Riv. Mat. Univ. Parma (6) **1** (1998), 13-30

G. L. Caraffini, C. E. Catalano and G. Spiga (*)

On the small mean free path asymptotics of the transport equation with inelastic scattering (**)

Dedicated to the memory of G. L. Braglia

1 - Introduction

Inelastic phenomena have recently been given some attention in kinetic theory [**1**], [**2**], [**3**]. As pointed out in [**4**], there are several similarities with the allied field of electron transport in semiconductors, in which several interesting results have been obtained in a recent past [**5**], [**6**], [**7**]. One of the crucial problem, ever, is the so called hydrodynamic limit, namely an approximate equation, at a macroscopic level, but still properly describing the phenomenon, which could be consistently derived as asymptotic limit when the Knudsen number tends to zero [**8**]. In this respect a rigorous procedure, called «compressed Chapman-Enskog expansion» has been proposed by Mika and Banasiak, allowing an error estimate of the approximate versus the exact solution. Such a procedure has been successfully applied to several standard problems (thus, with elastic scattering only) of kinetic and transport theory (see [**9**] and the bibliography therein). The modified asymptotic expansion has been extended later to the transport of test particles in a fixed background of inelastically scattering field particles at equilibrium. There are physical situations in which a collision with a test particle may lead in fact to excitation (de-excitation) of the field particle, and the relevant energy jump enters the

(*) Dept. Mathematics, Univ. of Parma, Via D'Azeglio 85, 43100 Parma, Italy.
(**) Received April 7, 1998. AMS classification 82 C 40, 76 P 05. Work performed in the frame of the activities sponsored by MURST and by CNR (GNFM and Project 96.03861.CT01).

overall energy conservation, leading to a loss (gain) of the total kinetic energy, and, in this sense, to an inelastic down-scattering (up-scattering) collision. When, for some reason, the excited field particles play a negligible role and may be considered as non-participating (as it occurs, for instance, if the background is in equilibrium at low temperature), up-scattering can be disregarded, and this implies a progressive loss of kinetic energy for the mixture of participating species (field particles in the ground state and test particles). A model problem and a simplified version relevant to negligible up-scattering have already been examined [**10**], [**11**], while the additional mathematical difficulties related to the presence of up-scattering are still under investigation [**12**], [**13**].

All previous inelastic work is relevant to the Lorentz gas model [**14**], which corresponds to the limiting case of vanishing ratio between the masses of test and field particles. Consequently, the energy jump in an inelastic scattering affects now only the kinetic energy of test particles, and, in particular, down-scattering slows them down by a fixed amount of energy at each collision. This case raises interesting mathematical questions due to the fact that energy gets partitioned in equivalence classes which are closed with respect to scattering [**3**] and thus equilibria are not uniquely determined (see also [**5**]). This peculiar situation breaks easily down when more than one excited energy level becomes significant for field particles. This generalization is also scheduled as future work.

The present note is aimed at studying, in the previous frame, the combined effects of elastic and inelastic scattering on the asymptotic limit, still under the simplifying assumption that down-scattering is the only effective inelastic mechanism. Indeed, it is well known that the standard diffusion approximation is recovered in the elastic case [**9**], whereas a limiting equation of the streaming type was obtained in [**11**] in the inelastic case. Three physical situations will be examined in this paper, according to whether elastic and inelastic terms are equally important or only one of them is dominant. All situations are relevant to collision dominated processes and then to small values of the proper Knudsen number. After derivation and discussion of the governing transport equation, all necessary mathematical properties are investigated and the compressed asymptotic expansion is applied to the three different scalings, and worked out up to a first order accuracy in the small parameter, including initial layer analysis.

2 - The physical problem

We consider the transport equation for the distribution function $\tilde{f}$ of a rarefied gas of test particles interacting with a given bath of heavy field particles by three types of binary collisions: elastic scattering, inelastic scattering, and absorption. For simplicity, all independent microscopic differential collision frequencies ν are taken to be constant with respect to both relative speed and deflection angle, and only two energy levels are allowed to field particles, separated by a fixed gap $\Delta E > 0$. Particles in the ground and excited states (labeled by subscripts 1 and 2, respectively) are assumed to be in equilibrium at a given temperature T, thus with densities related by the Boltzmann factor

$$\frac{n_2}{n_1} = \exp\left(-\frac{\Delta E}{kT}\right) < 1 \tag{1}$$

where k is the Boltzmann constant.

The (linear) Boltzmann transport equation to be dealt with reads as

$$\frac{\partial \mathfrak{f}}{\partial t} + v\Omega \cdot \frac{\partial \mathfrak{f}}{\partial \boldsymbol{x}} = J_e[\mathfrak{f}] + J_i[\mathfrak{f}] + J_a[\mathfrak{f}] \tag{2}$$

where indices e, i, and a denote elastic scattering, inelastic scattering, and absorption, respectively. Omitting all dependences except on speed $v \in (0, +\infty)$ and direction $\Omega \in S^2$, the collision terms J, in the limit when the ratio of t.p. mass to f.p. mass $m_1 = m_2$ tends to zero, are given by [**1**], [**3**]

$$\begin{aligned}
J_e[\mathfrak{f}] &= \int_{S^2} (n_1 \nu_{11} + n_2 \nu_{22})\, \mathfrak{f}(v\Omega')\, d\Omega' - \mathfrak{f}(v\Omega) \int_{S^2} (n_1 \nu_{11} + n_2 \nu_{22})\, \mathrm{d}\Omega' \\
J_i[\mathfrak{f}] &= \int_{S^2} \left[n_1 \frac{v_+}{v} \nu_{12}\, \mathfrak{f}(v_+ \Omega') + n_2 U(v-\delta)\, \nu_{12}\, \mathfrak{f}(v_- \Omega') \right] \mathrm{d}\Omega' \\
&\quad - \mathfrak{f}(v\Omega) \int_{S^2} \left[n_1 U(v-\delta)\, \nu_{12} + n_2 \frac{v_+}{v} \nu_{12} \right] \mathrm{d}\Omega' \\
J_a[\mathfrak{f}] &= -\, \mathfrak{f}(v\Omega) \int_{S^2} (n_1 \nu_{1a} + n_2 \nu_{2a})\, \mathrm{d}\Omega'
\end{aligned} \tag{3}$$

where U stands for the unit step function and

$$v_{\pm} = [v^2 \pm \delta^2]^{1/2}, \qquad \delta^2 = 2\,\Delta E/m\,. \tag{4}$$

