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R. A. ABD EL-BAKY (*)

The relation among Darboux vectors of ruled surfaces in a line congruence (**)

1 - Introduction

Let $R = R(u_1, u_2)$ be a line congruence in the Euclidean space E^3. We consider a ruled surface $R_1 = R(u_1(t), u_2(t))$ and the parameter ruled surfaces $R_{11} = R(u_1(t), c_2)$, $R_{21} = R(c_1, u_2(t))$, where c_1, c_2 are appropriate real constants.

Choosing as parameter ruled surfaces the principal ruled surfaces, we obtain a relation among the dual Darboux vectors (dual instantaneous axes) of the ruled surfaces R_1, R_{11}, R_{21} (Theorem 2, Sec. 5).

Using this relation, a formula, corresponding in line geometry to the classical J. Liouville's formula, is obtained and other formulas are proved.

Furthermore, new relations between the dual angles of pitches of the closed ruled surfaces generated by Darboux and Blaschke trihedrons are introduced.

2 - Preliminaries

The set of *dual numbers*, defined by

$$\boldsymbol{D} = \{a + \varepsilon a^*;\ a,\ a^* \in \boldsymbol{R},\ \varepsilon \neq 0 \text{ and } \varepsilon^2 = 0\}$$

is a commutative ring. The set

$$\boldsymbol{D}^3 = \{X = x + \varepsilon x^*;\ x,\ x^* \in \boldsymbol{R}^3,\ \varepsilon \neq 0 \text{ and } \varepsilon^2 = 0\}$$

(*) Dept. of Math., Fac. of Science, Univ. of Assiut, 71516 Assiut, Egypt.
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is a $\boldsymbol{D}$-module, called *dual space.* The elements of $\boldsymbol{D}^3$ are said *dual vectors* [5].

Definition 1. The *inner product* of the dual vectors X and Y is defined by

$$\langle X, Y\rangle = \langle x, y\rangle + \varepsilon(\langle x^*, y\rangle + \langle x, y^*\rangle).$$

and the *norm* of X is defined by

$$\|X\| = \|x\| + \varepsilon \frac{\langle x, x^*\rangle}{\|x\|} \qquad x \neq 0 .$$

Definition 2. The *dual unit sphere* Σ of $\boldsymbol{D}^3$ is defined by

$$\Sigma = \{X = x + \varepsilon x^*;\ \|X\| = 1,\ x,\ x^* \in \boldsymbol{R}^3,\ \varepsilon \neq 0,\ \varepsilon^2 = 0\}.$$

Theorem 1. (E. Study) [5]. *There exists a one-to-one correspondence between the set of points on the dual unit sphere in the* $\boldsymbol{D}$*-module* $\boldsymbol{D}^3$ *and the set of oriented lines in the Euclidean space* E^3.

According to this theorem, a dual unit vector $R = r + \varepsilon r^*$ corresponds to one and only to one oriented line of E^3. The real vector r gives the *direction* of this line and the real vector r^* is the *vectorial moment* of the unit vector r with respect to the origin point 0, i.e. $r^* = 0p \times r$, where p is a point on the r-oriented line and $\times$ denotes the cross product in $\boldsymbol{R}^3$.

Definition 3. The *dual angle* between the dual unit vectors X and Y is given by

$$\langle Y, X\rangle = \cos \Phi$$

where $\Phi = \varphi + \varepsilon\varphi^*$ $(0 \leq \varphi \leq \pi,\ \varphi^* \in \boldsymbol{R})$ is a dual number. Here φ and φ^* are the *angle* and the *minimal distance* between the two lines represented by X and Y, respectively.

3 - Blaschke derivatives formulas

According to E. Study's map, the dual unit vector

$$R = R(u_1, u_2) \tag{3.1}$$

depending on two real parameters u_i $(i = 1, 2)$ (the end points of R fill a domain on the dual unit sphere Σ) represents a *line congruence* in E^3, i.e. a two parameters differentiable family of oriented lines in E^3.

A *ruled surface* and the *parameter ruled surfaces* of the congruence (3.1) are given by the dual unit vectors $R_1 = R(u_1(t), u_2(t))$ and $R_{11} = R(u_1(t), c_2)$, $R_{21} = R_{21}(c_1, u_2(t))$, respectively. Therefore, the terms *dual curve* (dual unit vector depending on one real parameter) and *ruled surface* are synonymous in this paper.

