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ALESSANDRA DE ROSSI (*)

A posteriori error estimates for hierarchical methods (**)

1 - Introduction

A posteriori error estimates have been largely used in the last years in the numerical resolution of partial differential equations. Such estimates are not only important to determine the reliability of the method (an algorithm is defined reliable when the quantitativity control of error is guaranteed) but also to provide an adaptive optimization tool of the grid and for this reason they are at the basis of the local refinement schemes (*h* and *p refinement*). The literature on the subject is vast (see for instance [**12**], [**7**], [**20**], [**3**]).

In the numerical resolution of physical or technological problems it is possible that the global accuracy of the numerical approximation is deteriored by local singularity. An obvious remedy consists in refining the discretization near the critical regions, that is, to set more grid-points where the solution is less regular.

A priori error estimates are often insufficient and it is been proved the necessity of an error indicator which could be calculated a posteriori through the numerical solution of the problem.

Thanks to the analysis developed in [**7**], it is proved in this paper that if the discrete solution of an elliptic boundary value problem is expanded according to a multilevel decomposition, then its higher level component is an a-posteriori error indicator.

Let us suppose to solve a problem in a discretization space $\mathcal{V}_h$, corresponding

(*) Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy.
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to a mesh $\mathcal{T}_h$ (h is the discretization parameter). If we refine the mesh, that is we consider a $\overline{h}$ such that $\overline{h} < h$, we will obtain a new mesh $\mathcal{T}_{\overline{h}}$ and thus the discretization space will be enriched by the addition of new hierarchical basis functions to the set of functions already used for $\mathcal{V}_h$. Named $\mathcal{V}_{\overline{h}}$ the new space, we suppose that $\mathcal{V}_{\overline{h}}$ admits a hierarchical decomposition

$$\mathcal{V}_{\overline{h}} = \mathcal{V}_h \oplus \mathcal{W}_h,$$

where $\mathcal{W}_h$ is a space not necessarily orthogonal to $\mathcal{V}_h$ generated by the additional basis functions (the symbol $\oplus$ means a direct sum which may not be orthogonal). Moreover, the error indicator used is simply the component of the solution $u_{\overline{h}} \in \mathcal{V}_{\overline{h}}$ in the space $\mathcal{W}_h$, that is, written $u_{\overline{h}}$ in the form

$$u_{\overline{h}} = \widehat{u}_h + \widehat{e}_h,$$

where $\widehat{u}_h$ is the component of $u_{\overline{h}}$ in $\mathcal{V}_h$ and $\widehat{e}_h$ is the component of $u_{\overline{h}}$ in $\mathcal{W}_h$, we obtain estimates of the form

$$C_1 \,|||\, u - u_h \,||| \leq |||\, \widehat{e}_h \,||| \leq C_2 \,|||\, u - u_h \,|||\,,$$

where C_1, $C_2 > 0$ are constants of order one.

In this paper three cases are analysed:

- the case of a selfadjoint, positive definite variational form;
- the case of a variational form that is nonselfadjoint and indefinite;
- the case of a modified variational form.

We make the *saturation assumption*. This states that in energy norm the solution $u_{\overline{h}} \in \mathcal{V}_{\overline{h}}$ constitutes a better approximation to the exact solution u than $u_h \in \mathcal{V}_h$. A demonstration in the case of linear finite elements for the stabilized convection-diffusion problem is given in [**9**].

We also assume the *strengthened Cauchy-Schwarz inequality*. Such inequalities are widely used in the analysis of iterative methods based on hierarchical finite element bases (see for instance [**21**], [**5**], [**11**]) and in the analysis of a posteriori error estimates. A quite number of forms and proofs have been given in the case of finite elements, corresponding to different uses (see for example [**5**], [**11**], [**18**]).

In the case of not selfadjoint problems we must add the *continuity assumption* and the *inf-sup condition*.

These results are applied to the stabilized convection-diffusion problem in the last paragraph and they justify the wavelet-based adaptive finite element method ([8]).

2 - The selfadjoint case

We consider the solution of the selfadjoint variational problem

$$\begin{cases} \text{find } u \in \mathcal{V} \text{ such that} \\ a(u, v) = f(v), \qquad \forall v \in \mathcal{V}, \end{cases}$$

where $\mathcal{V}$ is an appropriate Hilbert space, $a(\cdot, \cdot)$ is a positive definite bilinear form and $f(\cdot)$ is a linear functional. The energy norm associated with $a(\cdot, \cdot)$ is denoted by

$$|||u|||^2 = a(u, u).$$

Let $\mathcal{V}_h \subset \mathcal{V}$ be a finite dimensional subspace and consider the approximate problem

$$\begin{cases} \text{find } u_h \in \mathcal{V}_h \text{ such that} \\ a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathcal{V}_h. \end{cases} \tag{2.1}$$

The solution of (2.1) satisfies the *best approximation* property

$$|||u - u_h||| = \inf_{v_h \in \mathcal{V}_h} |||u - v_h|||.$$

Now we define a larger space $\mathcal{V}_{\bar{h}}$, $\bar{h} < h$, that is a space such that $\mathcal{V}_h \subset \mathcal{V}_{\bar{h}} \subset \mathcal{V}$. In this space we have the approximate solution $u_{\bar{h}}$ satisfying

$$a(u_{\bar{h}}, v_{\bar{h}}) = f(v_{\bar{h}}), \qquad \forall v_{\bar{h}} \in \mathcal{V}_{\bar{h}} \tag{2.2}$$

and

$$|||u - u_{\bar{h}}||| = \inf_{v_{\bar{h}} \in \mathcal{V}_{\bar{h}}} |||u - v_{\bar{h}}|||.$$

