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A note on *YJ*-injectivity (**)

Throughout R is an associative ring with identity and modules are unitary. The left and right annihilators of a subset X of R are denoted by $\mathbf{l}(X)$ and $\boldsymbol{r}(X)$, respectively. The Jacobson redical of R is denoted by $J(R)$. The terminology and notation not defined here can be found in Anderson and Fuller [**AF**].

A right R-module M is called $p-$*injective* [**Y7**], [**Y8**] if every right R-homomorphism $aR \to M$, $a \in R$, extends to $R \to M$; and it is called $YJ-$*injective* [**Y7**], [**Y8**] ($= GP-$*injective* in [**NKK**]) if for every $a \in R$ there exists $n \in \mathbb{N}$ with $a^n \neq 0$ and every right R-homomorphism $a^n R \to M$ extends to $R \to M$. The ring R is called *right* $p-$*injective* [**Y1**], [**Y5**], [**NY1**], [**PWY**], if the right R-module R_R is p-injective; equivalently if $\mathbf{l}\boldsymbol{r}(a) = Ra$ for each $a \in R$; and R is called *right* YJ $-$*injective* if the right R-module R_R is YJ-injective; equivalently if for every $a \in R$ there exists $n \in \mathbb{N}$ with $a^n \neq 0$ and $\mathbf{l}\boldsymbol{r}(a^n) = Ra^n$ (see [**Y3**], Lemma 3).

YJ-injective rings heve been studied in many papers such as [**Y2**], [**Y3**], [**Y4**], [**Y6**], [**Y7**], [**Y8**], [**C1**], [**DC**], and [**NKK**], but an example of a right *YJ*-injective ring which is not right p-injective is lacked in the literature. In this note we give such an example to show that the notion of *YJ*-injective rings is indeed a proper generalization of p-injective rings. Recently, Yue Chi Ming [**Y8**] has shown that R is a right self-injective regular ring if R contains an injective maximal right ideal and every simple right R-module is *YJ*-injective, thus answering a question of [**Y7**] in the affirmative. We shall give a different proof of Yue Chi Ming's result. To answer two other questions of Yue Chi Ming [**Y5**], [**Y7**], we present a right p-injective ring R with maximum condition on right annihlators but R is not right

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artinian; and a *PI*-ring R with an injective maximal right ideal but R is not right self-injective.

The ring R in the next example was essentially given by Clark [**C2**], who proved that R is left hereditary but $\operatorname{Soc}(R_R)$ is not projective, in response to a question of Xue [**X2**]. Here we have a different purpose.

Example 1. Let $\mathbb{Z}_2 = \{0, 1\}$ be the field with two elements.

Let A be the subring of $\mathbb{Z}_2^{\mathbb{N}}$ consisting of elements of the form

$$(a_1, a_2, \ldots, a_n, a, a, a, \ldots)$$

i.e., A is obtained by adjoining the identity of $\mathbb{Z}_2^{\mathbb{N}}$ to its ideal $\mathbb{Z}_2^{(\mathbb{N})}$. Then A is a commutative countable regular ring with each element an idempotent. If $k \in \mathbb{Z}_2$ and $(a_1, a_2, \ldots, a_n, a, a, a, \ldots) \in A$ we let $k(a_1, a_2, \ldots, a_n, a, a, a, \ldots) = ka$ then $\mathbb{Z}_2$ is a right A-module.

Let

$$R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ & A \end{bmatrix}.$$

(1) Take $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R$. Then $a^2 = 0$ and $\boldsymbol{r}(\mathbf{l}(a)) = \boldsymbol{r}\left(R\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R \neq aR$. Hence R is not left *YJ*-injective. However $\boldsymbol{r}(a) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ & \mathbb{Z}^{(\mathbb{N})} \end{bmatrix}$ and $\mathbf{l}(\mathbf{r}(a)) = \mathbf{l}\left(\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ & \mathbb{Z}_2^{(\mathbb{N})} \end{bmatrix}\right) = Ra$.

