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LUIGI VERDIANI (*)

Curvature homogeneous metrics of cohomogeneity one on vector spaces (**)

1 - Introduction

Let V be a real n-dimensional vector space, and let $G \subset SO(n)$ be a compact connected Lie group acting transitively on $S^{n-1} \subset V$. That is, according to the H. Borel list [5]:

$$G \in \{SO(n), G_2, \mathrm{Spin}(7), U(\frac{n}{2}), SU(\frac{n}{2}), \mathrm{Sp}(\frac{n}{4}),$$

$$\mathrm{Sp}(1)\cdot\mathrm{Sp}(\frac{n}{4}), U(1)\cdot\mathrm{Sp}(\frac{n}{4}), \mathrm{Spin}(9)\}.$$

The aim of this paper is to classify curvature homogeneous G-invariant metrics g on V.

Recall that a Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for all points $p, q \in M$, there exists a linear isometry $f: T_p M \to T_q M$, which preserves the curvature tensor R of g, that is $f^* R_q = R_p$ [11].

A locally homogeneous Riemannian manifold is curvature homogeneous, but the converse is not true in general: the first examples of irreducible complete Riemannian manifolds which are curvature homogeneous, but are not locally homogeneous, were produced by K. Sekigava [10] and H. Takagi [12]. Now many other

(*) Dip. di Matem. U. Dini, Univ. Firenze, V.le Morgagni 67/A, 50134 Firenze, Italia.
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examples are known and the investigation on curvature homogeneous manifolds is related to other problems (for example a conjecture of M. Gromov): for a basic reference on this topic we refer to [4]. In particular, in 1989, K. Tsukada [**15**] showed that there exists a unique explicitely given curvature homogeneous hypersurface M of an n-dimensional space form ($n \geq 5$), which is not homogeneous and has non constant principal curvatures (for $n = 4$ the problem is still open). More precisely, M is an hypersurface of the 5-dimensional hyperbolic space $\boldsymbol{H}^5(-1)$, and it is a cohomogeneity one manifold, that is, it admits an isometry group with a codimension 1 orbit (for more references on cohomogeneity one manifolds, see [**1**], [**2**]).

The problem of finding curvature homogeneous manifolds in the class of cohomogeneity one manifolds is motivated by the fact that, while in general one has to solve partial differential equations, in these spaces the problem reduces to ordinary differential equations, which are, in principle, easier to manage. The study of curvature homogeneous metrics of cohomogeneity one on vector spaces is the first step in the intrinsic classification of curvature homogeneous metrics of cohomogeneity one. In fact, if G' is a compact Lie group, any cohomogeneity one G'-manifold M, in a neighbourhood of a singular orbit, is diffeomorphic to the twisted product: $G' \times_G V$, where G is the stabilizer of a singular point, and V is an n-dimensional real vector space such that G acts transitively on the standard sphere $S^{n-1} \subset V$ [**6**]. The study of G'-invariant metrics on M reduces to the description of G'-invariant metrics on $\frac{G'}{G}$ and G-invariant metrics on V [**3**], [**16**].

In [**16**] the explicit description of any G-invariant metric g on the set $V \backslash \{0\}$ is given, together with a necessary and sufficient condition for the extendibility to all V. Using this description, we write the ordinary differential equation on a metric to be curvature homogeneous, and state the initial conditions. Lemma 2, shows that the extension of a curvature homogeneous metric on $V \backslash \{0\}$ to V is again curvature homogeneous.

Denote by g_0 the standard Euclidean metric in V, by t the radial coordinate, by $\frac{\partial}{\partial t}$ the corresponding radial unit vector field, and let $dt = g_0(\cdot, \frac{\partial}{\partial t})$ be the corresponding 1-form on $V \backslash \{0\}$. Our main result can be stated as follows

Theorem 1. *Let $V = \boldsymbol{R}^n$ be the n-dimensional Euclidean vector space with the standard metric g_0, and let $G \subset SO(n)$ be a compact connected Lie group acting transitively on the unit sphere $S^{n-1} \subset V$. Any curvature homogeneous, G-invariant metric g on $V \backslash \{0\}$, that admits an extension to a smooth, complete metric on V, is homogeneous, Einstein and is G-diffeomorphic to one of the metrics g_i, $(i = 0, \ldots, 4)$ which are described below:*

1. *For* $G = SO(n)$, G_2, Spin(7):

$$g \simeq g_0 \quad or \quad g \simeq g_1 = dt^2 + \frac{\sinh^2(t)}{t^2} g_{0|S^{n-1}}$$

where $\simeq$ *means G-diffeomorphic.*

2. *Let* $n = 2m$, J *the standard complex structure on* V *and let*

$$G = U(m) = \{A \in SO(n): [A, J] = 0\} \quad or \quad G = SU(m) = \{A \in U(n): \det(A) = 1\}.$$

Define the 1*-form* θ, *on* $V \setminus \{0\}$, *by:* $\theta = g_0(\cdot, J(\frac{\partial}{\partial t}))$. *Denote by* W_2 *the codimension 2 distribution on* $V \setminus \{0\}$ g_0*-orthogonal to* $\mathrm{span}(\frac{\partial}{\partial t}, J(\frac{\partial}{\partial t}))$. *Then:*

$$g \simeq g_0, \quad or \quad g \simeq g_1 = dt^2 + \frac{\sinh^2(t)}{t^2} g_{0|S^{n-1}}$$

$$or \quad g \simeq g_2 = dt^2 + \frac{\sinh^2(2t)}{4t^2}\theta^2 + \frac{\sinh^2(t)}{t^2} g_{0|W_2}.$$

The metrics g_0 *and* g_1 *are* $SO(n)$*-invariant.*

3. *Let now* $n = 4m$ *and* J_1, J_2, $J_3 = J_1 J_2$, *be three anticommuting complex structures on* V, *which generate a quaternionic structure and*

$$G = \mathrm{Sp}(m) = \{A \in SO(V), [A, J_i] = 0,\ i = 1, 2, 3\} \quad or \quad G = U(1)\cdot \mathrm{Sp}(m).$$

Define the 1*-forms* $\theta_i (i = 1, 2, 3)$, *on* $V \setminus \{0\}$, *by* $\theta_i = g_0(\cdot, J_i(\frac{\partial}{\partial t}))$, *and let* W_4 *be the codimension* 4 *distribution* g_0*-orthogonal to the* $\mathrm{span}(\frac{\partial}{\partial t}, J_1(\frac{\partial}{\partial t}), J_2(\frac{\partial}{\partial t}), J_3(\frac{\partial}{\partial t}))$ *in* $V \setminus \{0\}$. *Then:*

$$g \simeq g_0 \quad or \quad g \simeq g_1 = dt^2 + \frac{\sinh^2(t)}{t^2} g_{0|S^{n-1}},$$

$$or \quad g \simeq g_2 = dt^2 + \frac{\sinh^2(2t)}{4t^2}\theta_1^2 + \sum_{i=2}^{3} \frac{\sinh^2(t)}{t^2}\theta_i^2 + \frac{\sinh^2(t)}{t^2} g_{0|W_4},$$

$$or \quad g \simeq g_3 = dt^2 + \sum_{i=1}^{3} \frac{\sinh^2(2t)}{4t^2}\theta_i^2 + \frac{\sinh^2(t)}{t^2} g_{0|W_4}.$$

