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Six-dimensional almost-Kähler manifolds with pointwise constant antiholomorphic sectional curvature (**)

1 - Introduction

Let (M, g, J) be a $2n$-dimensional $(n \geqslant 2)$ *almost Hermitian manifold.* A 2-plane α in the tangent space $T_x M$ at a point x of M is antiholomorphic if it is orthogonal to $J\alpha$.

The manifold (M, g, J) has pointwise constant antiholomorphic sectional curvature (p.c.a.s.c.) ν if, at any point x, the Riemannian sectional curvature $\nu(x) = k_x(\alpha)$ is independent on the choice of the antiholomorphic 2-plane α in $T_x M$. If (g, J) is a Kähler structure, this condition means that (M, g, J) is a complex space-form, i.e. a Kähler manifold of constant holomorphic sectional curvature 4ν. Therefore,the Riemannian curvature tensor R is given by

$$R = \nu(\pi_1 + \pi_2)$$

π_1, π_2 denoting the algebraic curvature tensor fields defined by:

$$\pi_1(X, Y, Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W)$$

$$\pi_2(X, Y, Z, W) = 2g(X, JY)\, g(Z, JW) + g(X, JZ)g(Y, JW) - g(Y, JZ)g(X, JW). \tag{1.1}$$

More generally, in [7] G. Ganchev proves that an almost Hermitian manifold has p.c.a.s.c. ν iff R is given by

$$R = \frac{1}{2(n+1)}\psi(\varrho^*(R)) + \nu\pi_1 - \frac{2(n+1)\,\nu + \tau^*(R)}{2(n+1)(2n+1)}\pi_2 \tag{1.2}$$

$\varrho^*(R)$ denoting the $*$*-Ricci tensor*, $\tau^*(R)$ the $*$*-scalar curvature.*

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As in [**19**], ψ stands for the operator acting on any (0,2)-tensor field S as follows

$$
\begin{aligned}
\psi(S)(X, Y, Z, W) &= 2g(X, JY)\, S(Z, JW) + 2g(Z, JW)\, S(X, JY) \\
&\quad + g(X, JZ)\, S(Y, JW) + g(Y, JW)\, S(X, JZ) \\
&\quad - g(X, JW)\, S(Y, JZ) - g(Y, JZ)\, S(X, JW)\,.
\end{aligned} \tag{1.3}
$$

The formula (1.2) implies the J-invariance of the Ricci tensor $\varrho(R)$, i.e. $\varrho(R)(JX, JY) = \varrho(R)(X, Y)$.

Following [**19**], we denote by $\frac{1}{2}\varrho^*(R + L_3 R)$, $\frac{1}{2}\varrho^*(R - L_3 R)$ the symmetric and skew-symmetric components of $\varrho^*(R)$. Then, one has:

$$
\begin{aligned}
\varrho^*(R + L_3 R) &= \frac{2}{3}(n+1)\,\varrho(R) - \frac{(n+1)\,\tau(R) - 3\tau^*(R)}{3n}\, g \\
8n(n^2 - 1)\,\nu &= (2n+1)\,\tau(R) - 3\tau^*(R)
\end{aligned} \tag{1.4}
$$

$\tau(R)$ being the scalar curvature.

The classification of the almost Hermitian manifolds with p.c.a.s.c. is still an open problem, even if several results in this direction are well-known.

We recall the theorem due to O. T. Kassabov ([**13**]), concerning $\mathcal{R}_3$-manifolds, i.e. almost Hermitian manifolds whose curvature tensor R satisfies the condition:

$$
\mathcal{R}_3 \qquad R(JX, JY, JZ, JW) = R(X, Y, Z, W)\,.
$$

Theorem A. *Any $\mathcal{R}_3$-manifold (M, g, J) with p.c.a.s.c. is a real or a complex space-form, provided that* $\dim M \geq 6$.

Moreover, the theory developed in [**19**] allows to prove that, if R is given by (1.2), then the tensor field $\varrho^*(R - L_3 R)$ represents the obstruction to the $\mathcal{R}_3$-condition. Since $\varrho^*(R - L_3 R)$, a priori, does not vanish in the almost-Kähler case, the classification of the almost-Kähler manifolds with p.c.a.s.c. is meaningful. Such problem is partially solved in [**5**], where the following result is achieved.

Theorem B. *Any connected, almost-Kähler manifold (M, g, J) with p.c.a.s.c. is a complex space-form, provided that* $\dim M \geq 8$.

The technique employed in [**5**], which combines the second Bianchi identity with the theory developed in [**10**], is not fruitful in obtaining a meaningful result in the 6-dimensional case.