(2) Take $a = \begin{bmatrix} 0 & 1 \\ 0 & e \end{bmatrix} \in R$ with $0 \neq e \in \mathbb{Z}_2^{(\mathbb{N})}$. Then $\boldsymbol{r}(a) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ & 0 \end{bmatrix}$ and $\mathbf{l}(\boldsymbol{r}(a)) = R\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq Ra$. Hence R is not right p-injective. However, $a^2 = \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$ is a non-zero idempotent so $\mathbf{l}(\boldsymbol{r}(a^2)) = Ra^2$.

(3) If $a \in R$ is an idempotent then $\mathbf{l}(\boldsymbol{r}(a)) = Ra$.

(4) If $a \in R$ is left invertible then $\boldsymbol{r}(a) = 0$ and $\mathbf{l}(\boldsymbol{r}(a)) = \mathbf{l}(0) = R = Ra$.

Since each $a \in R$ is of the form (1), (2), (3), or (4), it follows from [**Y3**], Lemma 3, that R is right *YJ*-injective.

The right *YJ*-injective ring R in Example 1 is neither right p-injective nor left *YJ*-injective. This shows that the right *YJ*-injectivity is a proper generalization of the right p-injectivity, and that the notion of YJ-injectivity is not left-right symmetric.

It is known (see [**Y1**]) that R is regular if and only if every right R-module is p-injective. Therefore if R is regular then every right R-module is YJ-injective, but the converse is open [**Y3**], [**Y4**], [**Y7**], [**Y8**].

Call R *right quasi-duo* [**DC**], [**Y7**] if every maximal right ideal of R is two-sided. If R is right quasi-duo and every simple right R-module is YJ-injective then R is strongly regular by [**DC**], Theorem 3.7, [**NKK**], Theorem 10, [**Y7**], Theorem 2, or [**Y8**], Theorem 1. Yue Chi Ming raised the following question in [**Y7**], p. 554, Question 2: Is R strongly regular if R is right quasi-duo and every simple left R-module is YJ-injective? This has been answered in the affirmative in [**DC**], Theorem 3.7 or [**Y8**], Theorem 1.

If every simple right R-module is YJ-injective, Chen [**C1**], Lemma 1, proved that $J(R)$ is reduced, i.e., $J(R)$ contains no non-zero nilpotent elements, and Yue Chi Ming [**Y8**], Lemma 1, improved this by showing that $J(R) = 0$. The next result generalizes [**Y8**], Lemma 1 and answers [**NKK**], Question 1, partially.

Proposition 2. *Suppose every simple right R-module is YJ-injective. Then for each $0 \neq a \in R$ there exists $n \in \mathbb{N}$ with $a^n \neq 0$ such that $a^n \in a^n RaR$. Consequently, $J(R) = 0$.*

Proof. (1) Suppose a is nilpotent with $a^n \neq 0$ and $a^{n+1} = 0$. If $Ra^n R + \boldsymbol{r}(a^n) = R$ then $a^n Ra^n R = a^n R$ and $a^n \in a^n Ra^n R \subseteq a^n RaR$. If $Ra^n R + \boldsymbol{r}(a^n) \neq R$ let I be a maximal right ideal containing $Ra^n R + \boldsymbol{r}(a^n)$. Since R/I is YJ-injective and $(a^n)^2 = 0$, every R-homomorphism $a^n R \to R/I$ extends to $R_R \to R/I$. Define f: $a^n R \to R/I$ via $a^n r \mapsto r + I$. Then f is well-defined and an R-homomorphism. Therefore there exist $b + I \in R/I$ such that $r + I = f(a^n r) = (b + I)a^n r$. In particular, $1 + I = ba^n + I$. Since $ba^n \in I$ we obtain $1 \in I$, a contradiction.