The metrics g_0 *and* g_1 *are* $SO(n)$*-invariant, the metric* g_2 *is* $U(2m)$*-invariant.*

4. *In the assumption of* 3, *let* $G = \mathrm{Sp}(1)\cdot\mathrm{Sp}(m)$, *where* $\mathrm{Sp}(1)$ *is the centralizer of* $\mathrm{Sp}(m)$ *in* $SO(n)$. *Then:*

$$g \simeq g_0 \quad or \quad g \simeq g_1 = dt^2 + \frac{\sinh^2(t)}{t^2}\, g_{0|S^{n-1}}$$

$$or \quad g \simeq g_3 = dt^2 + \sum_{i=1}^{3} \frac{\sinh^2(2t)}{4t^2}\,\theta_i^2 + \frac{\sinh^2(t)}{t^2}\, g_{0|W_4}\,.$$

The metrics g_0 *and* g_1 *are homogeneous and* $SO(n)$*-invariant, the metric* g_3 *is* $\mathrm{Sp}(m)$*-invariant.*

5. *Let* $n = 16$ *and* $G = \mathrm{Spin}(9)$. *Denote by* W_8 *and* W_9 *the two mutually* g_0*-orthogonal G-invariant distributions (of codimension* 8 *an* 9 *respectively), tangents to the spheres* $\mathrm{Spin}(9)v$, $v \in V^{16}\setminus\{0\}$, *which are the vertical and the horizontal distributions of the Hopf fibration* $S^{15} \to S^8$ [3]. *Then:*

$$g \simeq g_0 \quad or \quad g \simeq g_4 = dt^2 + 2\,\frac{\sinh^2(\frac{\sqrt{2}}{2t})}{t^2}\; g_{0|W_8} + \frac{\sinh^2(\sqrt{2}t)}{2t^2}\; g_{0|W_9}\,.$$

The first metric is also $SO(n)$*-invariant.*

This theorem is proved in Section 3. The completeness of these metric follows from a criterium of completeness given in Section 2.

2 - Preliminary results

Definition 1. A Riemannian manifold (M, g), with Riemannian curvature tensor R, is *curvature homogeneous* if, for all $p, q \in M$, there exists a linear isometry f between the tangent spaces T_pM and T_qM such that $f^*(R_q) = R_p$.

In [13], F. Tricerri and L. Vanhecke prove

Theorem 2. *A Riemannian manifold* (M, g), *with curvature tensor* R, *is curvature homogeneous if and only if there exists a linear metric connection* ∇ *which preserves* R, *that is* $\nabla R = 0$.

Remark that the difference $A = D - \nabla$ of two connections is a tensor field of type $(1, 2)$. In terms of such field Theorem 2 can be reformulated as follows

Theorem 2′. *A Riemannian manifold (M, g), with Levi-Civita connection D and curvature tensor R, is curvature homogeneous if and only if there exists a tensor field A of type $(1, 2)$ such that, for any vector field X on M:*

$$D_X R = A_X \cdot R \quad \text{and} \quad A_X^t = -A_X \tag{2.1}$$

where A_X^t denotes the transposition of the $(1, 1)$-tensor A_X.

Definition 2. A *cohomogeneity one Riemannian manifold* is a Riemannian manifold (M, g), with a given group G of isometries that has an orbit of codimension 1.

The following lemma shows that, in the case of cohomogeneity one Riemannian G-manifolds, we can always solve equations (2. 1) along the regular orbits (that is the orbits of maximal dimension) (see also [9]).

Lemma 1. *Let (M, g) be a cohomogeneity one Riemannian G-manifold, denote by R the curvature tensor of the Levi-Civita connection D of g. Then there exists a G-invariant linear connection ∇ on M, which is metric ($\nabla g = 0$) and preserves the curvature R along the orbits (i.e. $\nabla_X R = 0$, for any vector X tangent to a regular orbit of G).*

Proof. Let K be the isotropy subgroup of a regular point P of M, and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let $\mathfrak{m}$ be an $\mathrm{Ad}(K)$ invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$, then we have the direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. There is a natural identification between $\mathfrak{m}$ and the tangent space to the orbits in P (see for example [3]). Using this identification we can define a new connection in M: let N be the unit normal vector field (which is globally defined when there is no singular orbit [1]), and, for any vector $Y \in T_P M$, let $\overline{Y}$ be the Killing vector field given by the element of $\mathfrak{m}$ corresponding to Y. If $\{Y_i, N(P)\}$ $(i = 1 \ldots, n-1)$ forms a basis of $T_P M$, then $\{\overline{Y}_i, N\}$ forms a local frame in a neighbourhood of P, hence, we can define a connection ∇ with:

$$\nabla_{\overline{Y}_i} \overline{Y}_j = [\overline{Y}_i, \overline{Y}_j] \qquad \nabla_{\overline{Y}_i} N = 0 \qquad \nabla_N \overline{Y}_i = D_N \overline{Y}_i \qquad \nabla_N N = 0\,.$$

This connection is G-invariant, in fact D and N are G-invariant and, for any $g \in G$, $g_*[X, Y] = [g_* X, g_* Y]$. This connection is also metric. We prove it just in one case, the proof in the other cases being similar:

$$Y_h\langle \overline{Y}_i, \overline{Y}_j\rangle = \langle D_{Y_h}\overline{Y}_i, \overline{Y}_j\rangle + \langle D_{Y_h}\overline{Y}_j, \overline{Y}_i\rangle$$

$$= \langle [\overline{Y_h}, \overline{Y}_i], \overline{Y}_j\rangle - \langle D_{\overline{Y}_i}\overline{Y_h}, \overline{Y}_j\rangle + \langle [\overline{Y_h}, \overline{Y}_j], \overline{Y}_j\rangle - \langle D_{\overline{Y}_j}\overline{Y_h}, \overline{Y}_i\rangle$$

$$= \langle \nabla_{Y_h}\overline{Y}_i, \overline{Y}_j\rangle + \langle D_{\overline{Y}_j}\overline{Y_h}, \overline{Y}_i\rangle + \langle \nabla_{Y_h}\overline{Y}_j, \overline{Y}_j\rangle - \langle D_{\overline{Y}_j}\overline{Y_h}, \overline{Y}_i\rangle$$

$$= \langle \nabla_{Y_h}\overline{Y}_i, \overline{Y}_j\rangle + \langle \nabla_{Y_h}\overline{Y}_j, \overline{Y}_j\rangle .$$

Note that $\nabla_{\overline{Y}_i}\overline{Y}_j = [\overline{Y}_i, \overline{Y}_j]$ implies that, with respect to this connection, any parallel vector field tangent to the orbit $G(P)$ at P, is tangent to the orbit at any point. That is the orbits are totally geodesic. But ∇ defines on the orbits the canonical connection associated to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (see **[13]**), and this connection has the property that the parallel translation along geodesics (which are the integral curves of Killing fields) coincides with the differential of the flow associated to the tangent field (see [7], p. 191-193). Since N is G-invariant and parallel, this is true for any vector field in M.

Using the definition of covariant derivative in terms of parallel translation it follows that any G-invariant tensor field in M is parallel with respect to ∇ along any geodesic which is tangent to the orbits. In particular we have $\nabla_X R = 0$.