The collision frequency ν_{21} for the f.p. transition from the excited to the ground state (yielding up-scattering for t.p.) has been eliminated from (3) by using the microreversibility condition. After introducing the dimensionless energy variable $\xi = v^2/\delta^2$ and the corresponding new dependent variable f (proportional to the scalar flux $v\dagger$), it is convenient to measure distances and time in units of a typical macroscopic length L and of the related characteristic time L/δ, to get

$$\frac{\partial f}{\partial t} + \xi^{1/2}\,\Omega \cdot \frac{\partial f}{\partial \boldsymbol{x}} \tag{5}$$

$$= \frac{n_1 \nu_{11} L}{\delta} F(\xi) + \frac{n_1 \nu_{12} L}{\delta} F(\xi+1) + \frac{n_2 \nu_{12} L}{\delta} \left(\frac{\xi}{\xi-1} \right)^{1/2} U(\xi-1)\, F(\xi-1) + \frac{n_2 \nu_{22} L}{\delta} F(\xi)$$

$$- \frac{4\pi L}{\delta} f(\xi,\Omega) \left[n_1 \nu_{11} + n_1 \nu_{12} U(\xi-1) + n_2 \nu_{12} \left(\frac{\xi+1}{\xi} \right)^{1/2} + n_2 \nu_{22} + n_1 \nu_{1a} + n_2 \nu_{2a} \right]$$

with

$$F(\boldsymbol{x}, \xi, t) = \int_{S^2} f(\boldsymbol{x}, \xi, \Omega, t)\, \mathrm{d}\Omega\,. \tag{6}$$

Since the absorption term is deleted from the transport equation by simply replacing $f \exp(-\eta t)$ for f, with $\eta = 4\pi L (n_1 \nu_{1a} + n_2 \nu_{2a})/\delta$, we shall consider from now on, without loss of generality, the case $\eta = 0$.

The dimensionless factors appearing in (5) represent several possible inverse Knudsen numbers, measuring the collisionality relevant to the different events. They are in fact the ratio of the macroscopic characteristic time L/δ to a mean collision time of the kind $(n_i \nu_{ij})^{-1}$ or $(n_j \nu_{ij})^{-1}$, relevant to elastic $(i=j)$ or inelastic $(i \neq j)$ collisions. Their order of magnitude is crucial as a label of the importance of the underlying physical process and associated collision operator. Bearing in mind also the ratios between different collision frequencies and between the f.p. populations n_2 and n_1, several small parameters could be singled out. In the sequel we shall confine ourselves to the case of n_2 negligible with respect to n_1 (thermal energy kT small enough, if compared to the jump ΔE) and examine the various options, arising according to the mutual relationship between ν_{11} and ν_{12}, which give rise to different asymptotic scalings.

We refer to the discussion in [**11**] for the implications of the presence of down-scattering only in the inelastic part: in particular, we shall take again a bounded energy interval (as it occurs in any slowing down problem), which means that, for some integer N, the initial distribution f vanishes for any $\xi \geqslant N+1$.

Specifically, we will examine in this paper the following three physical cases.

i. If ν_{11} and ν_{12} are of the same order, it is appropriate to introduce as small parameter a global Knudsen number accounting for all scattering collisions

$$\varepsilon = \frac{\delta}{4\pi n_1 (\nu_{11} + \nu_{12})\, L}\,. \tag{7}$$

Upon introducing the fraction of inelastic contribution, $\alpha = \nu_{12}/(\nu_{11} + \nu_{12})$, $0 < \alpha < 1$, the transport equation reads as

$$\frac{\partial f}{\partial t} + \xi^{1/2}\,\Omega \cdot \frac{\partial f}{\partial \boldsymbol{x}} = \frac{1}{\varepsilon}\, C[f]\,, \qquad C[f] = \alpha C_i[f] + (1-\alpha)\, C_e[f]\,, \tag{8}$$

$$C_i[f] = \frac{1}{4\pi}\, U(N-\xi)\, F(\xi+1) - U(\xi-1)\, f(\xi,\Omega)\,, \qquad C_e[f] = \frac{1}{4\pi}\, F(\xi) - f(\xi,\Omega)\,. \tag{9}$$

If one introduces now the equivalence relation connecting each energy ξ to those that can be reached by scattering from it, it is immediately seen that the equivalence class (which is constituted by all allowed energies $\xi \pm k$, k positive integer) is closed with respect to scattering. By taking quotient, it is sufficient to consider the interval $\xi \in [0, 1)$ and to define, for $n = 0, 1, 2 \ldots$,

$$f_n(\boldsymbol{x}, \xi, \Omega, t) = f(\boldsymbol{x}, \xi + n, \Omega, t), \qquad \xi \in [0, 1)\,. \tag{10}$$

In this way, $\xi \in [0, 1)$ becomes only a parameter in the governing equation, and different values of ξ in such an interval remain uncorrelated during the whole evolution. As physically expected, it is matter of simple calculation to verify that the quantity

$$\varrho(\boldsymbol{x}, \xi, t) = \frac{1}{4\pi} \sum_{k=0}^{N} \int_{S^2} f_k(\boldsymbol{x}, \xi, \Omega, t)\, \mathrm{d}\Omega\,, \qquad \xi \in [0, 1) \tag{11}$$

is conserved under scattering, namely it is a first integral of (8) in the absence of spatial gradients. Scattering, the dominant interaction, amounts to slowing down t.p. below the threshold $\xi = 1$ and to isotropizing directions.