The approximate solution $u_{\bar{h}}$ is not computed but it is important in the theoretical analysis of the a posteriori error estimate for u_h. In particular we assume that the approximate solution $u_{\bar{h}}$ converges to u more rapidly than u_h. This is expressed in terms of the *saturation assumption*

$$|||u - u_{\bar{h}}||| \leq \beta \, |||u - u_h||| \tag{2.3}$$

where $\beta < 1$ independent of h.

Now we decompose the space $\mathcal{V}_{\bar{h}}$ in the direct sum of $\mathcal{V}_h$ and $\mathcal{W}_h$, where $\mathcal{W}_h$ is an appropriate supplementary not necessary orthogonal

$$\mathcal{V}_{\bar{h}} = \mathcal{V}_h \oplus \mathcal{W}_h . \tag{2.4}$$

Moreover we assume the *strengthened Cauchy-Schwarz inequality* for the decomposition

$$|a(v, w)| \leq \gamma ||| v ||| \; ||| w ||| , \qquad \forall v \in \mathcal{V}_h, \forall w \in \mathcal{W}_h, \tag{2.5}$$

where $\gamma < 1$ independent of h.

Now we want estimate the error $u - u_h$ by the component of $u_{\bar{h}}$ in $\mathcal{W}_h$. Decompose the approximate solution $u_{\bar{h}}$ as

$$u_{\bar{h}} = \widehat{u}_h + \widehat{e}_h ,$$

where $\widehat{u}_h \in \mathcal{V}_h$ and $\widehat{e}_h \in \mathcal{W}_h$. We want determine an estimate of the form

$$C_1 ||| u - u_h ||| \leq ||| \widehat{e}_h ||| \leq C_2 ||| u - u_h ||| , \tag{2.6}$$

where C_1, $C_2 > 0$ are constants independent of h. To make this we consider the decomposition

$$\mathcal{V}_{\bar{h}} = \mathcal{V}_h \oplus \mathcal{W}^*_{\bar{h}} , \tag{2.7}$$

where $\mathcal{W}^*_{\bar{h}}$ is the orthogonal space to $\mathcal{V}_h$ in $\mathcal{V}_{\bar{h}}$. By this decomposition we have

$$u_{\bar{h}} = P_{\mathcal{V}_h} u_{\bar{h}} + P_{\mathcal{W}^*_{\bar{h}}} u_{\bar{h}} ,$$

where $P_{\mathcal{V}_h}$ and $P_{\mathcal{W}^*_{\bar{h}}}$ indicate the orthogonal projections on $\mathcal{V}_h$ and $\mathcal{W}^*_{\bar{h}}$, respectively. Note that the unicity of the solution leads to

$$P_{\mathcal{V}_h} u_{\bar{h}} = u_h .$$

Infact, set $P_{\mathcal{V}_h} u_{\bar{h}} = z_h$, we have

$$a(z_h, v_h) = a(u_{\bar{h}}, v_h) = f(v_h), \qquad \forall v_h \in \mathcal{V}_h .$$

Denoted $P_{\mathcal{W}^*_{\bar{h}}} u_{\bar{h}}$ with e_h thus we obtain

$$e_h = u_{\bar{h}} - u_h .$$

After the demonstration of the following lemma, we will formulate an estimate of the form (2.6).

Lemma 2.1. Let $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ *be solution of* (2.2), *with* $u_{\bar{h}} = \widehat{u}_h + \widehat{e}_h$, *where* $\widehat{u}_h$

$\in \mathcal{V}_h$ and $\widehat{e}_h \in \mathcal{W}_h$. Moreover, let e_h be the orthogonal projection $u_{\bar{h}}$ in $\mathcal{W}_h^*$, where $\mathcal{W}_h^* \subset \mathcal{V}_{\bar{h}}$ is the orthogonal complement of $\mathcal{V}_h$. Then

$$|||\, e_h \,|||^2 \leqslant |||\, \widehat{e}_h \,|||^2 \leqslant \frac{1}{1-\gamma^2}\, |||\, e_h \,|||^2 .$$

Proof. By the definitions given above we compute

$$|||\, e_h \,|||^2 = |||\, u_{\bar{h}} - u_h \,|||^2 = |||\, \widehat{u}_h + \widehat{e}_h - u_h \,|||^2 = |||\, (\widehat{u}_h - u_h) + \widehat{e}_h \,|||^2$$

$$= |||\, \widehat{u}_h - u_h \,|||^2 + |||\, \widehat{e}_h \,|||^2 + 2a(\widehat{u}_h - u_h, \widehat{e}_h)$$

$$\geqslant |||\, \widehat{u}_h - u_h \,|||^2 + |||\, \widehat{e}_h \,|||^2 - 2\gamma\, |||\, \widehat{u}_h - u_h \,|||\ |||\, \widehat{e}_h \,|||$$

$$\geqslant |||\, \widehat{u}_h - u_h \,|||^2 + |||\, \widehat{e}_h \,|||^2 - 2\left[\frac{1}{2}\, |||\, \widehat{u}_h - u_h \,|||^2 + \frac{1}{2}\gamma^2\, |||\, \widehat{e}_h \,|||^2\right] = |||\, \widehat{e}_h \,|||^2 (1-\gamma^2).$$