This lemma implies that a cohomogeneity one manifold (M, g) is curvature homogeneous if and only if there exists a solution of (2. 1) when X is a vector field not tangent to the orbits.

We consider now the case when $(M, g) = (V, g)$ is a cohomogeneity one real vector space of dimension n, with the isometric action of a compact connected subgroup G of $SO(n)$, transitive on the unit sphere $S^{n-1} \subset V$. For any vector $v \in V \setminus \{0\}$, denote by t be the Euclidean norm of v and by $\frac{\partial}{\partial t}$ the radial vector field (note that $\frac{\partial}{\partial t}$ is trasversal to the orbits). Let $\{Y_i\}$, $(i = 1, \ldots, n)$ be a parallel g-orthonormal frame along the straight line tv (which in general is not a geodesic **[16]**). The problem of finding curvature homogeneous G-invariant metrics on $V \setminus \{0\}$ reduces then to determine functions $a_i^j(\frac{\partial}{\partial t})(v) = a_i^j(t)$ (using the

G-invariance of a_i^j) that satisfy:

$$\frac{\partial}{\partial t}(R_{Y_i, Y_j, Y_h, Y_k}) = \sum_{m=1}^{n}(a_m^i(t)\, R_{Y_m, Y_j, Y_h, Y_k} + a_m^j(t) R_{Y_i, Y_m, Y_h, Y_k}\cdot$$

$$+ a_m^h(t) R_{Y_i, Y_j, Y_m, Y_k} + a_m^k(t)\, R_{Y_i, Y_j, Y_h, Y_m}) \tag{2.2}$$

and $\quad a_i^j(t) = -a_j^i(t).$

To solve this problem, we have to find an explicit expression of the curvature tensor of any cohomogeneity one metric g, but we need first to fix more notations.

Remark 1. Denote by g_0 the standard euclidean metric on V, and by g_S its restriction to S^{n-1}. We recall, following [17], the description of the isotropy representation for all the groups we consider. Let G be a compact connected Lie group acting transitively on $S^{n-1} \subset V$, let $e \in S^{n-1}$ be a regular point and denote by K the isotropy subgroup of e. Then, for any $t \in \boldsymbol{R}^+$, as K-module, TV_{te} admits the decomposition:

1. For $G = SO(n)$, Spin(7), G_2:

$$T_{te} V = V = \boldsymbol{R}e + W_1\,,$$

where $K = SO(n-1)$, G_2, $SU(3)$ respectively, and $W_1 = T_{te} S^{n-1}$ is an irreducible K-module.

2. For $n = 2m$, and $G = U(m)$, $SU(m)$: denote by J the corresponding standard complex structure on V. Then the decomposition of V is given by

$$T_{te} V = V = \boldsymbol{R}e + W_0 + W_2 = \boldsymbol{R}e + \boldsymbol{R}J(e) + W_2\,,$$

where $K = U(m-1)$, $SU(m-1)$ respectively, W_0 and W_2 are two orthogonal distribution such that $T_{te} S^{n-1} = W_0 + W_2$, $\dim W_0 = 1$, $\dim W_2 = n-2$. K acts trivially on W_0 and irreducibly on W_2.

3. Let $n = 4m$ and $G = \mathrm{Sp}(m) = \{A \in SO(V), [A, J_a] = 0,\ a = 1, 2, 3\}$, where $J_1, J_2, J_3 = J_1 J_2$, are three anticommuting complex structures on V, that define a quaternionic structure. Then $K = \mathrm{Sp}(m-1)$, and V admits the K-invariant decomposition

$$T_{te} V = V = \boldsymbol{R}e + W_0 + W_4 = \boldsymbol{R}e + \boldsymbol{R}J_1(e) + \boldsymbol{R}J_2(e) + \boldsymbol{R}J_3(e) + W_4$$

where W_0 and W_4 are two mutually orthogonal distributions such that $T_{te} S^{n-1}$

is equal to $W_0 + W_4$, $\dim W_0 = 3$, $\dim W_4 = n - 4$. K acts trivially on W_0 and irreducibly on W_1.

4. For $n = 4m$ and $G = \mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$, using the notations of **3**, the decomposition of V is given by

$$T_{te} V = V = \boldsymbol{R}e + W_0 + W_4 = \boldsymbol{R}e + \boldsymbol{R}J_1(e) + \boldsymbol{R}J_2(e) + \boldsymbol{R}J_3(e) + W_4$$

where $K = \mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ acts irreducibly on W_0 and W_4.

5. For $n = 4m$ and $G = \mathrm{Sp}(m)\cdot U(1)$, using the notations of **3**, the decomposition of W is given by

$$T_{te} V = V = \boldsymbol{R}e + W_0 + W_{n-2} + W_4$$

where, for (i, j, k) fixed cyclic permutation of $(1, 2, 3)$: $W_0 = \boldsymbol{R}J_i(e)$, $W_{n-2} = \boldsymbol{R}J_j(e) + \boldsymbol{R}J_k(e)$, W_4, are mutually orthogonal distributions such that $T_{te} S^{n-1} = W_0 + W_{n-2} + W_4$. $K = \mathrm{Sp}(m)\cdot U(1)$ acts trivially on W_0, and irreducibly on W_{n-2} and W_4.

6. For $n = 16$ and $G = \mathrm{Spin}(9)$: $K = \mathrm{Spin}(7)$, V admits the K-invariant decomposition

$$T_{te} V = V = \boldsymbol{R}e + W_8 + W_9$$

where $W_8 + W_9 = T_e S^{15}$, $\dim W_8 = 8$, $\dim W_9 = 7$. K acts irreducibly on W_8 and W_9.

We recall here the main result of **[16]**, which gives the description of all G-invariant metrics of cohomogeneity one on V.

Theorem 3. *Let V be the Euclidean vector space of dimension n, and $G \subset SO(n)$ a compact connected Lie group acting transitively on the sphere $S^{n-1} \subset V$. Then, using the notations introduced in Remark* 1, *any smooth G-invariant metric g on $V \setminus \{0\}$ which admits a smooth extension to the origin, can be described as follows:*

1. *For* $G = SO(n)$, $\mathrm{Spin}(7)$, G_2:

$$g = \mu^2(t)\, dt^2 + \eta^2(t)\, g_S\,,$$

where $\mu(t)$, $\eta(t)$ are smooth, even and positive functions of t such that $\mu(0) = \eta(0)$.

2. *For $n=2m$ and $G=U(m)$, $SU(m)$:*

$$g=\mu^2(t)\,dt^2+a(t)\,dt\,\theta+\lambda^2(t)\,\theta^2+\eta^2(t)\,g_{S|W_2}$$

where θ is the 1-form defined, on $V\backslash\{0\}$, by: $\theta=g_0(\cdot, J(e))$, $\mu(t)$, $\lambda(t)$, $\eta(t)$ are smooth, even and positive functions of t such that $\mu(0)=\lambda(0)=\eta(0)$, and $a(t)$ is a smooth and even function of t, with $a(0)=0$, which satisfies: $a(t)^2<\mu(t)^2\lambda(t)^2$.