On the other hand, the investigation of the 6-dimensional almost-Kähler manifolds satisfying prescribed curvature conditions is still a topical subject. For the sake of brevity, we mention only the most recent result in this direction, due to J. Armstrong ([1]).

Theorem C. *Any 6-dimensional almost-Kähler manifold with constant sectional curvature is a (flat) Kähler manifold.*

This theorem helps in proving the analogous of the theorem B in the 6-dimensional case. In fact, in this paper we state the following results, whose proof is divided in several steps.

Theorem 1. (Schur's lemma of antiholomorphic type). *Let (M, g, J) be a connected 6-dimensional almost-Kähler manifold with p.c.a.s.c. ν. Then the function ν is constant.*

Theorem 2. *Any connected, 6-dimensional almost-Kähler manifold with p.c.a.s.c. is a $\mathcal{R}_3$-manifold.*

Theorem 3. *Any connected 6-dimensional almost-Kähler manifold with p.c.a.s.c. is a complex space-form.*

2 - Some preliminary lemmas

The aim of this section is to obtain all the basic formulas useful for the proof of the theorems 1, 2, 3.

Let (M, g, J) be a 6-dimensional *almost-Kähler manifold* with p.c.a.s.c. ν, Levi-Civita connection ∇ and 2-fundamental form ω, $\omega(X, Y) = g(JX, Y)$.

The almost-Kähler condition, i.e.

$$\underset{(V,X,Y)}{\sigma}(\nabla_V \omega)(X, Y) = 0$$

σ denoting the cyclic sum, implies

$$(\nabla_X J)\, Y + (\nabla_{JX} J)\, JY = 0\,. \tag{2.1}$$

In this case, the formulas (1.4) reduce to:

$$\begin{aligned} \varrho^*(R + L_3 R) &= \frac{8}{3}\varrho(R) - \frac{4\tau(R) - 3\tau^*(R)}{9} g \\ 64\nu &= \frac{7}{3}\tau(R) - \tau^*(R)\,. \end{aligned} \tag{2.2}$$

Therefore, the condition (1.2) is equivalent to

$$R = \psi(Q) + \nu\pi_1 - \frac{5}{3}\nu\pi_2 \tag{2.3}$$

where $\quad Q = \frac{1}{6}\varrho(R) + \frac{1}{16}\varrho^*(\tilde{R}) \quad$ and $\quad \tilde{R} = R - L_3 R$.

We remark that $\varrho(R)$, $\varrho^*(\tilde{R})$ respectively determine the symmetric and J-invariant, skew-symmetric and J-anti-invariant components of Q, i.e.

$$
\begin{aligned}
&Q(X, Y) - Q(Y, X) = Q(X, Y) - Q(JX, JY) = \frac{1}{8}\varrho^*(\tilde{R})(X, Y) \\
&Q(X, Y) + Q(Y, X) = Q(X, Y) + Q(JX, JY) = \frac{1}{3}\varrho(R)(X, Y).
\end{aligned} \tag{2.4}
$$

By means of (2.3), one computes the covariant derivative ∇R, which is given by

$$
\begin{aligned}
(\nabla_V R)&(X, Y, Z, W) \\
&= \psi(\nabla_V Q)(X, Y, Z, W) + V(\nu)(\pi_1 - \frac{5}{3}\pi_2)(X, Y, Z, W) \\
&-2\{\omega(X, Y)\, Q(Z, (\nabla_V J)\, W) + (\nabla_V \omega)(X, Y)\, Q(Z, JW)\} \\
&-2\{\omega(Z, W)\, Q(X, (\nabla_V J)\, Y) + (\nabla_V \omega)(Z, W)\, Q(X, JY)\} \\
&-\omega(X, Z)\, Q(Y, (\nabla_V J)\, W) - (\nabla_V \omega)(X, Z)\, Q(Y, JW) \\
&+\omega(X, W)\, Q(Y, (\nabla_V J)\, Z) + (\nabla_V \omega)(X, W)\, Q(Y, JZ) \\
&-\omega(Y, Z)\, Q(X, (\nabla_V J)\, W) + (\nabla_V \omega)(Y, Z)\, Q(X, JW) \\
&-\omega(Y, W)\, Q(X, (\nabla_V J)\, Z) - (\nabla_V \omega)(Y, W)\, Q(X, JZ) \\
&-\frac{10}{3}\nu\{\omega(X, Y)(\nabla_V \omega)(Z, W) + \omega(Z, W)(\nabla_V \omega)(X, Y)\} \\
&-\frac{5}{3}\nu\{\omega(X, Z)(\nabla_V \omega)(Y, W) - \omega(X, W)(\nabla_V \omega)(Y, Z)\} \\
&+\frac{5}{3}\nu\{\omega(Y, Z)(\nabla_V \omega)(X, W) - \omega(Y, W)(\nabla_V \omega)(X, Z)\}.
\end{aligned} \tag{2.5}
$$

