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FERENC MÁTYÁS (*)

On a bound for the zeros of polynomials defined by special linear recurrences of order k (**)

1 - Introduction

Let $k \geqslant 2$ be an integer. The polynomial sequence of order k $\{G_n(x)\}$ is defined for every $n \geqslant 2$ by the recursion

$$G_n(x) = P_1(x)\, G_{n-1}(x) + P_2(x)\, G_{n-2}(x) + \ldots + P_k(x)\, G_{n-k}(x), \tag{1}$$

where $P_i(x)$ $(1 \leqslant i \leqslant k)$ and $G_j(x)$ $(2-k \leqslant j \leqslant 1)$ are given polynomials with complex coefficients and $P_k(x)\, G_1(x)$ is not equal to the zeropolynomial. If it is necessary then we will use the formula

$$G_n(x) = G_n(P_1(x), P_2(x), \ldots, P_k(x), G_{2-k}(x), G_{3-k}(x), \ldots, G_1(x)).$$

Recently, some papers have been publicated on the zeros of polynomials defined by second order linear recursions, that is, when $k = 2$ in (1). These results are in close relation with the well-known Fibonacci-polynomials $G_n(x, 1, 0, 1)$ [**4**] and the Chebyshev-polynomials $G_n(2x, -1, 0, 1)$. For example, M. N. S. Swamy ([8], [**9**]) and R. André-Jeannin ([**2**], [**3**]) have proved explicit formulae for the zeros of polynomials $G_n(x+2, -1, 1, x+t)$ and $G_n(x+p, -q, 1, x+p \pm \sqrt{q})$, where $p \in \boldsymbol{R}$, $q \in \boldsymbol{R}^+$ and $t = 1, 2, 3$. Similar, but not explicit, results have been proved in [**6**] for the polynomials $G_n(P_1(x), P_2(x), 0, 1)$, $G_n(P_1(x), q, c_0, c_1)$ and $G_n(P_1(x), q, c, cp(x)+e)$, where $q, c_0, c_1, c, e \in \boldsymbol{C}$.

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Using another method, P. E. Ricci [**7**] has given a common upper bound for the absolute values of zeros of polynomials $G_n(x, 1, 1, x+1)$, namely if $G_n(x, 1, 1, x+1) = 0$ then $|x| < 2$. We generalized his result in [**5**], and afterwards it was proved in [**6**] that if z was a zero of the polynomial $G_n(ax+b, q', c, dx+e)$ with some $n \geqslant 1$, then

$$|z| \leqslant \frac{1}{|ad|}\big(\max\big(|ac\sqrt{q'}| + |ae - bd|, 2|d\sqrt{q'}|\big) + |bd|\big), \tag{2}$$

where $a, b, c, d, e, q' \in \boldsymbol{C}$ and $aq'cd \neq 0$.

G. B. Djordjevic [**1**] has proved an explicit formula for the polynomials $G_n(x+p, 0, -q, 0, 0, 1)$ $(p, q \in \boldsymbol{R}, q \neq 0)$, that is, for the terms of a third order Morgan–Voyce-type polynomial sequence, but that is not a suitable formula even to determine the zeros of these very special polynomials.

The purpose of this paper is to investigate the zeros of polynomials

$$G_n(px+q, 0, 0, \dots, 0, e, a_{2-k}, a_{3-k}, \dots, a_0, rx+s),$$

where $p, q, r, s, a_j \in \boldsymbol{C}$ $(2-k \leqslant j \leqslant 0)$, $pr \neq 0$, $e = 1$ or $e = -1$. We are going to construct a common upper bound for the absolute values of zeros of above polynomials, which does not depend on n.

The following theorem will be proved.

Theorem. *Let $k \geqslant 2$ be an integer, $p, q, r, s, a_j \in \boldsymbol{C}$ $(2-k \leqslant j \leqslant 0)$, $e = 1$ or $e = -1$, $pr \neq 0$. With some $n \geqslant 1$ and $x = z$ complex number, if*

$$G_n(px+q, 0, 0, \dots, 0, e, a_{2-k}, a_{3-k}, \dots, a_0, rx+s) = 0$$

then

$$|z| \leqslant \frac{1}{|pr|}\left(\max\left(|ps - rq| + |p|\sum_{j=2-k}^{0}|a_j|, 2|r|\right) + |rq|\right).$$

It is obvious, that from the above Theorem one can get (2) if $k = 2$ and $q' = \pm 1$.