3. *For $n=4m$ and $G=\mathrm{Sp}(m)$:*

$$g=\mu^2(t)\,dt^2+\sum_{i=1}^{3}a_i(t)\,dt\theta_i+\sum_{i\neq j=1}^{3}b_{ij}(t)\,\theta_i\theta_j+\sum_{i=1}^{3}\lambda_i^2(t)\,\theta_i^2+\eta^2(t)\,g_{S|W_4}$$

where θ_i are the 1-forms defined, on $V\backslash\{0\}$, by: $\theta_i=g_0(\cdot, J_i(e))$ $(i=1,2,3)$, $\mu(t)$, $\lambda_i(t)$, $\eta(t)$ are smooth, even and positive functions of t such that $\mu(0)=\lambda_i(0)=\eta(0)$. $a_i(t)$, $b_{ij}(t)=b_{ji}(t)$ are smooth and even function of t, with $a_i(0)=b_{ij}(0)=0$, for $i<j=1,2,3$, and such that the matrix

$$\begin{pmatrix}\lambda_1^2 & b_{12} & b_{13} & a_1\\ b_{12} & \lambda_2^2 & b_{23} & a_2\\ b_{13} & b_{23} & \lambda_3^2 & a_3\\ a_1 & a_2 & a_3 & \mu^2\end{pmatrix}$$

is positive definite.

4. *For $n=4m$ and $G=\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$, the G-invariant metrics g are metrics from 3 with: $\lambda_i(t)=\lambda(t)$,$a_i(t)=b_{ij}(t)=0$, for $i<j=1,2,3$.*

5. *For $n=4m$ and $G=\mathrm{Sp}(m)\cdot U(1)$, the G-invariant metrics are metrics from 3 with: $\lambda_i(t)=\lambda_j(t)=\lambda(t)\neq\lambda_k(t)$, $a_i(t)=a_j(t)=b_{ij}(t)=b_{jk}(t)=b_{ik}(t)=0$, where (i,j,k) is a fixed permutation of $(1,2,3)$.*

6. *For $G=\mathrm{Spin}(9)$:*

$$g=\mu^2(t)\,dt^2+\eta_1^2(t)\,g_{S|W_8}+\eta_2^2(t)\,g_{S|W_9}\,,$$

where $\mu(t)$, $\eta_i(t)$ are smooth, even and positive functions of t, such that $\mu(0)=\eta_i(0)$, for $i=1,2$.

The following lemma shows that the extension of a curvature homogeneous metric on $V\backslash\{0\}$ to V is curvature homogeneous.

Lemma 2. *Let (M, g) be a Riemannian manifold. If M admits a smooth, open dense and curvature homogeneous (with respect to the induced metric) submanifold M^*, then M is curvature homogeneous.*

Proof. Let V be an n-dimensional vector space. Denote by $\mathfrak{R}(V)$ the set of curvature like tensors on V (the tensors which have the same simmetry properties of the curvature tensor of a Riemannian manifold, see [7]). Then the orthogonal group $O(n)$ acts on $\mathfrak{R}(V)$: if $K \in \mathfrak{R}(V)$ and $a \in O(n)$:

$$(aK)(X_1, \ldots, X_n) = K(a^{-1}X_1, \ldots, a^{-1}X_n).$$

Denote by $\mathfrak{R}^*(V)$ the orbit space of this action and by π': $\mathfrak{R}(V) \to \mathfrak{R}^*(V)$ the corresponding projection.

Let $P \in M$, then any element $u = (u_1, \ldots, u_n)$ of the orthonormal frame bundle $O(M)$ defines an isometry between (V, g_0) and $(T_P M, g_{|P})$ by:

$$u(v_1, \ldots, v_n) = \sum_{i=1}^{n} v_i u_i .$$

Using this fact, we can view the curvature tensor R of M, in a point P, as an element of $\mathfrak{R}(V)$. Let π be the projection of $O(M)$ on M, then we can define the equivariant map:

$$\tilde{R}\colon O(M) \to \mathfrak{R}(V)$$

$$\tilde{R}(u)(X_1, \ldots, X_n) = R_{\pi(u)}(uX_1, \ldots, uX_n).$$

Then the following diagram is commutative:

$$\begin{array}{ccc} O(M) & \xrightarrow{\tilde{R}} & \mathfrak{R}(V) \\ \pi\downarrow & & \pi'\downarrow \\ M & \xrightarrow{\overline{R}} & \mathfrak{R}^*(V) \end{array}$$

and defines a continuous map $\overline{R}$: $M \to \mathfrak{R}^*(V)$. A metric in M is curvature homogeneous if and only if $\overline{R}(M) \subset \mathfrak{R}^*(V)$ is a point (see [8]). Since $\overline{R}$ is continuous and constant in the dense subset $M^* \subset M$, it must be constant in M.

We can give a necessary and sufficient condition on metrics described in Theorem 3, to obtain the completeness on V (we prove it here with the additional hypothesis $b_{ij}=0$, $i<j=1,2,3$, when $G=\mathrm{Sp}(\frac{n}{4})$, $\mathrm{Sp}(\frac{n}{4})\cdot\mathrm{Sp}(1)$, $\mathrm{Sp}(\frac{n}{4})\cdot U(1)$, since the proof in the general case is similar):

Proposition 1. *Let V be a real n-dimensional vector space, and let $G\subset SO(n)$, be one of the linear connected Lie groups which act transitively on the sphere $S^{n-1}\subset V$. Let g be one of the G-invariant metrics on V described in Theorem 3, and let:*

$$\sigma(t)=\mu(t) \qquad \text{if } G=SO(N),\ \mathrm{Spin}(7),\ G_2,\ \mathrm{Spin}(9)$$

$$\sigma(t)=\sqrt{\mu(t)^2-\frac{a(t)^2}{\lambda(t)^2}}, \qquad \text{if } G=U(\frac{n}{2}),\ SU(\frac{n}{2})$$

$$\sigma(t)=\sqrt{\mu(t)^2-\sum_{i=1}^{3}\frac{a_i(t)^2}{\lambda_i(t)^2}}, \qquad \text{if } G=\mathrm{Sp}(\frac{n}{4}),\ \mathrm{Sp}(\frac{n}{4})\cdot\mathrm{Sp}(1),\ \mathrm{Sp}(\frac{n}{4})\cdot U(1)$$

$$\text{and } b_{ij}=0,\ i<j=1,2,3\,.$$

Then g is complete if and only if

$$\int_0^{+\infty}\sigma(t)\,dt=+\infty\,.$$

Proof. The function $\sigma(t)$ is the norm of the vector:

$$N=\frac{\partial}{\partial t} \qquad \text{if } G=SO(N),\ \mathrm{Spin}(7),\ G_2,\ \mathrm{Spin}(9)$$

$$N=\frac{\partial}{\partial t}-\frac{a(t)}{\lambda(t)^2}J_1(\frac{\partial}{\partial t}) \qquad \text{if } G=U(\frac{n}{2}),\ SU(\frac{n}{2})$$

$$N=\frac{\partial}{\partial t}-\sum_{i=1}^{3}\frac{a_i(t)}{\lambda_i(t)^2}J_i(\frac{\partial}{\partial t}) \qquad \text{if } G=\mathrm{Sp}(\frac{n}{4}),\ \mathrm{Sp}(\frac{n}{4})\cdot\mathrm{Sp}(1),\ \mathrm{Sp}(\frac{n}{4})\cdot U(1)$$

$$\text{and } b_{ij}=0,\ i<j=1,2,3$$

which is orthogonal to the regular orbits with respect to the corresponding G-invariant metric g (see [16]). Since a curve which is orthogonal to the orbits at each point is, up to reparametrization, a geodesic, $\int_0^{\bar{t}}\sigma(t)\,dt$ is the radius of the sphere with center in the origin and radius $\bar{t}$ in the standard Euclidean metric g_0. We want to prove that the metric g is complete if and only if a normal geodesic has infinite length. This condition is necessary, in fact otherwise, since $\mu(t) \geqslant \sigma(t)$, if we fix a straight line through the origin, we have:

$$\int_0^{+\infty} \mu(t)\, dt = c < +\infty$$

that is a half line has finite length. In this case the metric cannot be complete.