(2) Suppose $a^n \neq 0$ for each $n \in \mathbb{N}$

(i) if $Ra^n R + \boldsymbol{r}(a^n) = R$ for some $n \in \mathbb{N}$ then $a^n Ra^n R = a^n R$ and $a^n \in a^n Ra^n R \subseteq a^n RaR$. So let

(ii) $Ra^n R + \boldsymbol{r}(a^n) \neq R$ for each $n \in \mathbb{N}$. If $\sum_{i=1}^{\infty} (Ra^i R + \boldsymbol{r}(a^i)) \neq R$ let I be a maximal right ideal containing $\sum_{i=1}^{\infty} (Ra^i R + \boldsymbol{r}(a^i))$. Since R/I is YJ-injective there exists $n \in \mathbb{N}$ such that every R-homomorphism $a^n R \to R/I$ extends to $R_R \to R/I$. Define f: $a^n R \to R/I$ via $a^n r \mapsto r + I$. Then as in (1) we obtain a contradiction. So we must have $\sum_{i=1}^{\infty} (Ra^i R + \boldsymbol{r}(a^i)) = R$. Since R_R is finitely generated $\sum_{i=1}^{n} (Ra^i R$

$+\mathbf{r}(a^i)) = R$ for some $n \in \mathbb{N}$. It follows that $RaR + \mathbf{r}(a^n) = R$. Then $a^n RaR = a^n R$ and $a^n \in a^n RaR$.

Suppose $0 \neq j \in J(R)$. Then there exists $n \in \mathbb{N}$ such that $0 \neq j^n \in j^n RjR$. Hence $j^n = j^n j'$ for some $j' \in J(R)$. Since $j^n(1 - j') = 0$ and $1 - j'$ is right invertible we obtain $j^n = 0$, a contradiction.

The following lemma has its own interest.

Lemma 3. *If M_R is p-injective and B_R is YJ-injective then $M \oplus B$ is YJ-injective.*

Proof. Let $0 \neq a \in R$. Since B is YJ-injective there exists $n \in \mathbb{N}$ such that $a^n \neq 0$ and every right R-homomorphism $a^n R \to B$ extends to $R \to B$. Suppose f: $a^n R \to M \oplus B$ is a right R-homomorphism. Let p_1: $M \oplus B \to M$ and p_2: $M \oplus B \to B$ be the projections. The right R-homomorphism $p_2 f$: $a^n R \to B$ extends to $R \to B$ so there exists $b \in B$ such that $p_2 f(a^n r) = ba^n r$ for each $a^n r \in a^n R$. Since M is p-injective the right R-homomorphism $p_1 f$: $a^n R \to M$ extends to $R \to M$, so there exists $m \in M$ such that $p_1 f(a^n r) = ma^n r$ for each $a^n r \in a^n R$. Then $(m, b) \in M \oplus B$ and for each $a^n r \in a^n R$ we have

$$f(a^n r) = (p_1 f(a^n r), p_2 f(a^n r)) = (ma^n r, ba^n r) = (m, b)\, a^n r$$

i.e., f: $a^n R \to M \oplus B$ extends to $R \to M \oplus B$.

Yue Chi Ming [**Y7**] call R a *right MI-ring* if R contains an injective maximal right ideal. The following question was raised in [**7**], p. 556, Question 4: Is R a right self-injective regular ring if R is a right MI-ring whose simple right R-modules are YJ-injective? Recently, Yue chi Ming ([**Y8**], Theorem 2)) has answered this in the affirmative. Here we give a different proof.

Theorem 4 [**Y8**]. *If R is a right MI-ring and every simple right R-module is YJ-injective then R is a right self-injective regular ring.*

Proof. Let M be an injective maximal right ideal. Then $R = M \oplus B$ where B is a simple right ideal. By Lemma 3, R is right YJ-injective. Hence R is right self-injective by [**Y7**], Proposition 4. It follows that $R/J(R)$ is regular. But $J(R) = 0$ by Proposition 2, so R is regular.

Yue Chi Ming ([**Y5**], p. 26) asked the following question: Is a right p-injective ring with maximum condition on right annihlators right artinian? (Such rings are left artinian by a symmetric version of [**R**], p. 205, Theorem). We answer this in the negative.