Via (2.5) and the second Bianchi identity, we obtain the relation (see proposition 2.1 in [5])

$$
\begin{aligned}
&Q(X, (\nabla_Y J)\, V + (\nabla_V J)\, Y) - Q(Y, (\nabla_X J)\, V + (\nabla_V J)\, X) \\
&= -\frac{1}{12}\tau(R)(\nabla_V \omega)(X, Y) + 2\{JX(\nu)\, \omega(JY, V) - JY(\nu)\, \omega(JX, V)\} \\
&\quad +2\{X(\nu)\, \omega(Y, V) - Y(\nu)\, \omega(X, V) - 2V(\nu)\, \omega(X, Y)\}.
\end{aligned} \tag{2.6}
$$

Now, one rewrites (2.6) considering the triplet (X, JY, JV); the resulting relation is added to (2.6); then, via (2.1) and (2.4), we have

$$\begin{aligned} &\varrho^*(R - L_3R)(Y, (\nabla_X J)V + (\nabla_V J)X) \\ &= 32\{V(\nu)\,\omega(X, Y) + X(\nu)\,\omega(V, Y) - JV(\nu)\,\omega(JX, Y) - JX(\nu)\,\omega(JV, Y)\}. \end{aligned} \tag{2.7}$$

Moreover, combining (2.6) with the formulas stated in [5] (see the Lemmas 2.1, 2.2 in [5]), we can express the covariant derivative ∇Q as follows

$$\begin{aligned} (\nabla_V Q)(X, Y) &= \frac{3}{8} Q((\nabla_Y J)X - (\nabla_X J)Y, JV) + \frac{3}{4} Q((\nabla_V J)Y, JX) \\ &+ \frac{1}{4} Q(JY, (\nabla_V J)X) + \frac{5}{3}(\nu - \frac{\tau(R)}{32})(\nabla_V \omega)(X, JY) \\ &- \frac{1}{6}\{5V(\nu) + \frac{1}{8}V(\tau(R))\}\,\omega(JX, Y) + \frac{1}{4}\{Y(\nu)\,\omega(JX, V) - JX(\nu)\omega(Y, V)\} \\ &- \frac{1}{12}[\{5X(\nu) + \frac{1}{4}X(\tau(R))\}\,\omega(JY, V) - \{5JY(\nu) + \frac{1}{4}JY(\tau(R))\}\,\omega(X, V)]. \end{aligned} \tag{2.8}$$

Lemma 2.1. *For any quintuplet (V, X, Y, Z, W) of vector fields on M, one has:*

$$\begin{aligned} &(\nabla_V \omega)(X, Z)\,\varrho^*(\tilde{R})(Y, JW) - (\nabla_V \omega)(X, JZ)\,\varrho^*(\tilde{R})(Y, W) \\ &+ \omega(JY, V)\,\varrho^*(\tilde{R})(W, (\nabla_{JZ} J)X) + \omega(Y, V)\varrho^*(\tilde{R})(W, (\nabla_Z J)X) \\ &- \omega(JW, V)\,\varrho^*(\tilde{R})(Y, (\nabla_{JZ} J)X) - \omega(W, V)\,\varrho^*(\tilde{R})(Y, (\nabla_Z J)X) \\ &= -\frac{32}{3}\{V(\nu)[\pi_1(Z, X, Y, W) - \pi_1(Z, X, JY, JW)] \\ &+ JV(\nu)[\pi_1(Z, X, JY, W) + \pi_1(Z, X, Y, JW)] \\ &+ X(\nu)[\pi_1(Z, V, Y, W) - \pi_1(Z, V, JY, JW)] \\ &+ JX(\nu)[\pi_1(Z, V, JY, W) + \pi_1(Z, V, Y, JW)] \\ &+ 2Z(\nu)[\pi_1(X, V, Y, W) - \pi_1(X, V, JY, JW)] \\ &+ 2JZ(\nu)[\pi_1(X, V, JY, W) + \pi_1(X, V, Y, JW)]\} \end{aligned} \tag{2.9}$$

where $\tilde{R} = R - L_3 R$.

Proof. We apply the second Bianchi identity in the version

$$\begin{aligned}(2.10)\qquad & \underset{(V,X,Y)}{\sigma}\{(\nabla_V R)(X,Y,Z,W)+(\nabla_V R)(X,Y,JZ,JW)\}\\ &+\underset{(V,JX,JY)}{\sigma}\{(\nabla_V R)(JX,JY,Z,W)+(\nabla_V R)(JX,JY,JZ,JW)\}=0\,.\end{aligned}$$

The complete expression of the first member in (2.10), evaluated by means of (2.5), is a tensor field which contains five blocks of terms, respectively depending on $\omega\otimes\nabla Q$, $d\nu\otimes(\pi_1-\frac{5}{3}\pi_2)$, $\omega\otimes Q(\cdot,\nabla J)$, $\nabla\omega\otimes Q$, $\omega\otimes\nabla\omega$.