2 - Auxiliary results

To prove our Theorem we need some lemmas.

Lemma 1. *Let $G_n(x)$ be defined by* (1), *and let $k \geqslant 2$. Then, for every*

$n \geqslant 2-k$ *and* $c \in \boldsymbol{C} \setminus \{0\}$,

$$G_n(x) = cG_n^{\star}\left(P_1(x),\ \ldots,\ P_k(x),\ \frac{G_{2-k}(x)}{c},\ \frac{G_{3-k}(x)}{c},\ \ldots,\ \frac{G_1(x)}{c}\right). \tag{3}$$

Proof. It is obvious that (3) holds for every $2-k \leqslant n \leqslant 1$. Let us suppose that (3) holds for $n-k,\ n+1-k,\ \ldots,\ n-1$ if $n \geqslant 2$. By (1) and our induction hipothesis we have

$$G_n(x) = P_1(x)\, G_{n-1}(x) + P_2(x)\, G_{n-2}(x) + \ldots + P_k(x)\, G_{n-k}(x)$$

$$= P_1(x)\, cG_{n-1}^{\star}(x) + P_2(x)\, cG_{n-2}^{\star}(x) + \ldots + P_k(x)\, cG_{n-k}^{\star}(x)$$

$$= c\left(P_1(x)\, G_{n-1}^{\star}(x) + P_2(x)\, G_{n-2}^{\star}(x) + \ldots + P_k(x)\, G_{n-k}^{\star}(x)\right) = cG_n^{\star}(x).$$

So, (3) holds for every $n \geqslant 2-k$.

Now, let $\{G_n(x)\}$ be a polynomial sequence satisfying the conditions of the Theorem. Then, substituting

$$y = px + q \qquad \left(x = \frac{y-q}{p}\right), \tag{4}$$

we have

$$\begin{aligned} &G_n(px+q,\ 0,\ 0,\ \ldots,\ 0,\ e,\ a_{2-k},\ a_{3-k},\ \ldots,\ a_0,\ rx+s) \\ &= G_n\left(y,\ 0,\ 0,\ \ldots,\ 0,\ e,\ a_{2-k},\ a_{3-k},\ \ldots,\ a_0,\ \frac{r}{p}\, y - \frac{rq-ps}{p}\right), \end{aligned} \tag{5}$$

which can be easily verified.

For $n \geqslant 2-k$, applying Lemma 1, we have

$$\begin{aligned} &G_n\left(y,\ 0,\ 0,\ \ldots,\ 0,\ e,\ a_{2-k},\ a_{3-k},\ \ldots,\ a_0,\ \frac{r}{p}\, y - \frac{rq-ps}{p}\right) \\ &= \frac{r}{p}\, G_n^{\star}(y,\ 0,\ 0,\ \ldots,\ 0,\ e,\ \alpha_{2-k},\ \alpha_{3-k},\ \ldots,\ \alpha_0,\ y-\alpha), \end{aligned} \tag{6}$$

where

$$\alpha_j = \frac{pa_j}{r} \quad (2-k \leqslant j \leqslant 0) \qquad \text{and} \qquad \alpha = \frac{rq-ps}{r}. \tag{7}$$

The following step is to determine a matrix $\boldsymbol{A}_n$ with the characteristic polynomial $G_n^\star(y, 0, 0, \ldots, 0, e, \alpha_{2-k}, \alpha_{3-k}, \ldots, \alpha_0, y-\alpha)$.