Example 5. Let K be a field with a ring monomorphism f into its subfield $f(K)$ such that K is infinite dimensional over $f(K)$; e.g., $K = F(x_1, x_2, x_3, \dots)$ with F a field, $f(x_i) = x_{i+1}$ and $f =$ identity on F. Define a K-bimodule ${}_K V_K$ as follows: ${}_K V = {}_K K$ as a left K-module; V_K is given as $vk = f(k)\,v$ for $v \in V$ and $k \in K$. Then $\dim({}_K V) = 1$ and $\dim(V_K) = \infty$. Let $R = K \propto_K V_K$ be the trivial extension. Then R is a local left artinian ring which is not right artinian. Hence R has maximum condition on right annihilators. We see that R has only three left ideals R, $J = J(R)$, and 0. Since $J = \mathbf{l}(\mathbf{r}(J))$, every left ideal is a left annihilator, so R is right p-injective.

The following question was raised in [**Y7**], p. 556, Question 3: Is R right self-injective if R is a *PI*-ring which is right *MI*? The answer is «No» by the next example.

Example 6. Let K be a field. Then $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ is an artinian hereditary ring which is a *PI*-ring since $R/J(R)$ is commutative. Now R is a right *MI*-ring since R has an injective maximal right ideal $\begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix}$ but R is not right self-injective. Similarly, one notes that R is also left *MI* but R is not left self-injective.

The ring R is *right mininjective* [**NY2**] if every R-homomorphism from a simple right ideal $aR \to R$ extends to $R_R \to R_R$. If aR is a simple right ideal of R then either $(aR)^2 = 0$ or $aR = eR$ for some idempotent $e \in R$. Hence every right *YJ*-injective ring is right mininjective. The coverse is not true since any ring R with $\mathrm{Soc}(R_R) = 0$ is right mininjective but need not be right *YJ*-injective (e.g., the ring $\mathbb{Z}$ of integers). We obtain the following two strict containments:

$$\{\text{right } p\text{-injective rings}\} \subset \{\text{right } YJ\text{-injective rings}\}$$

$$\subset \{\text{right mininjective rings}\}.$$

According to Rada and Saorin [**RS**], we call R a *right CF-ring* if every cyclic right R-module embeds in a free module. If I is a right ideal and R/I embeds in a free module then $I = \mathbf{r}\{a_1, \dots, a_n\}$ for $a_1, \dots, a_n \in R$. In particular, every right *CF*-ring is left p-injective. It is not known whether or not a right *CF*-ring is right artinian. The next result is [**RS**], Theorem 3.5. Here we give a short proof. The reader is referred to [**AF**] for a presentation of *QF* rings.

Theorem 7 [**RS**]. *Let R be a semiperfect ring. If R is right CF and right mininjective then R is QF.*

Proof. Since R is right *CF*, R is right Kasch (i.e., every simple right R-module embeds in R) and left p-injective. By [**NY2**], Corollary 2.6, we have $\mathrm{Soc}\,({}_R R) = \mathrm{Soc}\,(R_R)$. Using [**NY2**], Proposition 3.3 we see that $\mathrm{Soc}\,({}_R Re)$ is simple for any primitive idempotent e. So R is left minful [**NY2**], p. 563. By a version of [**NY2**], Theorem 3.7. R is left Kasch. By [**NY1**], Corollary 1.1 and Theorem 1.3 (2), R_R is finitely cogenerated. Since R is right *CF*, R is right artinian. By [**NY2**], Corollary 4.8, R is *QF*.

Corollary. 8. *Let R be a semiperfect ring. If R is right CF and right YJ-injective then R is QF.*

Call R *right duo* if every right ideal of R is two-sided. Puninski, Wisbauer and Yousif [**PWY**], Theorem 3.1, proved the following result for a right $p-$injective right duo ring R. Our result is a generalization since every idempotent in a right duo ring is central by [**X1**], Lemma 12.2.

Proposition 9. *Let e and f be two idempotents of R. If $ef = fe$ then both $eR \cap fR$ and $eR + fR$ are summands of R_R.*

Proof. Since $ef = fe$ one notes that

(1) ef is an idempotent and $eR \cap fR = efR$; and

(2) $e + f - ef$ is an idempotent and $eR + fR = (e + f - ef)\,R$.

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Abstract

Yue Chi Ming introduced the notions of p-injectivity and YJ-injectivity to study von Neumann regular rings. A right YJ-injective ring which is not right p-injective is given to show that the former is a proper generalization of the latter. Some results related to YJ-injective modules are also obtained

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