This proves the right-hand side inequality. The left-hand side one is obtained by the definitions of $\mathcal{W}_h$ and $\mathcal{W}_h^*$. ■

Theorem 2.2. *Let $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ be the solution of (2.2). Decompose $u_{\bar{h}}$ as $u_{\bar{h}} = u_h + e_h$ and $u_{\bar{h}} = \widehat{u}_h + \widehat{e}_h$, where $u_h \in \mathcal{V}_h$, $e_h \in \mathcal{W}_h^*$, $\widehat{u}_h \in \mathcal{V}_h$, $\widehat{e}_h \in \mathcal{W}_h$. We have*

$$(1-\beta^2)\, |||\, u - u_h \,|||^2 \leqslant |||\, \widehat{e}_h \,|||^2 \leqslant \frac{1}{1-\gamma^2}\, |||\, u - u_h \,|||^2 . \tag{2.8}$$

Proof. First we prove that

$$(1-\beta^2)\, |||\, u - u_h \,|||^2 \leqslant |||\, e_h \,|||^2 \leqslant |||\, u - u_h \,|||^2 . \tag{2.9}$$

Considering the left-hand side inequality we have

$$|||\, u - u_h \,|||^2 = |||\, (u - u_{\bar{h}}) + (u_{\bar{h}} - u_h) \,|||^2 = |||\, u - u_{\bar{h}} \,|||^2 + |||\, e_h \,|||^2 + a(u - u_{\bar{h}}, e_h)$$

$$= |||\, u - u_{\bar{h}} \,|||^2 + |||\, e_h \,|||^2 \leqslant \beta^2\, |||\, u - u_h \,|||^2 + |||\, e_h \,|||^2 .$$

and with regard to the right-hand side one

$$|||\, e_h \,|||^2 = a(u_{\bar{h}} - u_h, u_{\bar{h}} - u_h) = a(u - u_h, u_{\bar{h}} - u_h) \leqslant |||\, u - u_h \,|||\ |||\, e_h \,||| .$$

Lemma 2.1 implies the (2.8). ■

3 - The nonselfadjoint, indefinite case

Now we want to generalize the results above to the nonselfadjoint and indefinite problem

$$\begin{cases} \text{find } u \in \mathcal{V} \text{ such that} \\ A(u, v) = f(v), \qquad \forall v \in \mathcal{V}, \end{cases}$$

where $A(\cdot, \cdot)$ is a bilinear form and $f(\cdot)$ is a linear functional. The energy norm $|||\cdot|||$ is associated with a positive definite bilinear form $a(\cdot, \cdot)$. Let $\mathcal{V}_h$ and $\mathcal{V}_{\bar{h}}$ be two finite-dimensional subspaces of $\mathcal{V}$ with $\mathcal{V}_h \subset \mathcal{V}_{\bar{h}}$. Let us consider the approximate problems

$$\begin{cases} \text{find } u_h \in \mathcal{V}_h \text{ such that} \\ A(u_h, v_h) = f(v_h), \qquad \forall v_h \in \mathcal{V}_h \end{cases} \tag{3.1}$$

and

$$\begin{cases} \text{find } u_{\bar{h}} \in \mathcal{V}_{\bar{h}} \text{ such that} \\ A(u_{\bar{h}}, v_{\bar{h}}) = f(v_{\bar{h}}), \qquad \forall v_{\bar{h}} \in \mathcal{V}_{\bar{h}}. \end{cases} \tag{3.2}$$

We suppose that $A(\cdot, \cdot)$ satisfies the *continuity condition*: $\exists \mu > 0$ such that

$$|A(u, v)| \le \mu\, |||u|||\ |||v|||, \qquad \forall u, v \in \mathcal{V} \tag{3.3}$$

and the *inf-sup condition*: $\exists \nu$ such that

$$\inf_{\substack{u \in \mathcal{S} \\ |||u||| = 1}} \ \sup_{\substack{v \in \mathcal{S} \\ |||v||| \le 1}} A(u, v) \ge \nu > 0 \tag{3.4}$$

where $\mathcal{S} = \mathcal{V}, \mathcal{V}_h, \mathcal{V}_{\bar{h}}$. This ensures that all variational problems considered will have unique solutions. We suppose that the solutions u_h e $u_{\bar{h}}$ converge to u and that the saturation assumption (2.3) holds. Moreover let $\mathcal{W}_h$ and $\mathcal{W}_{\bar{h}}^*$ be defined as above and assume the strengthened Cauchy-Schwarz inequality (2.5).