ii. If $\nu_{11} \gg \nu_{12}$, it is convenient to define as small parameter a Knudsen number relevant to elastic collision only

$$\varepsilon = \frac{\delta}{4\pi n_1 \nu_{11} L}\,. \tag{12}$$

Under the assumption that the inverse Knudsen number for inelastic collision,

$\beta_i = 4\pi n_1 \nu_{12} L/\delta$, is of order unity, the transport equation takes the form

$$\frac{\partial f}{\partial t} + \xi^{1/2}\Omega\cdot\frac{\partial f}{\partial \boldsymbol{x}} - \beta_i C_i[f] = \frac{1}{\varepsilon} C_e[f]. \tag{13}$$

Now elastic scattering is the dominant event and it leaves energy unchanged, its effect being only isotropization. The equivalence class of any $\xi \in [0, N+1)$, defined as before but with only elastic collisions, is constituted only by ξ itself. The quantity which is conserved under elastic scattering is

$$\varrho(\boldsymbol{x}, \xi, t) = \frac{1}{4\pi}\int_{S^2} f(\boldsymbol{x}, \xi, \Omega, t)\, d\Omega, \qquad \xi \in [0, N+1), \tag{14}$$

whereas a first integral of (13) in space homogeneous conditions (under all collisions) is given by (11) again: of course, the different mechanism of energy exchange, and the corresponding different structure of the equivalence classes, make the variable ξ appearing in (11) and (14) not comparable.

iii. If finally $\nu_{11} \ll \nu_{12}$, a proper small parameter is given by the Knudsen number restricted to inelastic collisions

$$\varepsilon = \frac{\delta}{4\pi n_1 \nu_{12} L}. \tag{15}$$

Now, under the assumption that the inverse Knudsen number for elastic collision, $\beta_e = 4\pi n_1 \nu_{11} L/\delta$, is of order unity, the transport equation may be written as

$$\frac{\partial f}{\partial t} + \xi^{1/2}\Omega\cdot\frac{\partial f}{\partial \boldsymbol{x}} - \beta_e C_e[f] = \frac{1}{\varepsilon} C_i[f]. \tag{16}$$

The dominant collision operator moves down energy by a unit step at each interaction and isotropizes directions, but it leaves unscattered all t.p. which have energy below the threshold $\xi = 1$. The equivalence classes as above, but relevant only to inelastic collisions, are now the same as for problem (8), and ξ may be restricted to the interval [0, 1). Here the quantity which is conserved under inelastic scattering is

$$\varrho(\boldsymbol{x}, \xi, \Omega, t) = f_0(\boldsymbol{x}, \xi, \Omega, t) + \frac{1}{4\pi}\sum_{k=1}^{N}\int_{S^2} f_k(\boldsymbol{x}, \xi, \Omega, t)\, d\Omega, \qquad \xi \in [0, 1) \tag{17}$$

as it is easily verified from (16) itself, with neither spatial gradients nor elastic scattering. There is here an angle dependence in the conserved quantity, due to the fact that particles with energy $\xi < 1$ remain uncollided for ever.

The asymptotic analysis, with respect to the small parameter ε, of the problems (8), (13) and (16) will be the object of the following Sections.

3 - Mathematical setting

Let X be the Banach space $L_1(R^3 \times (0, N+1) \times S^2)$ with the usual norm. Define the operator

$$Sf = -\,\xi^{1/2}\,\Omega \cdot \frac{\partial f}{\partial \boldsymbol{x}}\,, \qquad D(S) = \{f : f \in X,\ Sf \in X\}\,. \tag{18}$$

Each of the equations (8), (13), and (16) can be cast in abstract form as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = Sf + \gamma_i C_i[f] + \gamma_e C_e[f] \tag{19}$$

for suitable γ_e and γ_i, with initial condition $f(0) = f^0 \in X$, $f^0 \geq 0$. By standard methods from the theory of semigroups [**15**], it is easily verified that $S + \gamma_i C_i + \gamma_e C_e$ generates a positive semigroup of contractions, and then existence and uniqueness of a nonnegative solution for the Cauchy problem in the L_1-setting can be readily inferred.

It is clear that in all problems (8), (13), and (16), the t.p. evolution, after grouping together those which have the same energy ξ modulo 1 at time $t = 0$, is governed by an integrodifferential equation with respect to the independent variables $\boldsymbol{x}$, Ω, t, containing ξ only as a parameter. It seems thus appropriate, according to the discussion in the previous Section, to introduce as new unknown function the $(N+1)$-dimensional vector $\boldsymbol{f} = (f_0, f_1, \ldots, f_N)$ depending on a parameter ξ. We are led consequently to change the mathematical setting and to consider all evolution problems in the Banach space $Y = [L_1(R^3 \times S^2)]^{N+1}$; the collision operators will be accordingly considered as acting on the Banach space $Z = [L_1(S^2)]^{N+1}$. From now on, it is then implicitly understood that the symbols S, C_e, C_i denote the matrix form of the streaming and collision operators previously defined. Notice that the new setting avoids nonuniqueness of collision equilibria [**5**], [**3**], since ξ dependence has been eliminated from the unknowns, and evolution problems relevant to different values of the parameter ξ are uncorrelated.

One of the crucial points of the compressed asymptotic method [**9**] is the determination, for any fixed value of the parameter ξ, of the null space of the dominant operator, and the corresponding spectral decomposition of the pertinent Banach space Z. We have:

Lemma 3.1. *For any value of* $\xi \in [0, 1)$, $\lambda = 0$ *is isolated eigenvalue of multiplicity one of the operator* $C = \alpha C_i + (1-\alpha) C_e$, *and the corresponding eigenspace consists of all elements* $\boldsymbol{f} \in Z$ *that are of the form* $(f_0, 0, \ldots, 0)$, *with constant* f_0.