Since (g, J) is an almost Kähler structure, the whole block in $\omega\otimes\nabla\omega$ vanishes. Moreover, only the J-anti-invariant component of Q, i.e. $\varrho^*(\tilde R)$,is involved in the blocks depending on $\nabla\omega\otimes Q$ and on $\omega\otimes Q(\cdot,\nabla J)$. Then, applying (2.8) and (2.7) after a long computation, (2.10) turns out to be equivalent to the condition $D=A+B+C=0$ where the (0, 5)-tensor fields A, B, C, are defined by:

$$\begin{aligned}&A(V,X,Y,Z,W)\\
&=-2\omega(X,Y)\{\varrho^*(\tilde R)(Z,(\nabla_V J)W)-\varrho^*(\tilde R)(W,(\nabla_V J)Z)\}+\\
&\quad+2\omega(Z,W)\{\varrho^*(\tilde R)(X,(\nabla_V J)Y)-\varrho^*(\tilde R)(Y,(\nabla_V J)X)\}\\
&\quad+\omega(X,Z)\ \{\varrho^*(\tilde R)(W,(\nabla_V J)Y)+\varrho^*(\tilde R)(Y,(\nabla_V J)W)\}\\
&\quad-\omega(Y,Z)\ \{\varrho^*(\tilde R)(W,(\nabla_V J)X)+\varrho^*(\tilde R)(X,(\nabla_V J)W)\}\\
&\quad+g(X,Z)\ \{\varrho^*(\tilde R)(JW,(\nabla_V J)Y)+\varrho^*(\tilde R)(Y,(\nabla_V J)JW)\}\\
&\quad-g(Y,Z)\ \{\varrho^*(\tilde R)(JW,(\nabla_V J)X)+\varrho^*(\tilde R)(X,(\nabla_V J)JW)\}\\
&\quad-\omega(X,W)\ \{\varrho^*(\tilde R)(Z,(\nabla_V J)Y)+\varrho^*(\tilde R)(Y,(\nabla_V J)Z)\}\\
&\quad+\omega(Y,W)\ \{\varrho^*(\tilde R)(Z,(\nabla_V J)X)+\varrho^*(\tilde R)(X,(\nabla_V J)Z)\}\\
&\quad-g(X,W)\ \{\varrho^*(\tilde R)(JZ,(\nabla_V J)Y)+\varrho^*(\tilde R)(Y,(\nabla_V J)JZ)\}\\
&\quad+g(Y,W)\ \{\varrho^*(\tilde R)(JZ,(\nabla_V J)X)+\varrho^*(\tilde R)(X,(\nabla_V J)JZ)\}\end{aligned}$$

$$
\begin{aligned}
&B(V, X, Y, Z, W)\\
&\quad = \omega(Z, V)\{\varrho^*(\tilde{R})(X, (\nabla_W J)Y) - \varrho^*(\tilde{R})(Y, (\nabla_W J)X)\}\\
&\quad - g(Z, V)\{\varrho^*(\tilde{R})(X, (\nabla_{JW} J)Y) - \varrho^*(\tilde{R})(Y, (\nabla_{JW} J)X)\}\\
&\quad - \omega(W, V)\{\varrho^*(\tilde{R})(X, (\nabla_Z J)Y) - \varrho^*(\tilde{R})(Y, (\nabla_Z J)X)\}\\
&\quad + g(W, V)\{\varrho^*(\tilde{R})(X, (\nabla_{JZ} J)Y) - \varrho^*(\tilde{R})(Y, (\nabla_{JZ} J)X)\}\\
&\quad - \omega(Y, V)\{\varrho^*(\tilde{R})(Z, (\nabla_X J)W) - \varrho^*(\tilde{R})(W, (\nabla_X J)Z)\}\\
&\quad + g(Y, V)\{\varrho^*(\tilde{R})(Z, (\nabla_X J)JW) + \varrho^*(\tilde{R})(JW, (\nabla_X J)Z)\}\\
&\quad + \omega(X, V)\{\varrho^*(\tilde{R})(Z, (\nabla_Y J)W) - \varrho^*(\tilde{R})(W, (\nabla_Y J)Z)\}\\
&\quad - g(X, V)\{\varrho^*(\tilde{R})(Z, (\nabla_Y J)JW) + \varrho^*(\tilde{R})(JW, (\nabla_Y J)Z)\}\\
&\quad + \frac{64}{3} V(\nu)\{\pi_1(X, Y, Z, W) + \pi_1(X, Y, JZ, JW) - 2\omega(X, Y)\omega(Z, W)\}
\end{aligned}
$$