To obtain such an estimate as theorem 2.2 for the nonselfadjoint and indefinite case let us decompose the solution in the following way

$$u_{\bar{h}} = P_{\mathcal{V}_h} u_{\bar{h}} + P_{\mathcal{W}_{\bar{h}}^*} u_{\bar{h}}$$

(note that in this case u_h is not equal to $P_{\mathcal{V}_h} u_{\bar{h}}$) and set

$$e_h = P_{\mathcal{W}_{\bar{h}}^*} u_{\bar{h}}.$$

We observe that the hypotheses made ensure that the theorem 2.1 holds whenever $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ is the solution of (3.2) with $u_{\bar{h}} = \widehat{u}_h + \widehat{e}_h$, where $\widehat{u}_h \in \mathcal{V}_h$ and $\widehat{e}_h \in \mathcal{W}_h$ (let us remember that in this case $P_{\mathcal{V}_h} u_{\bar{h}}$ is not equal to u_h). Then we must prove an estimate of the type (2.9) of theorem 2.2.

Lemma 3.1. *Let $u_h \in \mathcal{V}_h$ and $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ be solutions of the approximate problems* (3.1) *and* (3.2), *respectively, and suppose for u_h and $u_{\bar{h}}$ the saturation assumption* (2.3). *Then we have*

$$(3.5) \qquad (1-\beta)\, ||| u - u_h ||| \leq ||| u_{\bar{h}} - u_h ||| \leq (1+\beta)\, ||| u - u_h |||.$$

Proof. Triangle inequality and saturation assumption lead easily to (3.5). Infact the left-hand side inequality is given by

$$||| u - u_h ||| \leq ||| u - u_{\bar{h}} ||| + ||| u_{\bar{h}} - u_h ||| \leq \beta\, ||| u - u_h ||| + ||| u_{\bar{h}} - u_h |||$$

while

$$||| u_{\bar{h}} - u_h ||| \leq ||| u_{\bar{h}} - u ||| + ||| u_h - u ||| \leq \beta\, ||| u - u_h ||| + ||| u_h - u |||$$

yields to the right-hand side one. ■

Lemma 3.2. *Let $\mathcal{V}_h$, $\mathcal{V}_{\bar{h}}$ and $\mathcal{W}_h$ be defined as above with $\mathcal{V}_{\bar{h}} = \mathcal{V}_h \oplus \mathcal{W}_{\bar{h}}^*$; moreover we assume* (2.3), (2.5), (3.3), (3.4). *Then we have*

$$\frac{\nu^4}{\mu^2(\mu+\nu)^2}(1-\gamma^2)(1-\beta)^2\, ||| u - u_h |||^2 \leq ||| e_h |||^2 \leq (1+\beta)^2\, ||| u - u_h |||^2.$$

Proof. We know that

$$||| e_h ||| = ||| u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}} ||| \leq ||| u_{\bar{h}} - u_h |||\,.$$

and then, by right-hand side inequality (3.5),

$$||| e_h |||^2 \leq ||| u_{\bar{h}} - u_h |||^2 \leq (1+\beta)^2\, ||| u - u_h |||^2.$$

To obtain the left-hand side inequality we consider the approximate equations

$$A(u_h, v_h) = f(v_h), \qquad \forall v_h \in \mathcal{V}_h,$$

and

$$A(u_{\bar{h}}, v_{\bar{h}}) = f(v_{\bar{h}}), \qquad \forall v_{\bar{h}} \in \mathcal{V}_{\bar{h}},$$

which implies

$$A(u_{\bar{h}} - u_h, v_h) = 0 , \qquad \forall v_h \in \mathcal{V}_h. \tag{3.6}$$

Moreover, by definition of e_h we have

$$A(e_h, v_{\bar{h}}) = A(u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}}, v_{\bar{h}}), \qquad \forall v_{\bar{h}} \in \mathcal{V}_{\bar{h}}, \tag{3.7}$$

and taking $v_h = v_{\bar{h}} \in \mathcal{V}_h$, and using (3.6), we obtain

$$A(u_h - P_{\mathcal{V}_h} u_{\bar{h}}, v_h) = A(e_h, v_h), \qquad \forall v_h \in \mathcal{V}_h. \tag{3.8}$$

Then observe that, if $v \in \mathcal{V}_h$ and $w \in \mathcal{W}_h$, with $|||\, v + w \,||| = 1$, we have

$$1 = |||\, v + w \,|||^2 = |||\, v \,|||^2 + |||\, w \,|||^2 + 2a(v, w)$$

$$\geqslant |||\, v \,|||^2 + |||\, w \,|||^2 - 2\gamma \,|||\, v \,|||\; |||\, w \,||| \geqslant (1 - \gamma^2) \,|||\, w \,|||^2.$$

Thus

$$|||\, w \,|||^2 \leqslant \frac{1}{1 - \gamma^2} . \tag{3.9}$$

Now, applying (3.6), (3.7) and (3.9), we obtain

$$\nu \,|||\, u_{\bar{h}} - u_h \,||| \leqslant \sup_{|||\, v + w \,||| = 1} A(u_{\bar{h}} - u_h, v + w) = \sup_{|||\, v + w \,||| = 1} A(u_{\bar{h}} - u_h, w)$$

$$= \sup_{|||\, v + w \,||| = 1} \{A(u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}}, w) + A(P_{\mathcal{V}_h} u_{\bar{h}} - u_h, w)\}$$

$$= \sup_{|||\, v + w \,||| = 1} \{A(e_h, w) + A(P_{\mathcal{V}_h} u_{\bar{h}} - u_h, w)\}$$

$$\leqslant \frac{\mu}{\sqrt{1 - \gamma^2}} \,|||\, e_h \,||| + \frac{\mu}{\sqrt{1 - \gamma^2}} \,|||\, u_h - P_{\mathcal{V}_h} u_{\bar{h}} \,|||.$$