Proof. For the determination of $N(C)$ it is sufficient to solve sequentially

$$(20) \quad [(1-\delta_{n0})\,\alpha + 1 - \alpha]\, f_n(\Omega) = \frac{1-\alpha}{4\pi} F_n + (1-\delta_{nN}) \frac{\alpha}{4\pi} F_{n+1}, \quad n = 0, 1, \ldots, N$$

with fixed $\xi \in [0, 1)$, which yields directly $f_0(\Omega) = \dfrac{1}{4\pi} F_0$, $f_n(\Omega) = 0\ \forall n > 0$, F_0 independent of Ω and arbitrary. As regards the spectrum of C, lengthy but standard calculations show that, for given $\boldsymbol{g} \in Z$, the equation $\lambda \boldsymbol{f} - C[\boldsymbol{f}] = \boldsymbol{g}$, or, componentwise,

$$(21) \qquad (\lambda + 1 - \alpha\delta_{n0})\, f_n(\Omega) = \frac{1-\alpha}{4\pi} F_n + (1-\delta_{nN}) \frac{\alpha}{4\pi} F_{n+1} + g_n(\Omega),$$

as a degenerate Fredholm integral equation, is uniquely solvable in Z for any λ in the complex plane, except for the set $\{0, -\alpha, -(1-\alpha), -1\}$, which makes up the point spectrum of C. The resolvent $(\lambda I - C)^{-1}$ is implicitly given by (21) itself plus

$$(22) \quad F_0 = \frac{1}{\lambda} \sum_{k=0}^{N} \left(\frac{\alpha}{\lambda + \alpha} \right)^k G_k, \quad F_n = \frac{1}{\lambda + \alpha} \sum_{k=n}^{N} \left(\frac{\alpha}{\lambda + \alpha} \right)^{k-n} G_k, \quad n = 1, 2, \ldots, N,$$

where, like in (6),

$$G_k = \int_{S^2} g_k(\Omega)\, d\Omega\, .$$

It should be noted that, in the standard setting $L_1[(0, N+1) \times S^2]$, the eigenspace of $\lambda = 0$ would be infinite dimensional, due to the presence of an arbitrary function of $\xi \in [0, 1)$, which is a consequence of the absence of correlation among energies in that range. Thus, taking sections with respect to the former variable ξ yields instead a simple leading eigenvalue on each section, as expressed in the previous formulae by the presence of only a multiplicative constant F_0 in the eigenfunction.

Due to the simplicity of $\lambda = 0$, we immediately have the spectral decomposition [**9**]

$$Z = N(C) \oplus R(C) \tag{23}$$

where $R(C)$ is the range of C. There results in addition:

Lemma 3.2. *The projection P onto $N(C)$ along $R(C)$ is given by*

$$(P\boldsymbol{f})_0 = \frac{1}{4\pi} \sum_{k=0}^{N} \int_{S^2} f_k(\Omega)\,\mathrm{d}\Omega\,, \qquad (P\boldsymbol{f})_n = 0 \quad n = 1, \dots, N, \tag{24}$$

again for any value of the parameter $\xi \in [0, 1)$.

Proof. The proof is straightforward, with $\|P\| = 1$, $P^2 = P$, $PC = CP = 0$.

For the sake of completeness, we report also on the mathematical properties of the other operators which play the dominant role in the considered problems, referring to the quoted bibliography for the proofs. Actually, all quoted results are relevant to only one space dimension, thus with a directional variable $\mu \in [-1, 1]$ instead of $\Omega \in S^2$, but the generalization does not present any difficulty. It is then omitted here.

In the case of C_e, energies are actually uncorrelated at all, the parameters ξ ranges thus all over $[0, N+1)$, and it is proper to consider the operator on the Banach space $L_1(S^2)$. The matter has been widely dealt with in the literature [**9**]. The conclusions are summarized by:

Lemma 3.3. *$\lambda = 0$ is an isolated eigenvalue of multiplicity one of the operator C_e, and the corresponding eigenspace $N(C_e)$ consists of the functions in $L_1(S^2)$ which are constant. The spectral decomposition $L_1(S^2) = N(C_e) \oplus R(C_e)$ holds and the projection P onto $N(C)$ is given by*

$$Pf = \frac{1}{4\pi} \int_{S^2} f(\Omega)\,\mathrm{d}\Omega\,. \tag{25}$$

In the case of C_i, the situation gets more complicated because $\lambda = 0$, still isolated eigenvalue of C_i, has infinite dimensional eigenspace also in the present setting. However, the spectral decomposition can be proved, upon introducing the adjoint operator C_i^* and showing that the range of C_i is closed [**11**]. Here again $\xi \in [0, 1)$ and $Z = [L_1(S^2)]^{N+1}$. We have:

Lemma 3.4. *$\lambda = 0$ is an isolated eigenvalue of the operator C_i, and the corresponding eigenspace consists of the elements of Z of the form $(f_0, 0, \ldots, 0)$, with f_0 arbitrary (summable) function of Ω. The spectral decomposition $Z = N(C_i) \oplus R(C_i)$ holds and the projection P onto $N(C_i)$ is given by*

$$(26) \qquad (P\boldsymbol{f})_0 = f_0(\Omega) + \frac{1}{4\pi} \sum_{k=1}^{N} \int_{S^2} f_k(\Omega')\, d\Omega', \qquad (P\boldsymbol{f})_n = 0 \quad n = 1, \ldots, N.$$

The connection between the projections (24), (25), (26) on one side, and the null spaces and first integrals (11), (14), (17) on the other, is apparent from the above formulae.

Now the compressed asymptotic expansion [**9**] resorts to the spectral decomposition (23) by introducing, for the unknown $\boldsymbol{f}$, hydrodynamic and kinetic parts

$$(27) \qquad \boldsymbol{\varphi} = P\boldsymbol{f}, \qquad \boldsymbol{\psi} = Q\boldsymbol{f}$$

with $Q = I - P$ and $\boldsymbol{f} = \boldsymbol{\varphi} + \boldsymbol{\psi}$; both projections P and Q are applied then to the pertinent evolution equation and initial condition. Each unknown is further separated into a bulk and an initial layer contribution

$$(28) \qquad \boldsymbol{\varphi}(t) = \overline{\boldsymbol{\varphi}}(t) + \tilde{\boldsymbol{\varphi}}(\tau), \qquad \boldsymbol{\psi}(t) = \overline{\boldsymbol{\psi}}(t) + \tilde{\boldsymbol{\psi}}(\tau), \qquad \tau = t/\varepsilon,$$

where the hydrodynamic bulk part $\overline{\boldsymbol{\varphi}}$ is left unexpanded, whereas all other parts are expanded into asymptotic power series with respect to ε, of the kind

$$(29) \qquad \overline{\boldsymbol{\psi}}(t) = \sum_{i=0}^{\infty} \overline{\boldsymbol{\psi}}^i(t)\, \varepsilon^i.$$

The initial layer contributions depend on the stretched variable τ and are correspondingly expanded. Initial conditions reads as

$$(30) \qquad \boldsymbol{\varphi}(0) = P\boldsymbol{f}^0 = \boldsymbol{u}, \qquad \boldsymbol{\psi}(0) = Q\boldsymbol{f}^0 = \boldsymbol{w}.$$

Initial conditions to be applied to the limiting asymptotic equation are determined by the initial layer analysis.