$$
\begin{aligned}
&C(V, X, Y, Z, W)\\
&\quad = 32\{Z(\nu)[\pi_1(X, Y, V, W) + \pi_1(X, Y, JV, JW)]\\
&\qquad + JZ(\nu)[\pi_1(X, Y, V, JW) - \pi_1(X, Y, JV, W)]\\
&\qquad - W(\upsilon)[\pi_1(X, Y, V, Z) + \pi_1(X, Y, JV, JZ)]\\
&\qquad - JW(\upsilon)[\pi_1(X, Y, V, JZ) - \pi_1(X, Y, JV, Z)]\}\\
&\quad + \frac{32}{3}\{X(\upsilon)[\pi_1(Y, V, Z, W) + \pi_1(JY, JV, Z, W) + 8\omega(V, Y)\omega(Z, W)]\\
&\qquad + JX(\upsilon)[\pi_1(JY, V, Z, W) - \pi_1(Y, JV, Z, W) + 8g(V, Y)\omega(Z, W)]\\
&\qquad - Y(\upsilon)[\pi_1(X, V, Z, W) + \pi_1(JX, JV, Z, W) + 8\omega(V, X)\omega(Z, W)]\\
&\qquad - JY(\upsilon)[\pi_1(JX, V, Z, W) - \pi_1(X, JV, Z, W) + 8g(V, X)\omega(Z, W)]\}\\
&\quad - 2\{(\nabla_Z \omega)(X, V)\varrho^*(\tilde{R})(Y, JW) - (\nabla_{JZ}\omega)(X, V)\varrho^*(\tilde{R})(Y, W)\\
&\qquad - (\nabla_Z \omega)(Y, V)\varrho^*(\tilde{R})(X, JW) + (\nabla_{JZ}\omega)(Y, V)\varrho^*(\tilde{R})(X, W)
\end{aligned}
$$

$$-(\nabla_W\omega)(X,V)\varrho^*(\tilde{R})(Y,JZ)+(\nabla_{JW}\omega)(X,V)\varrho^*(\tilde{R})(Y,Z)$$

$$+(\nabla_W\omega)(Y,V)\varrho^*(\tilde{R})(X,JZ)-(\nabla_{JW}\omega)(Y,V)\varrho^*(\tilde{R})(X,Z)\}$$

where $\tilde{R}=R-L_3R$.

Now, we use the condition

$$
\begin{aligned}
\frac{1}{16}\{&D(V,X,Y,Z,W) &&+D(V,Z,W,X,Y)\\
&+D(V,Y,Z,X,W) &&+D(V,X,W,Y,Z)\\
&-D(V,JX,Y,JZ,W) &&-D(V,JZ,W,JX,Y)\\
&-D(V,Y,JZ,JX,W) &&-D(V,JX,W,Y,JZ)\\
&+D(JV,X,Y,Z,JW) &&+D(JV,Z,JW,X,Y)\\
&+D(JV,Y,Z,X,JW) &&+D(JV,X,JW,Y,Z)\\
&+D(JV,X,JY,Z,W) &&+D(JV,Z,W,X,JY)\\
&+D(JV,JY,Z,X,W) &&+D(JV,X,W,JY,Z)\}=0
\end{aligned}
\tag{2.11}
$$

Since $D=A+B+C$, we can separately evaluate the contribution given by A, B, C in (2.11).

The properties:

$$A(V,X,Y,Z,W)=-A(V,Z,W,X,Y)$$
$$B(V,X,Y,Z,W)=B(V,Z,W,X,Y)=B(V,JX,JY,Z,W)$$

imply that the tensor field A does not contribute in (2.11), while the contribution due to B reduces to

$$
\begin{aligned}
\frac{1}{8}\{&B(V,X,Y,Z,W) &&+B(V,Y,Z,X,W)\\
&-B(V,JX,Y,JZ,W) &&-B(V,Y,JZ,JX,W)\\
&+B(JV,X,Y,Z,JW) &&+B(JV,Y,Z,X,JW)\\
&+B(JV,X,JY,Z,W) &&+B(JV,JY,Z,X,W)\}
\end{aligned}
$$

$$= -\frac{1}{2}\{\omega(Y, V)\varrho^*(\tilde{R})(W, (\nabla_Z J)X - (\nabla_X J)Z)$$