The relation

$$\langle R_{u_1}, R_{u_2} \rangle = F = f + \varepsilon f^* = 0 \tag{3.2}$$

is satisfied when principal ruled surfaces in the line congruence (3.1) are taken as parameter ruled surfaces. We make this assumption in the present paper.

Blaschke trihedrons of these ruled surfaces of the congruence R, passing through the common line $R_0 = R_0\ (c_1, c_2)$, with $c_i = u_i(t_0)$, are [1]:

$$\{R_1, R_2, R_3\} \quad \text{and} \quad \{R_{i1}, R_{i2}, R_{i3}\}$$

where $R_1 = R_{11} = R_{21} = R_0$, $R_2 = \left(\dfrac{dR_1}{dt} \left| \dfrac{dR_1}{dt} \right|^{-1} \right)_{t = t_0}$, $R_3 = R_1 \times R_2$ and similarly for R_{i2} and R_{i3}.

Blaschke vectors of these trihedrons are [1]:

$$B = QR_1 + PR_3 \quad \text{and} \quad B_i = Q_i R_{i1} + P_i R_{i3}$$

where P, Q, P_i, Q_i are the magnitudes of R_1 and R_{i1}, respectively.

The *dual arc elements* of these ruled surfaces are:

$$dS = P\,dt \qquad dS_i = P_i\,du_i \qquad (i = 1, 2) \tag{3.3}$$

where $P = \|R'^2\|$, $P_1 = \|R^2_{u_1}\| = \sqrt{E}$, $P_2 = \|R^2_{u_2}\| = \sqrt{G}$.

From relation (3.2) we may write

$$\langle R_{12}, R_{22} \rangle = 0 \quad \text{and} \quad R_0 = R_{12} \times R_{22} = \frac{R_{u_1} \times R_{u_2}}{\sqrt{EG}}\,.$$

Let $\Psi = \psi + \varepsilon \psi^*$ be the dual angle between the edges R_2 and R_{12}, then we have

$$R_2 = R_{12} \cos \Psi + R_{22} \sin \Psi \tag{3.4}$$

and

$$\cos \Psi = \frac{dS_1}{dS} = \sqrt{E}\frac{du_1}{dS} \qquad \sin \Psi = \frac{dS_2}{dS} = \sqrt{G}\frac{du_2}{dS}. \tag{3.5}$$

4 - The dual Darboux trihedrons

From [4], for the ruled surface $R_1 = R(u_1(t), u_2(t))$ in the line congruence (3.1), the *dual Darboux trihedron* $\{R_1, N, G\}$ connected with Blaschke trihedron by the dual transformation:

$$\begin{bmatrix} R_1 \\ N \\ G \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta & -\sin \Theta \\ 0 & \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

where $\Theta = \theta + \varepsilon\theta^*$ is the dual angle between the edges N and R_2, and the derivative formula of $\{R_1, N, G = R_1 \times N\}$ is

$$\frac{d}{dt}\begin{bmatrix} R_1 \\ N \\ G \end{bmatrix} = \begin{bmatrix} 0 & P_n & -P_g \\ -P_n & 0 & Q_g \\ P_g & -Q_g & 0 \end{bmatrix} \begin{bmatrix} R_1 \\ N \\ G \end{bmatrix}. \tag{4.1}$$

Here $P_g = -P \sin \Theta$, $P_n = P \cos \Theta$ and $Q_g = Q - \dfrac{d\Theta}{dt}$ are named the *dual geodesic curvature*, the *dual normal curvature* and the *dual geodesic torsion*, respectively. So, the *dual Darboux's vector* of this trihedron is

$$D = Q_g R_1 + P_g N + P_n G \tag{4.2}$$

which satisfies the relations:

$$R_1' = D \times R_1 \qquad N' = D \times N \qquad G' = D \times G. \tag{4.3}$$

In accordance with last equations and based on [4] we can define the dual Darboux trihedrons and their derivatives, for the ruled surfaces R_{i1}, in the form:

$$\begin{bmatrix} R_{i1} \\ N_i \\ G_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta_i & -\sin \Theta_i \\ 0 & \sin \Theta_i & \cos \Theta_i \end{bmatrix} \begin{bmatrix} R_{i1} \\ R_{i2} \\ R_{i3} \end{bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}u_i}\begin{bmatrix} R_{i1} \\ N_i \\ G_i \end{bmatrix} = \begin{bmatrix} 0 & P_n^{(i)} & -P_g^{(i)} \\ -P_n^{(i)} & 0 & Q_g^{(i)} \\ P_g^{(i)} & -Q_g^{(i)} & 0 \end{bmatrix}\begin{bmatrix} R_{i1} \\ N_i \\ G_i \end{bmatrix}$$

where $\Theta_i = \theta_i + \varepsilon\theta_i^*$ is the dual angle between N_i and R_{i2}, $P_g^{(i)} = -P_i \sin\Theta_i$, $P_n^{(i)} = P_i \cos\Theta_i$, $Q_g^{(i)} = Q_i - \dfrac{\mathrm{d}\Theta_i}{\mathrm{d}u_i}$. The quantities $P_g^{(i)}$, $P_n^{(i)}$, $Q_g^{(i)}$ will be called as previously. The corresponding Darboux vectors are:

$$D_i = Q_g^{(i)} R_{i1} + P_g^{(i)} N_i + P_n^{(i)} G_i \qquad i = 1, 2. \tag{4.4}$$

From (3.4) and (1.2.7) of [3] we get

$$N = N_1 \cos\Psi + N_2 \sin\Psi. \tag{4.5}$$

Hence we may state the corollaries:

Corollary 1. *The edges of Darboux trihedrons of the parameter ruled surfaces R_{i1} coincide if and only if we have:*

$$G_1 = N_2 \qquad G_2 = -N_1 \tag{4.6}$$

Corollary 2. *The Darboux vectors* (4.2) *and* (4.4) *depend only on the vectors N_i and R_0.*

Corollary 3. *Under the assumption of Corollary* 1 *the edges of Darboux trihedrons* (4.6) *change as functions of both parameters u_i* ($i = 1, 2$).

Proof. Taking Corollary 1 into account, we have

$$\frac{\partial R_1}{\partial u_i} = D_i \times R_1 \qquad \frac{\partial R_{11}}{\partial u_i} = D_i \times R_{11} \qquad \frac{\partial R_{21}}{\partial u_i} = D_i \times R_{21}$$

$$\frac{\partial G_1}{\partial u_i} = \frac{\partial N_2}{\partial u_i} = D_i \times G_1 = D_1 \times N_2$$

$$\frac{\partial G_2}{\partial u_i} = -\frac{\partial N_1}{\partial u_i} = D_i \times G_2 = -D_i \times N_1.$$

Corollary 4. *For the ruled surfaces N_i we have:*

$$\langle N_1, \frac{\partial N_2}{\partial S_1} \rangle = \frac{Q_g^{(1)}}{\sqrt{E}} \qquad \langle N_2, \frac{\partial N_1}{\partial S_2} \rangle = \frac{Q_g^{(2)}}{\sqrt{G}} .$$

5 – The relation among Darboux vectors.

Theorem 2. *In the congruence $R(u_1, u_2)$, there exists a relation among Darboux vectors of the ruled surface R_1 and of the principal ruled surfaces R_{i1} $(i = 1, 2)$, passing through the common line R_0. More explicitly, we have*

$$D = P(\frac{\cos \Psi}{P_1} D_1 + \frac{\sin \Psi}{P_2} D_2 + \frac{d\Psi}{dS} R_0) . \tag{5.1}$$

Proof. From Corollary 3 and equation (3.3) we get:

$$\frac{dN_i}{dS} = \frac{\partial N_i}{\partial S_1}\frac{\partial S_1}{\partial S} + \frac{\partial N_i}{\partial S_2}\frac{\partial S_2}{\partial S} = \frac{1}{\sqrt{E}} D_1 \times N_i \cos \Psi + \frac{1}{\sqrt{G}} D_2 \times N_i \sin \Psi .$$

In other words we have

$$\frac{dN_i}{dS} = M \times N_i \tag{5.2}$$

where

$$M = \frac{D_1}{P_1} \cos \Psi + \frac{D_2}{P_2} \sin \Psi .$$

Moreover if we use (5.2), differentiating (4.5) we get

$$\frac{dN}{dS} = (M + R_0 \frac{d\Psi}{dS}) \times N \tag{5.3}$$

and from (4.3) we have

$$\frac{dN}{dS} = \frac{1}{P} D \times N \tag{5.4}$$

Hence, from (5.3) and (5.4), relation (5.1) is proved.