4 - Asymptotic analysis of equation (8)

We begin with case **i.**), in which elastic and inelastic scattering are equally important, the quantity (11) is conserved by collisions, and Lemmas 3.1 and 3.2 hold. For the components of the unknown vector $\boldsymbol{f}$ the governing equation, for any fixed

$\xi \in [0, 1)$, reads as

$$\frac{\partial f_n}{\partial t} = -(\xi + n)^{1/2} \Omega \cdot \frac{\partial f_n}{\partial \boldsymbol{x}} + \frac{1-\alpha}{4\pi\varepsilon} F_n + (1-\delta_{nN}) \frac{\alpha}{4\pi\varepsilon} F_{n+1} - \frac{1-\alpha\delta_{n0}}{\varepsilon} f_n . \tag{31}$$

The system for the hydrodynamic and kinetic parts takes the form

$$\begin{aligned} \frac{\partial \boldsymbol{\varphi}}{\partial t} &= PSP\boldsymbol{\varphi} + PSQ\boldsymbol{\psi} \\ \frac{\partial \boldsymbol{\psi}}{\partial t} &= QSP\boldsymbol{\varphi} + QSQ\boldsymbol{\psi} + \frac{1}{\varepsilon} QCQ\boldsymbol{\psi} \end{aligned} \tag{32}$$

with P given by eq. (24). The same system is in order for the bulk parts $\overline{\boldsymbol{\varphi}}$ and $\overline{\boldsymbol{\psi}}$ and, when the latter is expanded up to the first order in ε, there follows readily, since QCQ is invertible in $R(C)$,

$$\overline{\boldsymbol{\psi}}^0 = 0 , \qquad \overline{\boldsymbol{\psi}}^1 = -(QCQ)^{-1} QSP\overline{\boldsymbol{\varphi}} , \tag{33}$$

and thus, apart from $O(\varepsilon^2)$ corrections,

$$\frac{\partial \overline{\boldsymbol{\varphi}}}{\partial t} = PSP\overline{\boldsymbol{\varphi}} - \varepsilon PSQ(QCQ)^{-1} QSP\overline{\boldsymbol{\varphi}} , \tag{34}$$

where it is easy to check that $PSP = 0$. This is clearly a scalar equation since the elements of $N(C)$ have all vanishing components, but the first. More precisely, some manipulations yield

$$(QSP\boldsymbol{g})_0 = -\frac{1}{4\pi} \xi^{1/2} \Omega \cdot \frac{\partial}{\partial \boldsymbol{x}} \sum_{k=0}^{N} G_k , \qquad (QSP\boldsymbol{g})_n = 0 , \qquad n > 0$$

$$(PSQ\boldsymbol{g})_0 = -\frac{1}{4\pi} \sum_{k=0}^{N} (\xi + k)^{1/2} \int_{S^2} \Omega \cdot \frac{\partial g_k}{\partial \boldsymbol{x}} \, \mathrm{d}\Omega , \quad (PSQ\boldsymbol{g})_n = 0 , \qquad n > 0$$

and, for $\boldsymbol{g} \in R(C)$,

$$[(QCQ)^{-1}\boldsymbol{g}]_0 = -\frac{1}{1-\alpha} g_0 + \frac{1}{1-\alpha} \frac{1}{4\pi} G_0 + \frac{1}{\alpha} \frac{1}{4\pi} \sum_{k=1}^{N} kG_k$$

$$[(QCQ)^{-1}\boldsymbol{g}]_n = -g_n - \frac{1-\alpha}{\alpha} \frac{1}{4\pi} G_n - \frac{1}{\alpha} \frac{1}{4\pi} \sum_{k=n+1}^{N} G_k .$$

The only component of interest in (34) is thus

$$-[PSQ(QCQ)^{-1}QSP\overline{\boldsymbol{\varphi}}]_0 = \frac{\xi}{(1-\alpha)\,4\pi}\sum_{i,j=1}^{3}\frac{\partial^2\overline{\varphi}_0}{\partial x_i \partial x_j}\int_{S^2}\Omega_i\Omega_j \mathrm{d}\Omega = \frac{\xi}{3(1-\alpha)}\nabla^2\overline{\varphi}_0, \tag{35}$$

and the limiting first order asymptotic equation is given by the diffusive approximation

$$\frac{\partial\overline{\varphi}_0}{\partial t} = \frac{\varepsilon}{3}\frac{\xi}{1-\alpha}\nabla^2\overline{\varphi}_0 \tag{36}$$

where the dependent variable $\overline{\varphi}_0$ is easily seen to represent the bulk part of the quantity ϱ in (11), conserved under scattering. The diffusion coefficient depends (linearly) on the parameter $\xi \in [0, 1)$ and on $\alpha \in (0, 1)$. It still makes sense in the limit $\alpha \to 0$, but it degenerates in the opposite limiting case $\alpha \to 1$, since, in the latter, elastic scattering would vanish, and isotropization as well, changing the structure of the spectral decomposition.

