Riv. Mat. Univ. Parma (5) **6** (1997), 213-218

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# Totally umbilical submanifolds: some remarks (**)

## 1 - Introduction

The present paper collects some results concerning totally umbilical submanifolds of Riemannian manifolds.

In Section 2, we give a new characterization of totally umbilical submanifolds, that could be useful in further research.

In Section 4, we consider the Riemannian manifolds of constant sectional curvature and the almost Hermitian manifolds of constant holomorphic, anti-holomorphic curvature and prove some results about the sectional curvature of their totally umbilical submanifolds.

Similar results concerning bisectional curvature will appear in a next paper.

## 2 - A characterization theorem

Let $\widetilde{M} = \widetilde{M}(g)$ be an $\widetilde{m}$-dimensional *Riemannian manifold* and $g$ its metric. Let $M$ be an $m$-dimensional *submanifold of* $\widetilde{M}$ with induced metric, still denoted by $g$.

We refer to [4] Ch. 7, to [9] Ch. 2 and to [2] Ch. 2 for the basic facts about the geometry of submanifolds. In the sequel $B$ denotes the *second fundamental form* and $H = \frac{1}{m}$ trace $B$ the *mean curvature vector field* of $M$.

We recall also that $M$ is a *totally umbilical* submanifold of $\widetilde{M}$, if and only if we have

$$B(X, Y) = Hg(X, Y) \tag{1}$$

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(**) Received December 29, 1997. AMS classification 53 C 40.

for any vector fields $X$, $Y$ of $\mathfrak{X}(M)$. In particular, $M$ is a *totally geodesic* submanifold of $\widetilde{M}$ if and only if we have

$$B(X, Y) = 0 \tag{2}$$

for any vector fields $X$, $Y$ of $\mathfrak{X}(M)$.

Other conditions for the second foundamental form $B$ have been considered by several Authors (See for example [2] p. 43, [5], p. 907). Recently, the classical Gauss equation suggested us to introduce *condition*

$$g(B(X, Y), B(Z, W)) = g(H, H)\, g(X, Y)\, g(Z, W) \tag{3}$$

for any vector fields $X$, $Y$, $Z$, $W$ of $\mathfrak{X}(M)$.

Now we are able to prove

Theorem 1. *Conditions* (1) *and* (3) *are equivalent. In other words, condition* (3) *characterizes the totally umbilical submanifolds.*

We may remark that, using condition (3), it is immediate from Gauss equation to derive relations

$$\widetilde{\chi}_{rs} = \chi_{rs} - g(H, H) \cos rs \tag{4}$$

$$\widetilde{K}_r = K_r - g(H, H)\,. \tag{5}$$

linking the bisectional and the sectional curvatures of $M$ with the corresponding curvatures of $\widetilde{M}$ (see (8), (10) at p. 117 of [7]).

The above relations are true at any point $x$ of the totally umbilical submanifold $M$ and for any pair $r$, $s$ of oriented planes of $T_x(M)$. Of course $g(H, H)$ has to be evaluated at point $x$. For the definition of the angle of two oriented planes, due to E. Cartan, see for example [6], p. 149.

## 3 - Proof of Theorem 1

It is immediate to check that (1) implies (3); so we have only to prove the converse.

Let $x$ be any point of $M$ and $T_x(M)$ the space tangent to $M$ at $x$. It is sufficient to prove that (3) implies (1) for any vectors $X$, $Y$, $Z$, $W$ of $T_x(M)$. Of course now $g$, $B$ and $H$ must be considered at point $x$.

If we have $H = 0$ at $x$, the proof is immediate. So we assume that $H \neq 0$ at point $x$.

Let $\{e_1, \dots, e_m\}$ be an orthonormal basis in $T_x(M)$. From equation (3) we derive:

$$g(B(e_i, e_j), B(e_i, e_j)) = 0 \qquad \text{for } i \neq j \tag{6}$$

$$g(B(e_i, e_i), B(e_j, e_j)) = g(H, H). \tag{7}$$

Remember that $B : T_x(M) \times T_x(M) \to T_x(M)^\perp$; so $\operatorname{Im} B \subset T_x(M)^\perp$. It is immediate to realize that $\operatorname{Im} B \subset \operatorname{span}(B(e_i, e_j);\ i, j = 1, \dots, m)$. Since from (6) we have

$$B(e_i, e_j) = 0 \qquad \text{for } i \neq j \qquad (i, j = 1, \dots, m) \tag{8}$$

we obtain

$$\operatorname{Im} B \subset \operatorname{span}(B(e_i, e_i);\ \ i = 1, \dots, m). \tag{9}$$

Let $\{\nu_1, \dots, \nu_r\}$ with $r = \tilde{m} - m$ be an orthonormal basis of $T_x(M)^\perp$. Then we can write $B(e_i, e_i) = b_i^t \nu_t$ where $t$ runs over the integers $1, \dots, r$.