Now, we will investigate the consequences of relation (5.1). To this purpose we replace the values of D and D_i given by (4.2) and (4.4) in the relation. So, we have the followings equations:

$$\frac{Q_g}{P} = \frac{Q_g^{(1)}}{P_1} \cos \Psi + \frac{Q_g^{(2)}}{P_2} \sin \Psi + \frac{d\Psi}{dS}$$

(5.5) $$P_g \cos \Psi - P_n \sin \Psi = P(P_g^{(1)} \frac{\cos \Psi}{P_1} - P_n^{(2)} \frac{\sin \Psi}{P_2})$$

$$P_g \sin \Psi - P_n \cos \Psi = P(P_n^{(1)} \frac{\cos \Psi}{P_1} - P_g^{(2)} \frac{\sin \Psi}{P_2}) .$$

The first equation in (5.5) is the analog in line geometry of J. Liouville's formula in surfaces theory, and the last two equations reduce to:

$$\frac{P_g}{P} = (\frac{P_g^{(1)}}{P_1} \cos^2 \Psi + \frac{P_g^{(2)}}{P_2} \sin^2 \Psi) + (\frac{P_2 P_n^{(1)} - P_1 P_n^{(2)}}{2 P_1 P_2}) \sin 2\Psi$$

$$\frac{P_n}{P} = (\frac{P_n^{(1)}}{P_1} \cos^2 \Psi + \frac{P_n^{(2)}}{P_2} \sin^2 \Psi) + (\frac{P_1 P_g^{(1)} - P_2 P_g^{(2)}}{2 P_1 P_2}) \sin 2\Psi .$$

These equations give us the magnitudes of the ruled surface R_1 in terms of the magnitudes of the principal ruled surfaces R_{i1}.

6 - The dual angle of pitch

In a closed spatial euclidean motion the *pitch* L_r and the *angle of pitch* λ_r of a *closed ruled surface* r, generated by a fixed oriented line embedded in the moving euclidean line space, are useful integral invariants in the study of differential geometry of the closed ruled surfaces.

The *dual angle of pitch*, $\Lambda_r = \lambda_r - \varepsilon L_r$, of a closed ruled surface r has been defined in [6].

We intend now to find a relation among the dual angle of pitch of the ruled surfaces in the congruence (3.1). To this aim, let us taken in the dual plane (N, G) of the moving Darboux trihedron $\{R_1, N, G\}$, a dual unit vector

(6.1) $$U = N \cos \Phi + G \sin \Phi$$

such that during the closed motion when the edge $R_1 = R(u_1(t), u_2(t))$ generates the closed ruled surface R_1 the dual unit vector $U(t)$ generates a torus, along the orthogonal trajectory of the surface R_1.

The total change $\Lambda_{R_1} = \oint \mathrm{d}\Phi$ of $\Phi = \phi + \varepsilon\varphi^*$ is called the *dual angle of pitch* of the surface R_1.

Differentiating equations (6.1) and using (4.3) we have

$$\mathrm{d}U = (Q_g\,\mathrm{d}t - \mathrm{d}\Phi)\, R_1 \times U + (-P_n \cos \Phi + P_g \sin \Phi)\, \mathrm{d}t\, R_1 \,.$$

Condition for $U(t)$-torus along the orthogonal trajectory of surface R_1 implies $Q_g\,\mathrm{d}t = \mathrm{d}\Phi$. Integrating this equation and taking into account (5.5) we are able to express the dual angle of pitch as

$$\Lambda_{R_1} = \Lambda^{(1)}_{R_1} + \Lambda^{(2)}_{R_1} + \Psi \,. \tag{6.2}$$

Separating the real and dual part of (6.2), we get:

$$\lambda_{R_1} = \lambda^{(1)}_{R_1} + \lambda^{(2)}_{R_1} + \psi \qquad L_{R_1} = L^{(1)}_{R_1} + L^{(2)}_{R_1} + \psi^* \,.$$

Equation (6.2) give us the geometrical interpretation of J. Liouville's formula in theory of line congruences.