It remains to prove that there $\exists K > 0$ such that

$$|||\, u_h - P_{\mathcal{V}_h} u_{\bar{h}} \,||| \leqslant K \,|||\, e_h \,||| \;;$$

but, by (3.8), we have

$$\nu \, ||| u_h - P_{\mathcal{V}_h} u_{\bar{h}} ||| \leq \sup_{\substack{v_h \in \mathcal{V}_h \\ ||| v_h ||| = 1}} A(u_h - P_{\mathcal{V}_h} u_{\bar{h}}, v_h) = \sup_{\substack{v_h \in \mathcal{V}_h \\ ||| v_h ||| = 1}} A(e_h, v_h) \leq \mu \, ||| e_h ||| .$$

Thus

$$||| e_h |||^2 \geq \frac{\nu^4}{\mu^2} \frac{1-\gamma^2}{(\mu+\nu)^2} \, ||| u_{\bar{h}} - u_h |||^2$$

and, using the left-hand side inequality of (3.5), in conclusion we obtain

$$||| e_h |||^2 \geq \frac{\nu^4}{\mu^2(\mu+\nu)^2} (1-\gamma^2)(1-\beta)^2 \, ||| u - u_h |||^2 . \qquad \blacksquare$$

Theorem 3. *Let* $u_h \in \mathcal{V}_h$ *and* $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ *be solutions of the problems* (3.1) *and* (3.2), *respectively; let* $\hat{e}_h$ *be the component of* $u_{\bar{h}}$ *in* $\mathcal{W}_h$. *Moreover assume hypotheses* (2.3), (2.5), (3.3), (3.4). *Then we have*

$$\frac{\nu^4}{\mu^2(\mu+\nu)^2} (1-\gamma^2)(1-\beta)^2 \, ||| u - u_h |||^2 \leq ||| \hat{e}_h |||^2 \leq \frac{(1+\beta)^2}{1-\gamma^2} \, ||| u - u_h |||^2 .$$

Proof. It follows immediately from lemmas 2.1, 3.2. $\blacksquare$

4 - The case of a modified variational form

We consider the variational problem

$$\begin{cases} \text{find } u \in \mathcal{V} \text{ such that} \\ A(u, v) = f(v), \qquad \forall v \in \mathcal{V}, \end{cases}$$

where $A(\cdot, \cdot)$ is a bilinear form and $f(\cdot)$ is a linear functional. We assume the hypotheses (3.3), (3.4) with ν and μ continuity and inf-sup constants, respectively, of the form A. Let A_h be a bilinear form depending on the discretization parameter $h > 0$, more precisely

$$A_h \colon \mathcal{V}_h \times \mathcal{V}_h \to \boldsymbol{R} ,$$

$$A_h(\cdot, \cdot) = A(\cdot, \cdot) + \delta(h) \, a(\cdot, \cdot) , \tag{4.1}$$

where δ is a non-decreasing function, $a(\cdot, \cdot)$ is an inner product satisfying the strengthened Cauchy-Schwarz inequality (2.5), and the energy norm is associated to the inner product. We suppose that A_h satisfies the continuity hypothesis: $\forall h > 0\ \exists \mu_h > 0$ such that

$$|A_h(u_h, v_h)| \leq \mu_h \||| u_h \||| \||| v_h \|||, \qquad \forall u_h, v_h \in \mathcal{V}_h,$$

and the inf-sup condition: $\forall h > 0$, $\exists \nu_h > 0$ such that

$$\inf_{\substack{u_h \in \mathcal{S} \\ \||| u_h \||| = 1}} \sup_{\substack{v_h \in \mathcal{S} \\ \||| v_h \||| \leq 1}} A_h(u_h, v_h) \geq \nu_h,$$

where $\mathcal{S} = \mathcal{V}_h, \mathcal{V}_{\bar{h}}$. Now we consider the approximate problem for $h > 0$

$$\text{(4.2)} \qquad \begin{cases} \text{find } u_h \in \mathcal{V}_h \text{ such that} \\ A_h(u_h, v_h) = f(v_h), \qquad \forall v_h \in \mathcal{V}_h. \end{cases}$$

From now on u_h and $u_{\bar{h}}$ (with $\bar{h} < h$) will denote the solutions of two approximate problems as (4.2). We assume that u_h and $u_{\bar{h}}$ converge to u and that the saturation assumption (2.3) holds.

Then, for the problem (4.2), we can formulate a result similar to the one already proved for the previous cases.