From relation (7) we get:

$$g(H, H) = g(B(e_i, e_i), B(e_i, e_i)) = \sum_1^r (b_i^t)^2$$

$$g(H, H) = g(B(e_j, e_j), B(e_j, e_j)) = \sum_1^r (b_j^t)^2$$

$$g(H, H) = g(B(e_i, e_i), B(e_j, e_j)) = \sum_1^r b_i^t b_j^t$$

for any $i, j = 1, \dots, m$.

On the other hand, the identity

$$\sum_1^r (b_i^t)^2 \sum_1^r (b_j^t)^2 - (\sum_1^r b_i^t b_j^t)^2 = \sum_{s<t} (b_i^s b_j^t - b_i^t b_j^s)^2$$

is well known (cf. [8], p. 45). Therefore, in the present case, we have

$$b_i^s b_j^t - b_i^t b_j^s = 0 \qquad \text{for any } s, t = 1, \dots, r.$$

This equation shows that for any $i, j = 1, \dots, m$ the vectors $B(e_i, e_i)$ and $B(e_j, e_j)$ are linearly dependent. So we can conclude that

$$\dim \operatorname{span}(B(e_i, e_i);\ i = 1, \dots, m) \leqslant 1. \tag{10}$$

Consider now the vector $u = \dfrac{1}{\sqrt{m}} \sum_1^m e_i$ and note that

$$B(u, u) = \frac{1}{m} \sum_{i,j} B(e_i, e_j) = \frac{1}{m} \sum_1^m B(e_i, e_i) = H \neq 0.$$

Consequently

$$\mathrm{Im}\, B \supset \mathrm{span}\, H\,. \tag{11}$$

Finally, taking into account (10), from (9), (11) we derive

$$\mathrm{Im}\, B = \mathrm{span}\, H\,.$$

Remark that for any $i = 1, \ldots, m$ we can write $B(e_i, e_i) = a_i H$. So, putting $j = i$ in (7), we find $B(e_i, e_i) = \pm H$. Then, using the definition of $H$, we realize that

$$B(e_i, e_i) = H \qquad i = 1, \ldots, m\,. \tag{12}$$

The proof of Theorem 1 ends by remarking that for any vectors $X = X^i e_i$, $Y = Y^j e_j$ of $T_x(M)$ we have

$$B(X, Y) = X^i Y^j B(e_i, e_j) = \sum_1^m X^i Y^i B(e_i, e_i) = g(X, Y)\, H\,.$$

## 4 - Further results

In the sequel we always assume that $M$ *is a totally umbilical submanifold of the Riemannian manifold* $\widetilde{M}$.

We list here some results about sectional curvatures. We begin with

Proposition 1. *If* $\widetilde{M} = \widetilde{M}(\tilde{c})$ *has constant sectional curvature* $\tilde{c}$, *then* $M$ *has constant sectional curvature* $c = \tilde{c} + g(H, H)$ *at any of its points.*

*If* $M$ *is also connected and* $\dim_R M \geqslant 3$, *then* $M$ *is a manifold of constant sectional curvature,* $g(H, H)$ *is constant on* $M$ *and we have* $c \geqslant \tilde{c}$.

Before going further, we recall that a submanifold $M$ of an almost Hermitian manifold $\widetilde{M} = \widetilde{M}(g, J)$ is called *holomorphic* (or *J-invariant*) if and only if we have $JT_x(M) = T_x(M)$ at any point $x$ of $M$. A holomorphic submanifold $M$ of $\widetilde{M}$ is also an almost Hermitian manifold with the induced structures, so we can write $M = M(g, J)$.

We recall also that a submanifold $M$ of an almost Hermitian manifold $\widetilde{M} = \widetilde{M}(g, J)$ is called *anti-holomorphic* (or *J- anti-invariant* or *totally real*) if and only if we have $JT_x(M) \subset T_x(M)^{\perp}$ at any point $x$ of $M$.