This condition is also sufficient: by Hopf-Rinov theorem, it is sufficient to prove that any geodesic ball $B(0, r)$, centered in the origin $0 \in V$ is compact. Let x_n be a sequence in $B(0, r)$. Since g is G-invariant and G acts transitively on the unit sphere of V, the hypothesis guarantees that $B(0, r)$ is a ball of finite radius also with respect to the Euclidean metric. Hence there exists a subsequence y_n of x_n which converges to a point $y \in \overline{B(0, r)}$ with respect to the Euclidean metric. The metric g induces a symmetric bilinear form in $T_y V$. Denote by c_1 and c_2 respectively the smallest and the largest eigenvalues of this form and by $B_0(y, \bar{r})$, the ball centered in y with radius $\bar{r}$ with respect to the Euclidean metric. Then for any $\varepsilon > 0$, there exist a neighbourhood of y such that, if $\bar{r}$ is sufficiently small, $B(y, (c_1 + \varepsilon)\bar{r}) \subset B_0(y, \bar{r}) \subset B(y, (c_2 + \varepsilon)\bar{r})$. This implies that the subsequence y_n converges also with respect to the metric g.

3 - Curvature homogeneous metrics on V

Now we want to describe the metrics, among the ones described in Theorem 3, that are curvature homogeneous. By Lemma 2 it is sufficient to describe curvature homogeneous metrics on $V \backslash \{0\}$. According to the expressions of the G-invariant metrics in Theorem 3, we give a case by case proof of our result.

1. $G = SO(n)$, $G = \mathrm{Spin}(7)$, $G = G_2$.

Let $e \in S^{n-1}$, then any G-invariant metric on V is of the form:

$$g_{te} = \mu^2(t)\, dt^2 + \eta^2(t)\, g_{0|S^{n-1}},$$

where $\eta(t)$ and $\mu(t)$ are smooth, even and positive functions of t with $\mu(0) = \eta(0)$, and g_0 is the standard euclidean metric. Let $N = \dfrac{1}{\mu(t)} \dfrac{\partial}{\partial t}$ and let $\{Y_\alpha, N\}$, $(\alpha = 1, \ldots, n-1)$, be an orthonormal and parallel frame along an integral curve of $\frac{\partial}{\partial t}$ (N is parallel because, up to reparametrization, in this case, the integral curves of $\frac{\partial}{\partial t}$ are geodesics [16]). Let $f(t) = t\eta(t)$, $h(t) = \mu(t)$. We can compute the following sectional curvature (see [16]):

$$R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = \frac{1}{f^2} - \frac{f'^2}{f^2 h^2} \qquad R_{Y_\alpha, N, Y_\alpha, N} = \frac{-f''h + f'h'}{fh^3} .$$

Using formulas for the curvature tensor of a cohomogeneity one manifold (see, for example, [16]), one can prove that, for $X, Y, Z \in \{Y_\alpha, N\}$, $R_{X,Y,X,Z} = 0$, if $Z \neq Y$ (we omit the proof here). Hence equations (2.2) reduce to:

$$\frac{\partial}{\partial t}(R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta}) = 2a_\alpha^\alpha(t)\, R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} + 2a_\beta^\beta(t)\, R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = 0$$

$$\frac{\partial}{\partial t}(R_{Y_\alpha, N, Y_\alpha, N}) = 2a_\alpha^\alpha(t)\, R_{Y_\alpha, N, Y_\alpha, N} + 2a_n^n(t)\, R_{Y_\alpha, N, Y_\alpha, N} = 0$$

since $a_\alpha^\beta(t)$ are anti-symmetric. Hence the components of the curvature tensor must be constant.

A straightforward computation shows that

$$\lim_{t\to 0} R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = \lim_{t\to 0} R_{Y_\alpha, N, Y_\alpha, N} = \frac{\mu''(0) - 3\eta''(0)}{\mu(0)^3} = a .$$

Hence $R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = R_{Y_\alpha, N, Y_\alpha, N} = a$, and η must be one of the following functions:

1. $\eta(t) = \dfrac{\sin(\sqrt{a}H(t))}{\sqrt{a}t}$ $\qquad a > 0$
2. $\eta(t) = \dfrac{H(t)}{t}$ $\qquad a = 0$
3. $\eta(t) = \dfrac{\sinh(\sqrt{-a}H(t))}{\sqrt{-a}t}$ $\qquad a < 0$

where $H(t)$ is the primitive of $h(t)$ such that $H(0) = 0$.

Note that, up to a reparametrization of a normal geodesic (which is a G-diffeomorphism of V) we can suppose $h(t) = 1$ (hence $H(t) = t$) and $a \in \{-1, 0, 1\}$. In this case, the metrics have the form:

$$1. \quad g = dt^2 + \frac{\sin^2(t)}{t^2} g_{0|S^{n-1}} \qquad a = 1$$

$$2. \quad g = g_0 \qquad a = 0$$

$$3. \quad g = g_1 = dt^2 + \frac{\sinh(t^2)}{t^2} g_{0|S^{n-1}} \qquad a = -1$$

The metric 1 is not defined globally on all V, since $\eta(t)$ must be positive for all t (an open neighbourhood of the origin in V with this metric can be compactified to obtain the real projective space with the standard G-invariant metric). The metrics g_0 and g_1 are homogeneous. This is trivial for g_0, while for the metric g_1 this follows from the fact that g is a cohomogeneity one homogeneous metric on V, then (V, g) is a symmetric space of rank one (we omit the proof of this fact). Then (V, g_1) turns out to be isometric to the real hyperbolic space $\boldsymbol{H}^n(\boldsymbol{R})$ with the unique G-invariant Einstein metric (we omit the proof of these facts).

2. $n = 2m$, $G = U(m)$, $G = SU(m)$.

Let $e \in S^{n-1}$, then any G-invariant metric on V is of the form:

$$g_{te} = \mu^2(t)\, dt^2 + \lambda^2(t)\, \theta^2 + a(t)\, dt\, \theta + \eta^2(t)\, g_{0|W_2}\,.$$

$\lambda(t)$, $\eta(t)$, $\mu(t)$ must be smooth, even and positive functions of t, with the property $\lambda(0) = \eta(0) = \mu(0)$, and $a(t)$ must be smooth and even, with $a(0) = 0$ and $a(t)^2 < \mu(t)^2 \lambda(t)^2$.