Theorem 4.1. *The solution* $u_h \in \mathcal{V}_h$ *of the problem* (4.2) *satisfies the following a posteriori error estimate*

$$\text{(4.3)} \qquad \left(\frac{\nu_{\bar{h}} \nu_h}{2\mu_{\bar{h}}(\nu_h + \mu + \delta(h))} \right)^2 (1-\gamma^2)(1-\beta)^2 \||| u - u_h \|||^2 \leq \||| \hat{e}_h \|||^2$$

$$\leq \frac{(1+\beta)^2}{1-\gamma^2} \||| u - u_h \|||^2.$$

Proof. First we observe that also in this case the lemma 2.1 and 3.1 hold (formulated for $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ solution of the problem (4.2) written for $\bar{h}$, with $u_{\bar{h}} = \hat{u}_h + \hat{e}_h$). To prove the right-hand side inequality we use the lemmas 2.1 and 3.1. To demonstrate the left-hand side one, we apply (3.9) with $v \in \mathcal{V}_h$ and $w \in \mathcal{W}_h$, $\||| v + w \||| = 1$; we have

$$\nu_{\bar{h}} \||| u_{\bar{h}} - u_h \||| \leq \sup_{\||| v+w \||| = 1} A_{\bar{h}}(u_{\bar{h}} - u_h, v + w)$$

$$= \sup_{\||| v+w \||| = 1} \{ A_{\bar{h}}(u_{\bar{h}} - u_h, v) + A_{\bar{h}}(u_{\bar{h}} - u_h, w) \}$$

$$= \sup_{|||v+w|||=1} A_{\bar{h}}(u_{\bar{h}} - u_h, v) + \sup_{|||v+w|||=1} A_{\bar{h}}(u_{\bar{h}} - u_h, w)$$

$$= \sup_{|||v+w|||=1} \{A_{\bar{h}}(u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}}, v) + A_{\bar{h}}(P_{\mathcal{V}_h} u_{\bar{h}} - u_h, v)\}$$

$$+ \sup_{|||v+w|||=1} \{A_{\bar{h}}(u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}}, w) + A_{\bar{h}}(P_{\mathcal{V}_h} u_{\bar{h}} - u_h, w)\}$$

$$< 2 \frac{\mu_{\bar{h}}}{\sqrt{1-\gamma^2}} |||e_h||| + 2 \frac{\mu_{\bar{h}}}{\sqrt{1-\gamma^2}} |||u_h - P_{\mathcal{V}_h} u_{\bar{h}}|||.$$

Then we have to prove that: $\exists K > 0$ such that

$$|||u_h - P_{\mathcal{V}_h} u_{\bar{h}}||| \leq K |||e_h||| .$$

Let us observe that from problem (4.2) we obtain

$$A_{\bar{h}}(u_{\bar{h}}, v_h) = A_h(u_h, v_h), \qquad \forall v_h \in \mathcal{V}_h.$$

Then we have

$$\nu_h |||u_h - P_{\mathcal{V}_h} u_{\bar{h}}||| \leq \sup_{|||v_h|||=1} A_h(u_h - P_{\mathcal{V}_h} u_{\bar{h}}, v_h)$$

$$= \sup_{|||v_h|||=1} \{A_{\bar{h}}(u_{\bar{h}}, v_h) - A_h(P_{\mathcal{V}_h} u_{\bar{h}}, v_h)\} = \sup_{|||v_h|||=1} \{A(u_{\bar{h}}, v_h) - A(P_{\mathcal{V}_h} u_{\bar{h}}, v_h)$$

$$+ \delta(\bar{h}) a(u_{\bar{h}}, v_h) - \delta(h) a(P_{\mathcal{V}_h} u_{\bar{h}}, v_h)\},$$

and being

$$\max\{\delta(\bar{h}), \delta(h)\} = \delta(h),$$

we obtain

$$\nu_h |||u_h - P_{\mathcal{V}_h} u_{\bar{h}}||| \leq \nu |||e_h||| + \delta(h) \sup_{|||v_h|||=1} a(u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}}, v_h)$$

$$\leq \mu |||e_h||| + \gamma\delta(h) |||u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}}||| = (\mu + \gamma\delta(h)) |||e_h|||.$$

This leads to

$$|||u_{\bar{h}} - u_h||| \leq \frac{1}{\sqrt{1-\gamma^2}} \frac{2\mu_{\bar{h}}}{\nu_{\bar{h}}} \left(1 + \frac{\mu + \gamma\delta(h)}{\nu_h}\right) |||e_h||| ,$$

which implies

$$\frac{\nu_{\bar{h}}\nu_h\sqrt{1-\gamma^2}}{2\mu_{\bar{h}}(\nu_h+\mu+\gamma\delta(h))}\ |||\, u_{\bar{h}}-u_h\,|||\ \leqslant\ |||\, e_h\,|||.$$

To conclude the proof we apply to the lemmas 2.1 and 3.1. ■

Now we extend the results already proved considering a bilinear form B_h depending on h, more general than the one considered untill now. More precisely let $\mathcal{V}$, $\mathcal{H}$ be appropriate Hilbert spaces with norms $|||\cdot|||$ and $|\cdot|$, respectively, such that $\mathcal{V}\subset\mathcal{H}$, and let

$$B_h\colon \mathcal{V}_h\times\mathcal{U}_h\to\boldsymbol{R}$$

be a bilinear form, where $\mathcal{V}_h$ and $\mathcal{U}_h$ are finite-dimensional subspaces of $\mathcal{V}$ and $\mathcal{H}$, respectively, with

$$\dim\ \mathcal{U}_h=\dim\ \mathcal{V}_h.$$

Let us consider the variational approximate problem for $h>0$

$$\begin{cases}\text{find } u_h\in\mathcal{V}_h \text{ such that}\\ B_h(u_h,v_h)=f(v_h), \qquad \forall v_h\in\mathcal{U}_h,\end{cases} \tag{4.4}$$

where $f(\cdot)$ is a linear functional on $\mathcal{H}$.