Proposition 2. *Let* $\widetilde{M} = \widetilde{M}(g, J)$ *be an almost Hermitian manifold and* $M = M(g, J)$ *a holomorphic submanifold of* $\widetilde{M}$. *If* $\widetilde{M}$ *has constant holomorphic, anti-holomorphic curvature, then* $M$ *has constant holomorphic, anti-holomorphic curvature, at any of its points, respectively.*

Proposition 3. *Let $\widetilde{M} = \widetilde{M}(g, J)$ be an almost Hermitian manifold and $M$ an anti-holomorphic (totally real) submanifold of $\widetilde{M}$. If $\widetilde{M}$ has constant anti-holomorphic curvature $\tilde{c}_a$, then $M$ has constant sectional curvature $c = \tilde{c}_a + g(H, H)$ at any of its points.*

*If $M$ is also connected and $\dim_R M \geq 3$, then $M$ is a manifold of constant curvature, $g(H, H)$ is constant on $M$ and we have $c \geq \tilde{c}_a$.*

## 5 - Proofs and remarks

The proof of Proposition 1 is easy. Since at any point $x$ of $M \subset \widetilde{M}$ and for any plane $r$ of $T_x(M) \subset T_x(\widetilde{M})$ we have $\widetilde{K}_r = \tilde{c}$, then equation (5) gives $K_r = \tilde{c} + g(H, H)$. In other words $K_r$ does not depend on the plane $r$ of $T_x(M)$. We denote this constant by $c$ and the first part is proved.

In general $g(H, H)$ is a function of the point $x$; so we can assert only that $M$ has pointwise constant sectional curvature. But, under the additional assumptions of the second part of the statement, we can use the classical Schur theorem and conclude that now $c$ is constant on $M$.

Remark 1. Proposition 2 can be regarded as an almost Hermitian analogue of Proposition 1. However we must remark explicitly that the latter constants in the statement depend, in general, on the point $x$ of $M$. Since in the present case we have not a Schur-like theorem, so we cannot affirm that these quantities are constant all over $M$, even if $M$ is assumed to be a connected $m$-dimensional submanifold with $m \geq 4$.

To prove Proposition 2, consider first a point $\tilde{x}$ of $\widetilde{M}$ and recall that a plane $\tilde{h}$ of $T_{\tilde{x}}(\widetilde{M})$ is said *holomorphic* iff we have $\tilde{h} = J\tilde{h}$. A plane $\tilde{a}$ of $T_{\tilde{x}}(\widetilde{M})$ is said *anti-holomorphic (totally real)* iff we have $\tilde{a} \perp J\tilde{a}$.

As in the proof of Proposition 1, we start from equation (5) of Sect. 2. In the first case, at any point $x$ of $M \subset \widetilde{M}$ we know that $\widetilde{K}_h = \tilde{c}_0$ (constant) for any holomorphic plane $h$ of $T_x(M) \subset T_x(\widetilde{M})$. So we have $K_h = \tilde{c}_0 + g(H, H)$ at any point $x$ of $M$ and for any holomorphic plane $h$ of $T_x(M)$. In other words $M$ has pointwise constant holomorphic curvature. In the second case the proof is quite analogous.

To prove Proposition 3, note first that any plane tangent to $M$ is anti-holomorphic (totally real). On the other hand we know that at any point $\tilde{x}$ of $\widetilde{M}$ and for any anti-holomorphic plane $\tilde{a}$ of $T_{\tilde{x}}(\widetilde{M})$ we have $\widetilde{K}_{\tilde{a}} = \tilde{c}_a$. Consider now a point $x$ of $M \subset \widetilde{M}$ and a plane $r$ of $T_x(M) \subset T_x(\widetilde{M})$. Since $r$ is anti-holomorphic we have $\widetilde{K}_r = \tilde{c}_a$. From equation (5) of Sect. 2 we derive $K_r = \tilde{c}_a + g(H, H)$. In other words $M$ has pointwise constant sectional curvature.

The second part of the statement can be proved by using the classical Schur theorem as in Proposition 1.

We end the present section with two remarks.

Remark 2. In Propositions 1, 2, 3 the assumptions on $\widetilde{M}$ can be weakened. In effect you may limit yourself to consider only planes tangent to $M$.

Remark 3. Some Authors have recently considered submanifolds $M$ with $g(H, H)$ constant on $M$ (in particular, extrinsic spheres). This additional assumption permit us to prove the second part of Propositions 1, 3 without any assumption on connectedness and on dimension of $M$ and to obtain a global result also in the case of Proposition 2.

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## Sommario


*Il lavoro contiene una caratterizzazione delle sottovarietá totalmente ombelicali ed alcuni risultati ad esse relativi, nell'ipotesi che la varietá ambiente sia a curvatura sezionale costante, ovvero a curvatura olomorfa costante, oppure a curvatura anti-olomorfa costante.*


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