Upon resorting to the decompositions (27) and (30), and again to the first order approximation, initial conditions can be cast as

$$\begin{aligned} &\overline{\boldsymbol{\varphi}}(0) = \boldsymbol{u} - \widetilde{\boldsymbol{\varphi}}^0(0) - \varepsilon\widetilde{\boldsymbol{\varphi}}^1(0)\,,\\ &\widetilde{\boldsymbol{\psi}}^0(0) = \boldsymbol{w}\,, \qquad \widetilde{\boldsymbol{\psi}}^1(0) = (QCQ)^{-1}QSP\overline{\boldsymbol{\varphi}}(0)\,, \end{aligned} \tag{37}$$

where eq. (33), specialized at $t = 0$, has been taken into account. The initial layer equations

$$\begin{aligned} &\frac{\partial\widetilde{\boldsymbol{\varphi}}}{\partial\tau} = \varepsilon PSQ\widetilde{\boldsymbol{\psi}}\\ &\frac{\partial\widetilde{\boldsymbol{\psi}}}{\partial\tau} = \varepsilon QSP\widetilde{\boldsymbol{\varphi}} + \varepsilon QSQ\widetilde{\boldsymbol{\psi}} + QCQ\widetilde{\boldsymbol{\psi}} \end{aligned} \tag{38}$$

have to be solved by equating equal powers of ε, with vanishing limit conditions for $\tau \to +\infty$. This yields sequentially

$$\widetilde{\boldsymbol{\varphi}}^0 = 0\,, \qquad \frac{\partial\widetilde{\boldsymbol{\varphi}}^1}{\partial\tau} = PSQ\widetilde{\boldsymbol{\psi}}^0\,, \qquad \frac{\partial\widetilde{\boldsymbol{\psi}}^0}{\partial\tau} = QCQ\widetilde{\boldsymbol{\psi}}^0\,, \qquad \frac{\partial\widetilde{\boldsymbol{\psi}}^1}{\partial\tau} = QSQ\widetilde{\boldsymbol{\psi}}^0 + QCQ\widetilde{\boldsymbol{\psi}}^1$$

and then, if $G_*(\tau)$ denotes the bounded exponentially decaying semigroup generated by QCQ on $R(C)$ (its explicit expression is omitted for brevity), we may write

$\widetilde{\psi}^0 = G_*(\tau)\,\boldsymbol{w}$ and, by standard properties of semigroups,

$$\widetilde{\boldsymbol{\varphi}}^1 = \int_0^\tau PSQG_*(\tau')\,\boldsymbol{w}\,\mathrm{d}\tau' = -\int_\tau^\infty PSQG_*(\tau')\,\boldsymbol{w}\,\mathrm{d}\tau' = PSQ(QCQ)^{-1}G_*(\tau)\,\boldsymbol{w}$$

$$\widetilde{\boldsymbol{\psi}}^1 = G_*(\tau)(QCQ)^{-1}QSP\overline{\boldsymbol{\varphi}}(0) + \int_0^\tau G_*(\tau-\tau')\,QSQG_*(\tau')\,\boldsymbol{w}\,\mathrm{d}\tau'.$$

What we are mainly interested in is the first of (37), namely

$$\overline{\boldsymbol{\varphi}}(0) = \boldsymbol{u} - \varepsilon PSQ(QCQ)^{-1}\boldsymbol{w}\,, \tag{39}$$

which provides the initial condition for the hydrodynamic bulk part in (36) in terms of the hydrodynamic and kinetic parts of the actual initial condition. Again (39), as any equation in $N(C)$, has only one nontrivial scalar component, for which it gives, after some algebra,

$$\overline{\varphi}_0(0) = u_0 - \frac{\varepsilon}{4\pi}\left[\frac{1}{1-\alpha}\xi^{1/2}\int_{S^2}\Omega\cdot\frac{\partial w_0}{\partial \boldsymbol{x}}\,\mathrm{d}\Omega + \sum_{k=1}^{N}(\xi+k)^{1/2}\int_{S^2}\Omega\cdot\frac{\partial w_k}{\partial \boldsymbol{x}}\,\mathrm{d}\Omega\right]. \tag{40}$$

To first order accuracy, the asymptotic limit to the Cauchy problem (8) is thus the diffusion equation (36), with initial condition (40), for the bulk part of the hydrodynamic quantity ϱ given by (11).

5 - Asymptotic analysis of equation (13)

We are dealing here with case **ii**, in which elastic scattering is dominant over all other terms, including inelastic scattering, of the governing equation. Now the quantity conserved by the dominant events is given by (14), and the spectral decomposition is described by Lemma 3.3. Besides, Z is replaced by $L_1(S^2)$, the unknown is a scalar quantity, and ξ ranges all over $[0, N+1)$. With C_i and C_e defined by (9), the set of hydrodynamic and kinetic equations takes the form

$$\begin{aligned}\frac{\partial\varphi}{\partial t} &= PSP\varphi + PSQ\psi + \beta_i PC_i P\varphi + \beta_i PC_i Q\psi\\ \frac{\partial\psi}{\partial t} &= QSP\varphi + QSQ\psi + \beta_i QC_i P\varphi + \beta_i QC_i Q\psi + \frac{1}{\varepsilon}QC_e Q\psi\,,\end{aligned} \tag{41}$$

where P is defined now by (25) and the cross terms PC_iQ and QC_iP also vanish. Repeating the same steps as in the previous Section, we obtain for the hydrodynamic contributions, apart from $O(\varepsilon^2)$ corrections,

$$\overline{\psi} = -\varepsilon(QC_eQ)^{-1}QSP\overline{\varphi}\,, \tag{42}$$

where now $(QC_eQ)^{-1} = -I$ on $R(C_e)$, and once more $PSP = 0$. Therefore, we are simply left with

$$\frac{\partial\overline{\varphi}}{\partial t} = \varepsilon PSQQSP\overline{\varphi} + \beta_i PC_i P\overline{\varphi} \tag{43}$$

where $\overline{\varphi} \in N(C_e)$ is the bulk contribution to the quantity ϱ defined by (14) which is conserved by elastic scattering. Since

$$PC_iPg = \frac{1}{4\pi}U(N-\xi)\,G(\xi+1) - \frac{1}{4\pi}U(\xi-1)\,G(\xi)$$

$$QSPg = -\frac{1}{4\pi}\xi^{1/2}\,\Omega\cdot\frac{\partial G}{\partial \boldsymbol{x}}$$

$$PSQg = -\frac{1}{4\pi}\xi^{1/2}\int_{S^2}\Omega\cdot\frac{\partial g}{\partial \boldsymbol{x}}\,\mathrm{d}\Omega\,,$$