$$-g(Y, V)\varrho^*(\tilde{R})(W, (\nabla_{JZ} J)X - (\nabla_X J)JZ)$$

$$-\omega(W, V)\varrho^*(\tilde{R})(Y, (\nabla_Z J)X - (\nabla_X J)Z)$$

$$+g(W, V)\varrho^*(\tilde{R})(Y, (\nabla_{JZ} J)X - (\nabla_X J)JZ)\}$$

$$+\frac{32}{3}\{V(\nu)[\pi_1(X, Z, Y, W) - \pi_1(X, Z, JY, JW)]$$

$$+JV(\nu)[\pi_1(X, Z, Y, JW) + \pi_1(X, Z, JY, W)]\} \tag{2.12}$$

$$= -\omega(Y, V)\varrho^*(\tilde{R})(W, (\nabla_Z J)X) + g(Y, V)\varrho^*(\tilde{R})(W, (\nabla_{JZ} J)X)$$

$$+\omega(W, V)\varrho^*(\tilde{R})(Y, (\nabla_Z J)X) - g(W, V)\varrho^*(\tilde{R})(Y, (\nabla_{JZ} J)X)$$

$$+16\{Z(\nu)\ [\pi_1(V, X, Y, W) - \pi_1(V, X, JY, JW)]$$

$$+X(\nu)\ [\pi_1(V, Z, Y, W) - \pi_1(V, Z, JY, JW)]$$

$$+JZ(\nu)[\pi_1(V, X, Y, JW) + \pi_1(V, X, JY, W)]$$

$$+JX(\nu)[\pi_1(V, Z, Y, JW) + \pi_1(V, Z, JY, W)]\}$$

$$+\frac{32}{3}\{V(\nu)\ [\pi_1(X, Z, Y, W) - \pi_1(X, Z, JY, JW)]$$

$$+JV(\nu)[\pi_1(X, Z, Y, JW) + \pi_1(X, Z, JY, W)]\}$$

where $\tilde{R} = R - L_3 R$

We remark that the last equality follows by (2.7).

Moreover, a direct computation, combined with the almost Kähler condition and (1.1), yields to:

$$C(V, X, Y, Z, W) + C(V, Z, W, X, Y)$$

$$+C(V, Y, Z, X, W) + C(V, X, W, Y, Z)$$

$$= 2\{(\nabla_V \omega)(X, Y)\varrho^*(\tilde{R})(Z, JW) - (\nabla_V \omega)(X, JY)\varrho^*(\tilde{R})(Z, W)$$

$$+(\nabla_V \omega)(Z, W)\varrho^*(\tilde{R})(X, JY) - (\nabla_V \omega)(Z, JW)\varrho^*(\tilde{R})(X, Y)$$

$$+(\nabla_V\omega)(Y,Z)\varrho^*(\tilde{R})(X,JW)-(\nabla_V\omega)(Y,JZ)\varrho^*(\tilde{R})(X,W)$$

$$+(\nabla_V\omega)(X,W)\varrho^*(\tilde{R})(Y,JZ)-(\nabla_V\omega)(X,JW)\varrho^*(\tilde{R})(Y,Z)\}$$

$$-4\{(\nabla_V\omega)(Y,W)\varrho^*(\tilde{R})(X,JZ)-(\nabla_V\omega)(Y,JW)\varrho^*(\tilde{R})(X,Z)$$

$$+(\nabla_V\omega)(X,Z)\varrho^*(\tilde{R})(Y,JW)-(\nabla_V\omega)(X,JZ)\varrho^*(\tilde{R})(Y,W)\}$$

$$+\frac{64}{3}\{Z(\nu)(3\pi_1+\pi_2)(X,V,JY,JW)+X(\nu)(3\pi_1+\pi_2)(Z,V,JW,JY)$$

$$+JZ(\nu)(3\pi_1+\pi_2)(JX,V,Y,W)+JX(\nu)(3\pi_1+\pi_2)(JZ,V,W,Y)$$

$$+W(\nu)(3\pi_1+\pi_2)(Y,V,JX,JZ)+Y(\nu)(3\pi_1+\pi_2)(W,V,JZ,JX)$$

$$+JW(\nu)(3\pi_1+\pi_2)(JY,V,X,Z)+JY(\nu)(3\pi_1+\pi_2)(JW,V,Z,X)\}$$

where $\tilde{R}=R-L_3R$.