As an extra consequence of relation (5.1) we will obtain another result.

Let us define the dual vector W, given by the integration along the closed curve drawn by Darboux vector (4.2)

$$W = \oint (Q_g R_1 + P_g N + P_n G)\, \mathrm{d}t \tag{6.3}$$

which may be called, in analogy with the real case, *dual Steiner vector* of the trihedron $\{R_1, N, G\}$. The dual Steiner vectors W_i for the trihedrons $\{R_{i1}, N_i, G_i\}$ are:

$$W_i = \oint (Q^{(i)}_g R_{i1} + P^{(i)}_g N_i + P^{(i)}_n G_i)\, \mathrm{d}u_i \qquad i = 1, 2 \,. \tag{6.4}$$

Definition 4 [6]. Let $X(t) = x(t) + \varepsilon x^*(t)$, $\|X\| = 1$ represent an orientable closed ruled surface. The *dual angle of pitch* Λ_X is equal to the dual projection of $X(t)$ on the dual Steiner vector of the motion.

According to Definition (4), the dual Steiner's vectors, defined by equations (6.3), (6.4), rewritten as:

$$W = \Lambda_{R_1} R_1 + \Lambda_N N + \Lambda_G G \tag{6.5}$$

$$W_i = \Lambda_{R_{i1}} R_{i1} + \Lambda_{N_i} N_i + \Lambda_{G_i} G_i \tag{6.6}$$

where Λ_{R_1}, Λ_N, Λ_G and $\Lambda_{R_{i1}}$, Λ_{N_i}, Λ_{G_i} are the dual angles of pitch of the closed ruled surfaces, generated by the edges R_1, N, G and R_{i1}, N_i, G_i, $(i=1, 2)$ respectively.

Theorem 3. *In the congruence $R(u_1, u_2)$, there exists a relation among dual Steiner vectors of Darboux trihedrons $\{R_1, N, G\}$, $\{R_{i1}, N_i, G_i\}$, $(i=1, 2)$, for ruled surface R_1 and the principal ruled surfaces R_{i1}, passing through the common line R_0. More explicitly, we have*

$$W = W_1 + W_2 + \Psi R_0 \tag{6.7}$$

Proof. Taking into account equations (3.3) and (3.5), from relation (5.1) we derive

$$D\,\mathrm{d}t = D_1\,\mathrm{d}u_1 + D_2\,\mathrm{d}u_2 + R_0\,\mathrm{d}\Psi\,.$$

Integrating this relation along the closed curve drawn by the common line R_0 and using for W and W_i the expressions given by (6.5), (6.6), we obtain relation (6.7).

Using Corollary 1 and relation (6.7), we find again equation (6.2) and can easily establish the formulas:

$$\Lambda_N = \tfrac{1}{2}(\Lambda_{N_1} \cos \Psi + \Lambda_{N_2} \sin \Psi) \tag{6.8}$$

$$\Lambda_G = \tfrac{1}{2}(\Lambda_{G_1} \cos \Psi + \Lambda_{G_2} \sin \Psi) \tag{6.9}$$

Separating the real and dual parts in equations (6.8), (6.9), we are able to obtain the relations:

$$\begin{aligned} \lambda_N &= \tfrac{1}{2}(\lambda_{N_1} \cos \psi + \lambda_{N_2} \sin \psi) \\ &= \tfrac{1}{2}(-\lambda_{G_2} \cos \psi + \lambda_{G_1} \sin \psi) \end{aligned} \tag{6.10}$$

$$\begin{aligned} L_N &= \tfrac{1}{2}(L_{N_1} \cos \psi + L_{N_2} \sin \psi + \psi^*(\lambda_{N_1} \sin \psi + \lambda_{N_2} \cos \psi)) \\ &= \tfrac{1}{2}(-L_{G_2} \cos \psi + L_{G_1} \sin \psi - \psi^*(\lambda_{G_2} \sin \psi - \lambda_{G_1} \cos \psi)) \end{aligned} \tag{6.11}$$