the same steps as in (35) lead to the diffusive approximation

$$\frac{\partial\overline{\varphi}}{\partial t} = \frac{\varepsilon}{3}\xi\nabla^2\overline{\varphi} + \beta_i U(N-\xi)\,\overline{\varphi}_+ - \beta_i U(\xi-1)\,\overline{\varphi}\,, \tag{44}$$

with $\overline{\varphi} = \overline{\varphi}(\boldsymbol{x}, t;\, \xi)$ and $\overline{\varphi}_+ = \overline{\varphi}(\boldsymbol{x}, t;\, \xi+1)$. Thus the diffusion coefficient is $O(\varepsilon)$, as expected, but there is a $O(1)$ interaction term due to inelastic scattering (the non-dominant one), which links the unknown $\overline{\varphi}$ relevant to the chosen value of the parameter ξ to the unknown relevant to the value $\xi + 1$, and thus to all other values which are equal to it modulo 1. In other words, the previously introduced equivalence class enters the picture now, after performing the asymptotic procedure independently of it, since it did not affect the dominant operator C_e.

Notice that, on using the definition (10), summation of (44) with all other equations relevant to the same equivalence class eliminates inelastic scattering, and yields a global balance equation that, however, is not self-consistent for

$$\overline{\varphi}_*(\boldsymbol{x}, t;\, \xi) = \sum_{k=0}^{N}\overline{\varphi}_k(\boldsymbol{x}, t;\, \xi)\,, \qquad \xi \in [0, 1) \tag{45}$$

where $\overline{\varphi}_*$ is the bulk part of the quantity ϱ defined by (11), which is conserved under all kinds of scattering also for the present problem.

Regarding the initial layer analysis, it proceeds in the same manner as before. From the initial layer equations

$$\tag{46} \begin{aligned} \frac{\partial \widetilde{\varphi}}{\partial \tau} &= \varepsilon PSQ\widetilde{\psi} + \varepsilon\beta_i PC_i P\widetilde{\varphi} \\ \frac{\partial \widetilde{\psi}}{\partial \tau} &= \varepsilon QSP\widetilde{\varphi} + \varepsilon QSQ\widetilde{\psi} + \varepsilon\beta_i QC_i Q\widetilde{\psi} + QC_e Q\widetilde{\psi}, \end{aligned}$$

linear expansions with respect to ε yield

$$\widetilde{\varphi}^0 = 0, \qquad \widetilde{\psi}^0 = G_e(\tau)\, w, \qquad \widetilde{\varphi}^1 = PSQ(QC_e Q)^{-1} G_e(\tau)\, w$$

where G_e is the semigroup generated by $QC_e Q$ on $R(C_e)$, and $\widetilde{\psi}^0(0) = w$. The initial condition for the hydrodynamic bulk part turns out to be

$$\tag{47} \overline{\varphi}(0) = u - \varepsilon PSQ(QC_e Q)^{-1} w = u - \frac{\varepsilon}{4\pi}\, \xi^{1/2} \int_{S^2} \Omega \cdot \frac{\partial w}{\partial \boldsymbol{x}}\, \mathrm{d}\Omega$$

and provides $O(\varepsilon)$ correction to the hydrodynamic part u of the actual initial condition in terms of its kinetic part w. The approximate first order asymptotic Cauchy problem is given by (44)+(47), and involves the bulk part of the hydrodynamic quantity (14), via a set of diffusion equations which couple together parameters ξ that are equal modulo 1.

6 - Asymptotic analysis of equation (16)

We consider finally the case **iii**, in which inelastic scattering is the leading event, the quantity (17) is conserved by the dominant collisions, and Lemma 3.4 holds. We are back to the Banach space Z and to a parameter ξ ranging in $[0, 1)$, with C_i as dominant operator. With P given by (26), we have

$$\tag{48} \begin{aligned} \frac{\partial \boldsymbol{\varphi}}{\partial t} &= PSP\boldsymbol{\varphi} + PSQ\boldsymbol{\psi} + \beta_e PC_e P\boldsymbol{\varphi} \\ \frac{\partial \boldsymbol{\psi}}{\partial t} &= QSQ\boldsymbol{\psi} + \beta_e QC_e Q\boldsymbol{\psi} + \frac{1}{\varepsilon} QC_i Q\boldsymbol{\psi} \end{aligned}$$

where account has been given to the identities

$$\tag{49} QSP = PC_e Q = QC_e P = 0\,.$$

In spite of its complexity, this problem exhibits the simplifying feature that the second equation in (48) is decoupled from the first and allows an easy solution, after the usual linear expansion in ε. Since $QC_i Q$ is invertible on $R(C_i)$, we get at once $\overline{\boldsymbol{\psi}}^0 = 0$ and then $\overline{\boldsymbol{\psi}}^1 = 0$. With $\overline{\boldsymbol{\psi}} = 0$, the first equation is self-consistent and reads as

$$\frac{\partial \overline{\boldsymbol{\varphi}}}{\partial t} = PSP\overline{\boldsymbol{\varphi}} + \beta_e PC_e P\overline{\boldsymbol{\varphi}} \,. \tag{50}$$

This is equivalent again to a scalar equation and, after some manipulations, we end up with the explicit form

$$\frac{\partial \overline{\varphi}_0}{\partial t} = -\xi^{1/2}\Omega \cdot \frac{\partial \overline{\varphi}_0}{\partial \boldsymbol{x}} + \frac{\beta_e}{4\pi}\int_{S^2} \overline{\varphi}_0 \, \mathrm{d}\Omega - \beta_e \overline{\varphi}_0 \tag{51}$$

which is independent of ε, and involves the bulk part of the quantity ϱ defined by (17), conserved by inelastic scattering. The unknown $\overline{\varphi}_0$ depends on the variable Ω, in addition to $\boldsymbol{x}$ and t, as it occurred in [11]; along with the streaming process, which is $O(1)$, elastic scattering is present with contributions that are $O(1)$ as well. Indeed, these are the processes undergone by test particles, after the fast transient in which inelastic scattering slows them down below the energy threshold $\xi = 1$.