Let $\{Y_a, Y_{ai}, W, N\}$, with $i = 1, \ldots, m-1$ be an orthonormal basis in te, with:

$$Y_{ai} = J(Y_a) \qquad W = J\Big(\frac{\partial}{\partial t}\Big) \qquad N = \frac{1}{\sqrt{\mu(t)^2 - \dfrac{a(t)^2}{\lambda(t)^2}}}\Big(\frac{\partial}{\partial t} - \frac{a(t)}{\lambda(t)^2} J\Big(\frac{\partial}{\partial t}\Big)\Big).$$

Let $g(t) = t\eta(t)$, $f(t) = \lambda(t)$, $h(t) = \sqrt{\mu(t)^2 - \dfrac{a(t)^2}{\lambda(t)^2}}$. We can compute the following sectional curvatures (see [**16**]):

$$R_{Y_a, Y_\beta, Y_a, Y_\beta} = \frac{1}{g^2} - \frac{g'^2}{g^2 h^2} \qquad R_{Y_a, Y_{ai}, Y_a, Y_{ai}} = \frac{4g^2 - 3f^2}{g^4} - \frac{g'^2}{g^2 h^2}$$

$$R_{Y_{ai}, Y_{\beta i}, Y_{ai}, Y_{\beta i}} = R_{Y_a, Y_\beta, Y_a, Y_\beta} = R_{Y_a, Y_{\beta i}, Y_a, Y_{\beta i}} \qquad R_{W, N, W, N} = \frac{-f' h + fh'}{fh^3}$$

$$R_{W, Y_a, W, Y_a} = R_{W, Y_{ai}, W, Y_{ai}} = \frac{f^2}{g^4} - \frac{f' g'}{fgh^2}$$

$$R_{Y_a, N, Y_a, N} = R_{Y_{ai}, N, Y_{ai}, N} = \frac{-g' h + gh'}{gh^3}$$

A straightfoward computation shows that:

$$\lim_{t\to 0} R_{Y_a, Y_\beta, Y_a, Y_\beta} = \lim_{t\to 0} R_{Y_a, W, Y_a, W} = \lim_{t\to 0} R_{Y_a, N, Y_a, N} = \frac{\mu''(0) - 3\eta''(0)}{\mu(0)^3} = a$$

$$\lim_{t\to 0} R_{Y_a, Y_{ai}, Y_a, Y_{ai}} = \lim_{t\to 0} R_{N, W, N, W} = \frac{\mu''(0) - 3\lambda''(0)}{\mu(0)^3} = b\,.$$

As in case **1.**, if $X,\ Y,\ Z \in \{Y_a,\ Y_{ai},\ W,\ N\}$, then $R_{X, Y, X, Z} = 0$, if $Z \neq Y$, and (2. 2) implies that the sectional curvatures must be constant (we omit the proof, which is similar to the one in **1.**) Hence we have to solve:

$$R_{Y_a, Y_\beta, Y_a, Y_\beta} = R_{Y_a, W, Y_a, W} = R_{Y_a, N, Y_a, N} = a$$

$$R_{Y_a, Y_{ai}, Y_a, Y_{ai}} = R_{N, W, N, W} = b\,.$$

From $R_{Y_a, Y_\beta, Y_a, Y_\beta} = a$, we obtain $h = \dfrac{|g'|}{\sqrt{1 - ag^2}}$. $R_{Y_a, Y_{ai}, Y_a, Y_{ai}} = b$ implies then $f = g\sqrt{1 + \dfrac{a-b}{3g^2}}$.

Substituting in $R_{Y_a, W, Y_a, W} = a$, we obtain a relation which is satisfied if g is constant or $a = b$ or $b = 4a$, but since $g(0) = 0$, g cannot be constant. Hence, if $b = 4a$, the solutions are given by:

1. $\eta(t) = \dfrac{\sin(\sqrt{a}H(t))}{\sqrt{a}t} \qquad \lambda(t) = \dfrac{\sin(2\sqrt{a}H(t))}{\sqrt{a}t} \qquad a > 0$

2. $\eta(t) = \lambda = \dfrac{H(t)}{t} \qquad a = 0$

3. $\eta(t) = \dfrac{\sinh(\sqrt{-a}H(t))}{\sqrt{-a}t} \qquad \lambda(t) = \dfrac{\sinh(2\sqrt{-a}H(t))}{\sqrt{-a}t} \qquad a < 0\,.$

where $H(t)$ is the primitive of $h(t)$ such that $H(0)=0$, and $a=\dfrac{\mu''(0)-3\eta''(0)}{\mu(0)^3}$. Changing the parametrization of the normal geodesic, we can suppose that $h(t)=1$ (hence $H(t)=t$) and $a\in\{-1,0,1\}$, and, up to G-diffeomorphisms, that $a(t)=0$. Then the corresponding metrics have the form:

$$1.\quad g=dt^2+\frac{\sin^2(t)}{t^2}g_{0|W_2}+\frac{\sin^2(2t)}{t^2}\theta^2 \qquad a=1$$

$$2.\quad g=g_0 \qquad a=0$$

$$3.\quad g=g_2=dt^2+\frac{\sinh^2(t)}{t^2}g_{0|W_2}+\frac{\sinh^2(2t)}{t^2}, \qquad a=-1$$

The metric 1 is not globally defined on V, in fact $\eta(t)$ must be positive. The metric g_0 is the standard Euclidean metric, hence it is homogeneous. The metric g_2 admits a smooth extension in all V, in fact the functions:

$$\eta(t)=\frac{\sinh(t)}{t} \qquad \lambda(t)=\frac{\sinh(2t)}{t} \qquad \mu(t)=1$$

are smooth, even and positive functions of t in $V\backslash\{0\}$, with the property $\eta(0)=\lambda(0)=\mu(0)=1$. This metric is complete, in fact $\mu(t)$ satisfies the hypothesis of Proposition 1. The metric g_2 is also homogeneous. As in case 1 one can see that (V, g_2) is isometric to the complex hyperbolic space $\boldsymbol{H}^n(\boldsymbol{C})$ with the standard G-invariant metric.

If $a=b$, the curvature is constant with respect to the basis $\{Y_a, Y_{ai}, W, N\}$. Up to G-diffeomorphism we can suppose $a(t)=0$ and $\mu(t)=1$. Then g is one of the homogeneous $SO(n)$-invariant metrics described in **1.**

3. $n=4m,\quad G=\mathrm{Sp}(m)$.

Let $e\in S^{n-1}$. Any G-invariant metric on V is of the form

$$g_{te}=\mu^2(t)\,dt^2+\sum_{i=1}^{3}a_i(t)\,dt\,\theta_i+\sum_{i\neq j=1}^{3}b_{ij}(t)\,\theta_i\theta_j+\sum_{i=1}^{3}\lambda_i^2(t)\,\theta_i^2+\eta^2(t)\,g_{S|W_4}$$

where $\mu(t)$, $\lambda_i(t)$, $\eta(t)$ are smooth, even and positive functions of t such that $\mu(0)=\lambda_i(0)=\eta(0)$. $a_i(t)$, $b_{ij}(t)=b_{ji}(t)$ are smooth and even functions of t with,

$a_i(0) = b_{ij}(0) = 0$, for $i < j = 1, 2, 3$. Moreover the matrix

$$\begin{pmatrix} \lambda_1^2 & b_{12} & b_{13} & a_1 \\ b_{12} & \lambda_2^2 & b_{23} & a_2 \\ b_{13} & b_{23} & \lambda_3^2 & a_3 \\ a_1 & a_2 & a_3 & \mu^2 \end{pmatrix}$$

must be positive definite.