From this formula, using (2.1), the J-anti-invariance of $\varrho^*(\tilde{R})$ and (1.1), one has:

$$\frac{1}{16}\{C(V,X,Y,Z,W)+C(V,Z,W,X,Y)$$

$$+C(V,Y,Z,X,W)+C(V,X,W,Y,Z)$$

$$-C(V,JX,Y,JZ,W)-C(V,JZ,W,JX,Y)$$

$$-C(V,Y,JZ,JX,W)-C(V,JX,W,Y,JZ)$$

$$+C(JV,X,Y,Z,JW)+C(JV,Z,JW,X,Y)$$

$$+C(JV,Y,Z,X,JW)+C(JV,X,JW,Y,Z)$$

$$+C(JV,X,JY,Z,W)+C(JV,Z,W,X,JY)$$

$$+C(JV,JY,Z,X,W)+C(JV,X,W,JY,Z)\}$$

$$=-(\nabla_V\omega)(X,Z)\varrho^*(\tilde{R})(Y,JW)+(\nabla_V\omega)(X,JZ)\varrho^*(\tilde{R})(Y,W)$$

$$-\frac{16}{3}\{Z(\nu)\ [\pi_1(X,V,Y,W)-\pi_1(JX,JV,Y,W)]$$

$$+X(\nu)\ [\pi_1(Z,V,W,Y)-\pi_1(JZ,JV,W,Y)]$$

$$+JZ(\nu)[\pi_1(X,V,JY,W)+\pi_1(X,V,Y,JW)]$$

$$+JX(\nu)[\pi_1(Z,V,JW,Y)+\pi_1(Z,V,W,JY)]\}.$$

By means of this relation and (2.12), the condition (2.11) turns out to be equivalent to the statement.

3 - Proof of the Theorems 1, 2, 3

First of all, we prove that, in the hypothesis of Theorem 1, the differential of the function ν vanishes at any point of M. This condition is achieved when $\varrho^*(\tilde{R})=0$; in fact, in this case, (M, g, J) is a $\mathcal{R}_3$-manifold, therefore it satisfies the Schur lemma of antiholomorphic type ([**11**]).

Now, we are going to prove that $d\nu_{|U}=0$, for any open set U where $\varrho^*(\tilde{R})\neq 0$. To this aim, given U with $\varrho^*(\tilde{R})_{|U}\neq 0$, we can consider a local orthonormal frame $\{E_1, JE_1, E_2, JE_2, E_3, JE_3\}$ such that $\varrho^*(\tilde{R})(E_2, JE_3)\neq 0$ (on U).

Let X, Z be local vector fields. Applying (2.9) to the quintuplet (E_1, X, E_2, Z, E_3), we have

$$(3.1)\qquad \begin{aligned}&(\nabla_{E_1}\omega)(X, Z)\varrho^*(\tilde{R})(E_2, JE_3)-(\nabla_{E_1}\omega)(X, JZ)\varrho^*(\tilde{R})(E_2, E_3)\\&+\frac{32}{3}\{E_1(\nu)\,[\pi_1(Z, X, E_2, E_3)-\pi_1(Z, X, JE_2, JE_3)]\\&+JE_1(\nu)[\pi_1(Z, X, JE_2, E_3)+\pi_1(Z, X, E_2, JE_3)]\}=0.\end{aligned}$$

Putting in (3.1) $X=E_1$, $Z=(\nabla_{E_1}J)\,E_1$, we obtain

$$\|(\nabla_{E_1}J)\,E_1\|^2\varrho^*(\tilde{R})(E_2, JE_3)=0$$

and then $(\nabla_{E_1}J)E_1=0$. Combining with (2, 7), we get:

$0=2\varrho^*(\tilde{R})(E_1,(\nabla_{E_1}J)E_1)=64JE_1(\nu),\ 0=2\varrho^*(\tilde{R})(JE_1,(\nabla_{E_1}J)E_1)=64E_1(\nu)$,i.e.

$$(3.2)\qquad d\nu(E_1)=d\nu(JE_1)=0\,.$$

Thus, for any vector field X, putting in (3.1) $Z=(\nabla_{E_1}J)\,X$, we have $\|(\nabla_{E_1}J)X\|^2\varrho^*(\tilde{R})(E_2, JE_3)=0$ and this implies

$$(3.3)\qquad (\nabla_{E_1}J)_{|U}=0\,.$$

This condition, combined with (2.1), gives also: $(\nabla_{JE_1}J)_{|U}=0$.

Now, considering again arbitrary local vector fields X, Z and applying (2.9) to the quintuplet (Z, X, E_2, E_1, E_3), from (3.2) and (3.3) we have

$$(\nabla_Z\omega)(X, E_1)\,\varrho^*(\tilde{R})(E_2, JE_3)-(\nabla_Z\omega)(X, JE_1)\,\varrho^*(\tilde{R})(E_2, E_3)=0\,.$$

This relation implies $(\nabla_Z J)\, E_1 = 0$, and also, combining with (2.7) and (3.3),

$$0 = \varrho^*(\tilde{R})(JE_1, (\nabla_Z J)\, E_1 + (\nabla_{E_1} J)\, Z) = 32 Z(\nu).$$

Since Z is arbitrarily chosen, we have $d\nu_{|U} = 0$. Thus, Theorem 1 is proved.