Let $b_h(\cdot,\cdot)$ be an inner product on $\mathcal{H}$ and define *h-norm* the associated norm

$$|||\, v\,|||_h^2=b_h(v,v), \qquad \forall v\in\mathcal{H}.$$

Let us suppose that, for the decomposition $\mathcal{V}_{\bar{h}}=\mathcal{V}_h\oplus\mathcal{W}_h$, the strengthened Cauchy-Schwarz inequality holds: $\exists\gamma_b<1$ independent of h such that

$$|b_h(u_h,w_h)|\leqslant\gamma_b\,|||\, u_h\,|||_h\,|||\, w_h\,|||_h, \qquad \forall u_h\in\mathcal{V}_h,\ \forall w_h\in\mathcal{W}_h. \tag{4.5}$$

Moreover we assume verified the saturation assumption: $\exists\beta<1$ independent of h such that

$$|||\, u-u_{\bar{h}}\,|||_h\leqslant\beta\,|||\, u-u_h\,|||_h \tag{4.6}$$

and we hypothese that the bilinear form B_h satisfies the continuity condition in $\mathcal{V}_h\times\mathcal{U}_h$: $\forall h>0\ \exists\mu_h>0$ such that

$$|B_h(u_h,v_h)|\leqslant\mu_h\,|||\, u_h\,|||_h\,|||\, v_h\,|||_h, \qquad \forall u_h\in\mathcal{V}_h,\ \forall v_h\in\mathcal{U}_h, \tag{4.7}$$

and the inf-sup condition: $\forall h > 0,\ \exists \nu_h > 0$ such that

$$\nu_h ||| u_h |||_h \leq \sup_{\substack{v_h \in \mathcal{U}_h \\ ||| v_h |||_h = 1}} B_h(u_h, v_h), \qquad \forall u_h \in \mathcal{V}_h. \tag{4.8}$$

Then we have

Theorem 4.2. *For the solution $u_h \in \mathcal{V}_h$ of the problem* (4.4) *the a posteriori error estimate*

$$\left(\frac{\nu_{\bar{h}} \nu_h}{2\mu_{\bar{h}}(\nu_h + \mu_{\bar{h}})} \right)^2 (1-\gamma_b^2)(1-\beta)^2 ||| u-u_h |||_h^2 \leq ||| \hat{e}_h |||_h^2 \leq \frac{(1+\beta)^2}{1-\gamma_b^2} ||| u-u_h |||_h^2 \tag{4.9}$$

holds.

Proof. We have, by (4.5),

$$||| e_h |||_h^2 \leq ||| \hat{e}_h |||_h^2 \leq \frac{1}{1-\gamma_b^2} ||| e_h |||_h^2. \tag{4.10}$$

By triangle inequality and saturation assumption

$$(1-\beta)^2 ||| u - u_h |||_h^2 \leq ||| u_{\bar{h}} - u_h |||_h^2 \leq (1+\beta)^2 ||| u - u_h |||_h^2. \tag{4.11}$$

The right-hand side estimate of (4.9) is obtained applying (4.10) and (4.11). To have the left-hand side inequality the method is the same used in theorem 4.1. In particular to prove that $\exists K > 0$ such that

$$||| u_h - P_{\mathcal{V}_h} u_{\bar{h}} |||_h \leq K ||| e_h |||_h$$

we operate in the following way. We observe that

$$B_{\bar{h}}(u_{\bar{h}}, v_h) = B_h(u_h, v_h), \qquad \forall v_h \in \mathcal{U}_h,$$

and thus

$$\nu_h ||| u_h - P_{\mathcal{V}_h} u_{\bar{h}} |||_h \leq \sup_{||| v_h |||_h = 1} B_h(u_h - P_{\mathcal{V}_h} u_{\bar{h}}, v_h)$$

$$= \sup_{||| v_h |||_h = 1} \{B_{\bar{h}}(u_{\bar{h}}, v_h) - B_h(P_{\mathcal{V}_h} u_{\bar{h}}, v_h)\} \leq \mu_{\bar{h}} ||| u_{\bar{h}} |||_h - \nu_h ||| P_{\mathcal{V}_h} u_{\bar{h}} |||_h$$

but

$$\max\{\mu_{\bar{h}}, \nu_h\} = \mu_{\bar{h}},$$

and then we obtain

$$||| u_h - P_{\mathcal{V}_h} u_{\bar{h}} |||_h \leq \frac{\mu_{\bar{h}}}{\nu_h} ||| u_{\bar{h}} - P_{\mathcal{V}_h} u_{\bar{h}} |||_h = \frac{\mu_{\bar{h}}}{\nu_h} ||| e_h |||_h .$$

In conclusion we use again the (4.10) and (4.11). ■

5 - Examples: the stabilized convection-diffusion problem

We consider the numerical solution of the convection-diffusion equation

$$\text{(5.1)} \qquad \begin{cases} -\nu \Delta u + \underline{\beta} \cdot \nabla u = f & \text{in } \Omega , \\ u = 0 & \text{in } \partial\Omega , \end{cases}$$

where Ω is a polygonal region in $\boldsymbol{R}^2$, ν is a positive constant coefficient, named diffusion coefficient, $\underline{\beta} = \underline{\beta}(x) \in \boldsymbol{R}^2$ is a velocity vector, such that $\frac{1}{2} \nabla \cdot \underline{\beta} \geq 0$.