In [17] the author shows that, using simultaneous left and right multiplication by elements of $\mathrm{Sp}(1) \subset \mathrm{Sp}(m)$, for each $\bar{t} > 0$, there exists a diffeomorphism of the sphere $S_{\bar{t}}^{n-1}$ of radius $\bar{t}$, which preserves the decomposition $W_0 + W_4$ of the tangent space, and such that, for the induced metric, $b_{ij}(\bar{t}) = 0$. Using a curve in $\mathrm{Sp}(1)$, we can then build a global diffeomorphism $V \to V$, such that $b_{ij}(t) = 0$ for all $t > 0$. Then we only have to study metrics of the form

$$g_{te} = \mu^2(t)\, dt^2 + \sum_{i=1}^{3} a_i(t)\, dt\, \theta_i + \sum_{i=1}^{3} \lambda_i^2(t)\, \theta_i^2 + \eta^2(t)\, g_{S|W_4}.$$

Let $\{Y_\alpha, Y_{\alpha 1}, Y_{\alpha 2}, Y_{\alpha 3}, W_1, W_2, W_3, N\}$, with $\alpha = 1, \ldots, m-1$ be an orthonormal basis in te, with

$$Y_{\alpha i} = J_i(Y_\alpha) \qquad N = \frac{1}{\left(\mu^2 - \sum_{i=1}^{3} \frac{a_i^2}{\lambda_i^2}\right)} \left(\frac{\partial}{\partial t} - \sum_{i=1}^{3} \frac{a_i}{\lambda_i^2} J_i\left(\frac{\partial}{\partial t}\right)\right) \qquad W_i = J_i\left(\frac{\partial}{\partial t}\right)$$

and $i = 1, 2, 3$.

Let $g(t) = t\eta(t)$, $f_i(t) = t\lambda_i(t)$, $h(t) = \left(\mu^2 - \sum_{i=1}^{3} \frac{a_i^2}{\lambda_i^2}\right)$. Denote by (i, j, k) a cyclic permutation of $(1, 2, 3)$. We can compute the following sectional curvatures (see [16]):

$$R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = \frac{1}{g^2} - \frac{g'^2}{g^2 h^2} \qquad R_{Y_\alpha, Y_{\alpha i}, Y_\alpha, Y_{\alpha i}} = \frac{4g^2 - 3f_i^2}{g^4} - \frac{g'^2}{g^2 h^2}$$

$$R_{Y_{\alpha i}, Y_{\beta i}, Y_{\alpha i}, Y_{\beta i}} = R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = R_{Y_\alpha, Y_{\beta i}, Y_\alpha, Y_{\beta i}}$$

$$R_{Y_{\alpha i}, Y_{\alpha j}, Y_{\alpha i}, Y_{\alpha j}} = \frac{4g^2 - 3f_k^2}{g^4} - \frac{g'^2}{g^2 h^2}$$

$$R_{W_i, Y_\alpha, W_i, Y_\alpha} = R_{W_i, Y_{\alpha i}, W_i, Y_{\alpha i}} = \frac{f_i^2}{g^4} - \frac{g' f_i'}{g f_i h^2}$$

$$R_{W_i, Y_{aj}, W_i, Y_{aj}} = R_{W_i, Y_a, W_i, Y_a} = R_{W_i, Y_{ak}, W_i, Y_{ak}}$$

$$R_{W_i, W_j, W_i, W_j} = \frac{2}{f_i^2} + \frac{2}{f_j^2} - \frac{3f_k^2}{f_i^2 f_j^2} + \left(\frac{f_i}{f_j f_k} - \frac{f_j}{f_i f_k}\right)^2 - \frac{f_i' f_j'}{f_i f_j h^2} + \frac{a_k^2 (f_i^2 - f_j^2)^2}{f_i^2 f_j^2 f_k^4 h^2}$$

$$R_{Y_a, N, Y_a, N} = R_{Y_{ai}, N, Y_{ai}, N} = \frac{-g''h + g'h'}{gh^3}$$

$$R_{W_i, N, W_i, N} = \frac{-f_i''h + f_i'h'}{f_i h^3} - \frac{(f_j^2 - f_i^2)(f_i^2 + 3f_j^2)\, a_k^2}{f_i^2 f_j^2 f_k^4 h^2} - \frac{(f_k^2 - f_i^2)(f_i^2 + 3f_k^2) a_j^2}{f_i^2 f_j^4 f_k^2 h^2}\,.$$

A straightforward computation shows that:

$$\lim_{t\to 0} R_{Y_a, Y_\beta, Y_a, Y_\beta} = \lim_{t\to 0} R_{Y_a, W_i, Y_a, W_i} = \lim_{t\to 0} R_{Y_a, N, Y_a, N} = \frac{\mu''(0) - 3\eta''(0)}{\mu(0)^3} = a$$

$$\lim_{t\to 0} R_{Y_a, Y_{ai}, Y_a, Y_{ai}} = \lim_{t\to 0} R_{W_i, N, W_i, N} = \frac{\mu''(0) - 3\lambda_i''(0)}{\mu(0)^3} = b_i$$

$$\lim_{t\to 0} R_{Y_{ai}, Y_{aj}, Y_{ai}, Y_{aj}} = \lim_{t\to 0} R_{W_i, W_j, W_i, W_j} = \frac{\mu''(0) - 3\lambda_k''(0)}{\mu(0)^3} = b_k\,.$$

Equations (2.2) imply then (the calculations are similar to the ones in **1** and **2** and we omit them):

$$R_{Y_a, Y_\beta, Y_a, Y_\beta} = a \qquad R_{Y_a, Y_{ai}, Y_a, Y_{ai}} = b_i\,.$$

Hence: $h = \dfrac{|g'|}{\sqrt{1 - ag^2}}$ and $f_i = g\sqrt{1 + \dfrac{a - b_i}{3g^2}}$.

The other sectional curvatures are not necessarily constant, but (2.2) reduces to:

$$\frac{\partial}{\partial t}(R_{W_i, Y_a, W_i, Y_a}) = 2a_j^i R_{W_j, Y_a, W_i, Y_a} + 2a_k^i R_{W_k, Y_a, W_i, Y_a}$$

$$\frac{\partial}{\partial t}(R_{W_j, Y_a, W_j, Y_a}) = 2a_i^j R_{W_i, Y_a, W_j, Y_a} + 2a_k^j R_{W_k, Y_a, W_j, Y_a}$$

$$\frac{\partial}{\partial t}(R_{W_k, Y_a, W_k, Y_a}) = 2a_i^k R_{W_i, Y_a, W_k, Y_a} + 2a_j^k R_{W_j, Y_a, W_k, Y_a}.$$

Since a_j^i are antisymmetric:

$$\frac{\partial}{\partial t}(R_{W_i, Y_a, W_i, Y_a} + R_{W_j, Y_a, W_j, Y_a} + R_{W_k, Y_a, W_k, Y_a}) = 0\,,$$

that is: $$\frac{2a+b_i}{3+(a-b_i)g^2} + \frac{2a+b_j}{3+(a-b_j)g^2} + \frac{2a+b_k}{3+(a-b_k)g^2} = 0\,.$$

If $g \neq 0$ this is possible if and only if $b_i = a$ or $b_i = 4a$.

