione globale per le equazioni dei semiconduttori e la sua limitatezza nella norma* L^∞ *possono essere dimostrate in modo molto semplificato.*

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