$$\begin{aligned} \lambda_G &= \tfrac{1}{2}(\lambda_{G_1} \cos \psi + \lambda_{G_2} \sin \psi) \\ &= \tfrac{1}{2}(\lambda_{N_2} \cos \psi - \lambda_{N_1} \sin \psi) \end{aligned} \tag{6.12}$$

$$(6.13)\quad \begin{aligned} L_G &= \tfrac{1}{2}(L_{G_1}\cos\psi + L_{G_2}\sin\psi + \psi^*(\lambda_{G_1}\sin\psi + \lambda_{G_2}\cos\psi)) \\ &= \tfrac{1}{2}(L_{N_2}\cos\psi - L_{N_1}\sin\psi + \psi^*(\lambda_{N_2}\sin\psi - \lambda_{N_1}\cos\psi)) \end{aligned}$$

Relations (6.10) and (6.12) imply:

$$\lambda_{N_1} = -\lambda_{G_2} = 2(\lambda_N\cos\psi - \lambda_G\sin\psi)$$

$$\lambda_{N_2} = \quad \lambda_{G_1} = 2(\lambda_G\cos\psi + \lambda_N\sin\psi)$$

Similarly from (6.11) and (6.13) we get:

$$L_{N_1} = -L_{G_2} = 2(L_N\cos\psi - L_G\sin\psi) - \psi^*(\lambda_{N_1}\sin 2\psi + \lambda_{G_1}\cos 2\psi)$$

$$L_{N_2} = \quad L_{G_1} = 2(L_N\sin\psi + L_G\cos\psi) - \psi^*(\lambda_{G_2}\cos 2\psi + \lambda_{N_2}\sin 2\psi).$$

In other words, we have obtained some relations among the angles of pitches and the pitches of the ruled surfaces generated by the edges of Darboux's trihedrons for the ruled surface R_1 and the principal ruled surfaces R_{i1} $(i = 1, 2)$ in the congruence $R(u_1, u_2)$.

An interesting relation, concerning Blaschke vectors of the ruled surface R_1 and of the principal ruled surfaces R_{i1} $(i = 1, 2)$, was proved in [2]. In effect we have

$$(6.14)\qquad B = P\left(\frac{\cos\Psi}{P_1}B_1 + \frac{\sin\Psi}{P_2}B_2 + \frac{d\Psi}{dS}R_0\right).$$

This relation can be obtained as special case from our relation (5.1), if Darboux trihedron coincide with Blaschke trihedron for R_1 and for R_{i1}.

Moreover, if we follow the same way about dual Steiner vectors and apply (6.1) in relation (6.14), we obtain again equation (6.2) and

$$\Lambda_{R_3} = \Lambda_{R_{13}}\cos\Psi + \Lambda_{R_{23}}\sin\Psi$$

where Λ_{R_3}, $\Lambda_{R_{13}}$ and $\Lambda_{R_{23}}$ are the dual angles of pitches of the closed ruled surfaces generated by the edges R_3, R_{13} and R_{23}, respectively.

Separating the real and dual part we get

$$(6.15)\qquad \lambda_{R_3} = \lambda_{R_{13}}\cos\psi + \lambda_{R_{23}}\sin\psi$$

$$(6.16)\qquad L_{R_3} = L_{13}\cos\psi + L_{23}\sin\psi + \psi^*(\lambda_{R_{23}}\cos\psi - \lambda_{R_{13}}\sin\psi).$$

Equations (6.15), (6.16) give us the angle of pitch and the pitch of the closed ruled surface generated by the edge R_3 in terms of the angle of pitch and the pitch of the closed ruled surfaces generated by the edges R_{13} and R_{23}.

We conclude, remarking that from relation (5.1) we have derived many important new relations, as we announced in Sec. 1.

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Sommario

In una congruenza di rette in E^3 si considerano una rigata ed altre due rigate convenientemente associate alla prima. Si ottiene una interessante relazione fra i vettori di Darboux duali delle tre rigate, dalla quale derivano una formula analoga alla classica formula di Liouville ed altre formule. Vengono inoltre indicate alcune relazioni, concernenti rigate chiuse generate dai triedri di Darboux e di Blaschke.

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