The initial layer analysis goes through the same steps as before. The relationships (37) still hold, and the set (48) is valid also for the initial layer contributions provided t is replaced by τ and the left hand sides are multiplied by $\dfrac{1}{\varepsilon}$. Expanding both $\widetilde{\boldsymbol{\varphi}}$ and $\widetilde{\boldsymbol{\psi}}$, we get

$$\widetilde{\boldsymbol{\varphi}}^0 = 0 \,, \qquad \widetilde{\boldsymbol{\psi}}^0 = G_i(\tau)\,\boldsymbol{w} \,, \qquad \widetilde{\boldsymbol{\varphi}}^1 = PSQ(QC_iQ)^{-1}G_i(\tau)\,\boldsymbol{w}$$

where G_i is the semigroup generated by QC_iQ on $R(C_i)$. There follows

$$\overline{\boldsymbol{\varphi}}(0) = \boldsymbol{u} - \varepsilon PSQ(QC_iQ)^{-1}\boldsymbol{w} \,, \tag{52}$$

with the usual kinetic $O(\varepsilon)$ correction to the hydrodynamic part of the given initial condition. Again (52) corresponds to a scalar equation, which represents the initial condition to be applied to the hydrodynamic limit (51). Bearing in mind that

$$(PSQ\boldsymbol{g})_0 = \frac{1}{4\pi}\sum_{k=1}^{N}\left[\xi^{1/2}\Omega \cdot \frac{\partial G_k}{\partial \boldsymbol{x}} - (\xi + k)^{1/2}\int_{S^2} \Omega \cdot \frac{\partial g_k}{\partial \boldsymbol{x}}\,\mathrm{d}\Omega\right], \quad (PSQ\boldsymbol{g})_n = 0 \,, \quad n > 0$$

and, for $\boldsymbol{g} \in R(C_i)$,

$$[(QC_i Q)^{-1}\boldsymbol{g}]_0 = \frac{1}{4\pi} \sum_{k=1}^{N} kG_k, \qquad [(QC_i Q)^{-1}\boldsymbol{g}]_n = -g_n - \frac{1}{4\pi} \sum_{k=n+1}^{N} G_k, \qquad n > 0\,,$$

we conclude that

$$\overline{\varphi}_0(0) = u_0 + \frac{\varepsilon}{4\pi} \sum_{k=1}^{N} \left[\xi^{1/2} \Omega \cdot \frac{\partial}{\partial \boldsymbol{x}} \left(k \int_{S^2} w_k \, \mathrm{d}\Omega \right) - (\xi + k)^{1/2} \int_{S^2} \Omega \cdot \frac{\partial w_k}{\partial \boldsymbol{x}} \, \mathrm{d}\Omega \right]. \tag{53}$$

The asymptotic limit here is thus the non diffusive equation (51), describing streaming plus elastic scattering, with initial condition (53), for the bulk part of the hydrodynamic quantity (17), which depends here on direction Ω. Notice that integration of (51) over S^2 with respect to Ω eliminates elastic scattering, but it does not leave a self-consistent equation for the variable $\int_{S^2} \overline{\varphi}_0 \, \mathrm{d}\Omega$ (it is, indeed, the continuity equation).

References

[1] C. R. Garibotti and G. Spiga, *Boltzmann equation for inelastic scattering*, J. Phys. A: Math. Gen. **27** (1994), 2709-2717.

[2] A. Rossani, G. Spiga and R. Monaco, *Kinetic theory approach for two-level atoms interacting with monochromatic photons*, Mech. Res. Comm. **24** (1997), 237-242.

[3] A. Rossani and G. Spiga, *Kinetic theory with inelastic interactions*, Trans. Theory Statist. Phys. **27** (1998), 273-287.

[4] A. Rossani, *Fokker-Planck approximation of the linear Boltzmann equation with inelastic scattering: study of the distribution function for charged particles subjected to an external electric field*, Riv. Mat. Univ. Parma (5) **6** (1997), 35-45.

[5] A. Majorana, *Space homogeneous solutions of the Boltzmann equation describing electron-phonon interactions in semiconductors*, Trans. Theory Statist. Phys. **20** (1991), 261-279.

[6] A. Majorana, *Conservation laws from the Boltzmann equation describing electron-phonon interactions in semiconductors*, Trans. Theory Statist. Phys. **22** (1993), 849-859.

[7] P. Markowich, F. Poupaud and Ch. Schmeiser, *Diffusion approximation of nonlinear electron-phonon collision mechanisms*, RAIRO Model. Math. Anal. Numer. **29** (1995), 857-869.

[8] C. CERCIGNANI, *The Boltzmann Equation and its Applications*, Springer, New York 1988.

[9] J. R. MIKA and J. BANASIAK, *Singularly Perturbed Evolution Equations with Applications to Kinetic Theory*, World, Singapore 1995.

[10] A. V. BOBYLEV and G. SPIGA, *On a model transport equation with inelastic scattering*, SIAM J. Appl. Math. **58** (1998), 1128-1137.

[11] J. BANASIAK, G. FROSALI and G. SPIGA, *Asymptotic analysis for a particle transport equation with inelastic scattering in extended kinetic theory*, Math. Models Methods Appl. Sci. 8 (1998), 851-874.

[12] J. BANASIAK, *Mathematical properties of inelastic scattering models in the kinetic theory*, preprint.

[13] J. BANASIAK, G. FROSALI and G. SPIGA, *Asymptotic limit for particle transport in inelastically scattering media*, in preparation.

[14] G. L. BRAGLIA, *Teoria del trasporto elettronico in gas: processi di rilassamento*, Riv. Nuovo Cimento **18** (1995), 1-162.

[15] A. BELLENI-MORANTE, *A Concise Guide to Semigroups and Evolution Equations*, World, Singapore 1994.

Abstract

The asymptotic analysis of the linear Boltzmann equation with inelastic scattering is performed with respect to the proper Knudsen number for three physical situations characterized by different mutual relationships between elastic and inelastic collision terms and by negligible up-scattering. After establishing all necessary mathematical properties, the compressed asymptotic method by Mika and Banasiak is used to derive the first order approximate limiting equation, which does not always turn out to be of diffusive type.

* * *