A weak formulation of (5.1) is:

$$\text{(5.2)} \qquad \begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ A(u, v) = (f, v), \qquad \forall v \in H_0^1(\Omega) , \end{cases}$$

where $H_0^1(\Omega)$ is the usual subspace of the Sobolev space $H^1(\Omega)$ whose elements satisfy the homogeneous Dirichlet boundary condition, and

$$A(u, v) = \int_\Omega \nabla u \cdot (\nu \nabla v + \underline{\beta} v) \, \mathrm{d}x ,$$

$$(f, v) = \int_\Omega f v \, \mathrm{d}x .$$

The standard Galerkin method is known to be not a satisfying method if the exact solution is not regular. To remedy this situation we can make use of upwind discretization methods; such methods modify the bilinear form $A(\cdot, \cdot)$ through the addition of stabilization terms. In the *classical artificial diffusion method*, the bilinear form considered is

$$A_h : \mathcal{V}_h \times \mathcal{V}_h \to \boldsymbol{R} ,$$

$$\text{(5.3)} \qquad A_h(u_h, v_h) = \int_\Omega \nabla u_h \{ (\nu + \delta(h)) \nabla v_h + \underline{\beta} v_h \} \, \mathrm{d}x , \qquad \forall u_h, v_h \in \mathcal{V}_h$$

where $\delta(h)$ is the artificial diffusion coefficient depending on the discretization parameter h, $\mathcal{V}_h$ is a linear finite element space in $\Omega \subset \boldsymbol{R}^2$ associated with a triangulation $\mathcal{T}_h$ of the domain Ω.

Thus, the convection-diffusion problem stabilized with a diffusion term, is

$$\begin{cases} \text{find } u_h \in \mathcal{V}_h \text{ such that} \\ A_h(u_h, v_h) = f(v_h), \qquad \forall v_h \in \mathcal{V}_h \end{cases} \tag{5.4}$$

where A_h is furnished by (5.3).

The bilinear form A_h is continue and coercive ([**19**]). Moreover, set

$$a(u, v) = (\nabla u, \nabla v)$$

with $u, v \in \mathcal{V}$ and

$$|||\, u \,|||^2 = a(u, u)$$

the strengthened Cauchy-Schwarz inequality holds for finite elements (see for example [**21**]) and for unidimensional biorthogonal wavelets ([**10**]). We observe that we can write A_h in the form (4.1) where

$$A(u_h, v_h) = \int_\Omega \{\nu \nabla u_h \nabla v_h + \underline{\beta} v_h\}\, dx, \qquad \forall u_h, v_h \in \mathcal{V}_h.$$

Thus, if we prove the saturation assumption

$$\exists \alpha < 1 \text{ such that } |||\, u - u_{\bar{h}} \,||| \leq \alpha \,|||\, u - u_h \,||| \tag{5.5}$$

for $u_h \in \mathcal{V}_h$, $u_{\bar{h}} \in \mathcal{V}_{\bar{h}}$ solutions of the problem (5.1), with $\mathcal{V}_h \subset \mathcal{V}_{\bar{h}}$ linear finite element spaces, then we can use for this problem the a posteriori error estimate (5.2) proved in the theorem 4.1. In [**9**] the proof of the saturation assumption is given for unidimensional linear finite elements.

Now we consider the SUPG method for the convection-diffusion problem. The bilinear form B_h in such a case is

$$B_h\colon \mathcal{V}_h \times \mathcal{U}_h \rightarrow \boldsymbol{R},$$

$$B_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} (-\nu \Delta u_h, v_h)_T + (D_\beta u_h, v_h),$$

where τ_T is an opportune positive coefficient, $\mathcal{V}_h$ is a linear finite element space in $\Omega \subset \boldsymbol{R}^2$ associated with a triangulation $\mathcal{T}_h$ of the domain Ω, and

$$\mathcal{U}_h = \left\{ v_h \in L^2(\Omega) / \exists u_h \in \mathcal{V}_h\colon v_h = u_h + \sum_{T \in \mathcal{T}_h} \tau_T D_\beta u_h \,|_T \right\}.$$

Let

$$b_h(u, v) = \nu(\nabla u, \nabla v) + \sum_{T \in \mathcal{T}_h} \tau_T (D_\beta u, D_\beta v)_T$$

be the inner product in $\mathcal{H}$. We know that (4.7) and (4.8) are verified. Moreover, by theorem 3.2.5 in [**4**], also (4.5) holds. Thus, if the saturation assumption (4.6) is verified, we can formulate the a posteriori error estimate (4.9) for the approximate solution of the convection-diffusion problem stabilized with the SUPG method. The assumption (4.6) has been proved in [**9**] for unidimensional linear finite elements.

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Summary

This paper contains the proofs of some a posteriori error estimates based on hierarchical bases. The author analyzes the case of a selfadjoint and positive definite variational form and afterwards the nonselfadjoint and indefinite one. Moreover the case of a modified variational form is considered. Some examples are given for stabilized convection-diffusion equation. The estimates can be apply in the wavelet-based adaptive finite element method.

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