If $b_i = b_j = b_k = a$ the corresponding metric has constant curvature and, since, up to G-diffeomorphisms, we can suppose $a_i(t) = 0$, is one of the homogeneous metrics described in **1**.

If $b_i = b_j = a$, $b_k = 4a$, equation (2.2) for R_{W_i, W_j, W_i, W_j}, implies, with a straightforward computation, that the corresponding metric is curvature homogeneous only if $a_i(t) = a_j(t) = 0$ hence the solution is the metric g_2 described in **2**.

If $b_i = a$, $b_j = b_k = 4a$, equation (2.2) together with the fact that $g \in O(t^2)$ and $a_i(t) \in o(t^3)$, implies that there are no curvature homogeneous metrics (we omit the proof here).

If $b_i = b_j = b_k = 4a$ the components of the curvature tensor are constant, hence the metric is curvature homogeneous. As in **2**, if $a > 0$, the metric we obtain is not globally defined on V; if $a = 0$, the metric is homogeneous and $SO(n)$-invariant. If $a < 0$, up to G-equivariant diffeomorphisms, the solution is given by the metric

$$g = g_3 = dt^2 + \frac{\sinh^2(t)}{t^2} g_{0|W_4} + \sum_{i=1}^{3} \frac{\sinh^2(2t)}{t^2} \theta_i^2\,.$$

This metric admits a smooth extension to a complete metric on V. This metric is Einstein and (V, g_3) is isometric to the quaternionic hyperbolic space $\boldsymbol{H}^n(\boldsymbol{H})$ with the standard G-invariant metric.

4. $n = 4m$, $G = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$.

Any G-invariant metric is also $\mathrm{Sp}(m)$-invariant, and we obtain the curvature homogeneous metric g_3, corresponding to $b_i = b_j = b_k = 4a$, and the homogeneous $SO(n)$-invariant metrics g_0 and g_1.

5. $n = 4m$, $G = \mathrm{Sp}(m) \cdot U(1)$.

Any G-invariant metric is also $\mathrm{Sp}(m)$-invariant, and we obtain the curvature homogeneous metrics g_2 and g_3, and the homogeneous $SO(n)$-invariant metrics g_0 and g_1.

6. $G = \mathrm{Spin}(9)$.

Let $e \in S^{n-1}$, then any Spin(9)-invariant metric on $\boldsymbol{R}^{16}$ is of the form

$$g_{te} = \mu^2(t)\, dt^2 + \lambda_1^2(t)\, g_{0|W_8} + \lambda_2^2(t)\, g_{0|W_9}\,.$$

The metric is smooth if and only if $\mu(t)$, $\lambda_1(t)$, $\lambda_2(t)$ are smooth even and positive functions of t, with $\mu(0) = \lambda_1(0) = \lambda_2(0)$. Let $\{Y_i, N\}$ form an orthonormal basis in te, with $N = \dfrac{\partial}{\partial t}$ and $Y_i \in W_8$ for $i \leqslant 8$. Let $h(t) = \mu(t)$, $f(t) = t\lambda_1(t)$ and $g(t) = t\lambda_2(t)$. We can compute the following sectional curvatures (see [16]):

$$R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = \frac{4\lambda_1^2 - 3\lambda_2^2}{\lambda_1^4 t^2} - \frac{(\lambda_1' t + \lambda_1)^2}{\lambda_1^2 t^2 \mu^2} \qquad (\alpha \leqslant 8, \beta \leqslant 8)$$

$$R_{Y_\alpha, Y_p, Y_\alpha, Y_p} = \frac{\lambda_2^2}{\lambda_1^4 t^2} - \frac{(\lambda_1' t + \lambda_1)(\lambda_2' t + \lambda_2)}{\lambda_1 \lambda_2 \mu^2 t^2} \qquad (\alpha \leqslant 8, p \geqslant 9)$$

$$R_{Y_p, Y_q, Y_p, Y_q} = \frac{1}{\lambda_2^2 t^2} - \frac{(\lambda_2' t + \lambda_2)^2}{\lambda_2^2 t^2 \mu^2} \qquad (p \geqslant 9, q \geqslant 9)$$

$$R_{Y_\alpha, N, Y_\alpha, N} = \frac{(-\lambda_1'' t - 2\lambda_1')\,\mu + (\lambda_1' t + \lambda_1)\,\mu'}{\lambda_1 t \mu^3} \qquad (\alpha \leqslant 8)$$

$$R_{Y_p, N, Y_p, N} = \frac{(-\lambda_2'' t - 2\lambda_2')\,\mu + (\lambda_2' t + \lambda_2)\,\mu'}{\lambda_2 t \mu^3} \qquad (p \geqslant 9)$$

A straightforward computation shows that, for $\alpha, \beta \leqslant 8$, $p, q \geqslant 9$:

$$\lim_{t \to 0} R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta} = \lim_{t \to 0} R_{Y_p, Y_q, Y_p, Y_q} = \lim_{t \to 0} R_{Y_p, N, Y_p, N} = \frac{\mu''(0) - 3\lambda_2''(0)}{\mu(0)^3} = a$$

$$\lim_{t \to 0} R_{Y_\alpha, Y_p, Y_\beta, Y_p} = \lim_{t \to 0} R_{Y_\alpha, N, Y_\alpha, N} = \frac{\mu''(0) - 3\lambda_1''(0)}{\mu(0)^3} = b$$

As in the previous cases, (2.2) implies that the metric g is curvature homogeneous if and only if the components of the curvature tensor are constant. This condition, applied to R_{Y_p, Y_q, Y_p, Y_q} and $R_{Y_\alpha, N, Y_\alpha, N}$, implies

$$h = \frac{g'}{\sqrt{1 - ag^2}} = \frac{f'}{\sqrt{1 - bf^2}} .$$

From the expression of $R_{Y_\alpha, Y_\beta, Y_\alpha, Y_\beta}$, we obtain that the curvature is constant if and only if $f = g$ or $a = -2$, $b = -\frac{1}{2}$. In the first case we obtain the $SO(n)$-invariant metrics of **1**, in the second:

$$f = \sqrt{2}\sinh\left(\frac{\sqrt{2}}{2} H(t)\right) \qquad g = \frac{\sinh(\sqrt{2}H(t))}{\sqrt{2}}$$

where $H(t)$ is the primitive of $h(t)$ such that $H(0) = 0$. As in 2 we can assume that $h(t) = 1$. The corresponding metric admits a smooth extension to a complete metric on V. This metric is homogeneous and (V, g_4) is isometric to the Cayley hyperbolic plane with the standard G-invariant metric.

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Sommario

Sia V uno spazio vettoriale euclideo e G un sottogruppo di Lie di $SO(V)$ che operi transitivamente sulla sfera unitaria di V. Oggetto di questo articolo è la classificazione delle metriche riemanniane G-invarianti g su V che siano a curvatura omogenea, cioè tali che per ogni coppia di punti di V esista una isometria lineare fra i corrispondenti piani tangenti che preservi il tensore di curvatura di (V, g). Si prova che ogni metrica con queste proprietà è omogenea e, localmente, (V, g) è isometrico ad uno spazio simmetrico di rango 1.

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