As for as regards the proof of Theorem 2, we have to prove the condition $\varrho^*(\tilde{R}) = 0$ where $\tilde{R} = R - L_3 R$.

Thus, we assume that $\varrho^*(\tilde{R})$ does not vanish at least at a point p of M and then $\varrho^*(\tilde{R}) \neq 0$ in an open neighbourhood of p and derive a contradiction.

We consider a local orthonormal frame $\{E_1, JE_1, E_2, JE_2, E_3, JE_3\}$ defined in an open set U, with $\varrho^*(\tilde{R})(E_2, JE_3) \neq 0$ in U. We have just proved that $(\nabla_{E_1} J)_{|U} = (\nabla_{JE_1})_{|U} = 0$ and $(\nabla_Z J)\, E_1 = (\nabla_Z J)\, JE_1 = 0$, for any local vector field Z. In particular, this implies that the local vector fields $(\nabla_{E_3} J)\, E_3, (\nabla_{E_2} J)\, E_3$ can be expressed as a linear combination of E_2, JE_2, according to:

$$(3.4) \qquad (\nabla_{E_3} J)\, E_3 = (\nabla_{E_3} \omega)(E_3, E_2)\, E_2 + (\nabla_{E_3} \omega)(E_3, JE_2)\, JE_2$$

$$(3.5) \qquad (\nabla_{E_2} J)\, E_3 = (\nabla_{E_2} \omega)(E_3, E_2)\, E_2 + (\nabla_{E_2} \omega)(E_3, JE_2)\, JE_2.$$

The formula (3.5) implies:

$$(3.6) \qquad \begin{aligned} &\varrho^*(\tilde{R})(E_2, (\nabla_{JE_2} J)\, E_3) = -\varrho^*(\tilde{R})(JE_2, (\nabla_{E_2} J)\, E_3) = 0 \\ &\varrho^*(\tilde{R})(E_2, (\nabla_{E_2} J)\, E_3) = 0\,. \end{aligned}$$

Now, we apply twice Lemma 2.1, respectively considering the quintuplets $(E_3, E_3, E_2, E_2, E_3)$ and $(E_3, E_3, E_2, JE_2, E_3)$. Since ν is constant and (3.6) holds, from (2.9) we obtain:

$$(\nabla_{E_3} \omega)(E_3, E_2)\, \varrho^*(\tilde{R})(E_2, JE_3) - (\nabla_{E_3} \omega)(E_3, JE_2)\, \varrho^*(\tilde{R})(E_2, E_3) = 0$$

$$(\nabla_{E_3} \omega)(E_3, JE_2)\, \varrho^*(\tilde{R})(E_2, JE_3) + (\nabla_{E_3} \omega)(E_3, E_2)\, \varrho^*(\tilde{R})(E_2, E_3) = 0$$

and then, since $\varrho^*(\tilde{R})(E_2, JE_3) \neq 0$ in U, we have

$$(\nabla_{E_3} \omega)(E_3, E_2) = (\nabla_{E_3} \omega)(E_3, JE_2) = 0\,.$$

Comparing with (3.4), we have $(\nabla_{E_3} J)\, E_3 = 0$ and also

$$(\nabla_{E_3} J)\, E_2 = -(\nabla_{E_3} \omega)(E_3, E_2)\, E_3 - (\nabla_{E_3} \omega)(E_3, JE_2)\, JE_3 = 0\,.$$

Since also $(\nabla_{E_3} J)\, E_1 = 0$, one has $(\nabla_{E_3} J)_{|U} = 0$.

An analogous argument allows to prove that $(\nabla_{E_2}J)_{|U}=0$. Since $(\nabla_{E_1}J)_{|U}=0$ and (2.1) holds, the tensor field $(\nabla J)_{|U}$ vanishes. This means that (g, J) is a Kähler structure in U with constant antiholomorphic sectional curvature ν. Therefore, one has $R_{|U}=\nu(\pi_1+\pi_2)$ and then $\varrho^*(\widetilde{R})_{|U}=0$.

This contradicts the condition $\varrho^*(\widetilde{R})(E_2, JE_3)\neq 0$.

Finally, concerning the proof of Theorem 3, we remark that Theorem 2 allows to apply Theorem A to any 6-dimensional almost-Kähler manifold (M, g, J) with p.c.a.s.c. Thus, (M, g, J) turns out to be a complex space-form or, possibly, a real space-form. In the last case, Theorem C says that (M, g, J) is a flat Kähler manifold, i.e. again a complex space-form.

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Sommario

Si mostra che una varietà almost Kähler di dimensione 6, con curvatura sezionale antiolomorfa puntualmente costante, è una varietà di Kähler con curvatura sezionale olomorfa costante.

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