Let us consider the $n \times n$ matrix $\boldsymbol{A}_n = (a_{t,j})$ where $a_{1,1} = \alpha$, $a_{1,j} = \varepsilon^{j+1}\alpha_{j-k}$ $(2 \leqslant j \leqslant k)$, $a_{j+1,j} = \varepsilon^3$ $(1 \leqslant j \leqslant n-1)$, $a_{j,k+j-1} = \varepsilon^{k+1}$ $(2 \leqslant j \leqslant n+1-k)$ and the other entries are equal to 0. That is,

$$\text{(8)} \qquad \boldsymbol{A}_n = \begin{pmatrix} \alpha & \varepsilon^3\alpha_{2-k} & \varepsilon^4\alpha_{3-k} & \ldots & \varepsilon^{k+1}\alpha_0 & 0 & 0 & \ldots & 0 & 0 \\ \varepsilon^3 & 0 & 0 & \ldots & 0 & \varepsilon^{k+1} & 0 & \ldots & 0 & 0 \\ 0 & \varepsilon^3 & 0 & \ldots & 0 & 0 & \varepsilon^{k+1} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \varepsilon^3 & 0 \end{pmatrix},$$

where $\varepsilon = -1$ if $e = -1$ and $\varepsilon = -i$ if $e = 1$.

We prove that the matrix $\boldsymbol{A}_n$ has the expected property.

Lemma 2. *For every $n \geqslant 1$, the characteristic polynomial of $\boldsymbol{A}_n$ is the polynomial $G_n^\star(y) = G_n^\star(y, 0, 0, \ldots, 0, e, \alpha_{2-k}, \alpha_{3-k}, \ldots, \alpha_0, y-\alpha)$.*

Proof. Denote the characteristic polynomial of matrix $\boldsymbol{A}_n$ by $f_n(y)$. It is known that $f_n(y) = \det(y\boldsymbol{I}_n - \boldsymbol{A}_n)$, where $\boldsymbol{I}_n$ is the $n \times n$ unit matrix. Because of the entries of matrix $\boldsymbol{A}_n$, we need to separate the proof into the cases $1 \leqslant n \leqslant k$ and $n > k$.

First we consider the case $1 \leqslant n \leqslant k$. Then, for $n = 1$ $f_1(y) = \det(y\boldsymbol{I}_1 - \boldsymbol{A}_1) = y - \alpha = G_1^\star(y)$. If $n = 2$ or 3, then we have

$$f_2(y) = \begin{vmatrix} y-\alpha & -\varepsilon^3\alpha_{2-k} \\ -\varepsilon^3 & y \end{vmatrix} = y(y-\alpha) - \varepsilon^6\alpha_{2-k}$$

$$= yG_1^\star(y) + eG_{2-k}^\star(y) = G_2^\star(y)$$

and

$$f_3(y) = \begin{vmatrix} y-\alpha & -\varepsilon^3\alpha_{2-k} & -\varepsilon^4\alpha_{3-k} \\ -\varepsilon^3 & y & 0 \\ 0 & -\varepsilon^3 & y \end{vmatrix}$$

$$= yf_2(y) - \varepsilon^4\alpha_{3-k}\varepsilon^6 = yG_2^\star(y) - \varepsilon^2\alpha_{3-k} = yG_2^\star(y) + eG_{3-k}^\star(y) = G_3^\star(y).$$

Suppose that $f_{n-j}(y) = G_{n-j}^\star(y)$ $(j = 1, 2, 3)$ holds for an integer n, where

$4 \leqslant n < k$. Then, developing the determinant

$$\det(y\boldsymbol{I}_n - \boldsymbol{A}_n)$$

$$= \begin{vmatrix} y-\alpha & -\varepsilon^3\alpha_{2-k} & -\varepsilon^4\alpha_{3-k} & \dots & -\varepsilon^n\alpha_{n-1-k} & -\varepsilon^{n+1}\alpha_{n-k} \\ -\varepsilon^3 & y & 0 & \dots & 0 & 0 \\ 0 & -\varepsilon^3 & y & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -\varepsilon^3 & y \end{vmatrix}$$

with respect to the last column, we have

$$f_n(y) = \det(y\boldsymbol{I}_n - \boldsymbol{A}_n) = yf_{n-1}(y) - (-1)^{n+1}\varepsilon^{n+1}\alpha_{n-k}(-\varepsilon^3)^{n-1}$$

$$= yG^\star_{n-1}(y) + (-1)^{2n+1}\varepsilon^{4n-2}\alpha_{n-k} = yG^\star_{n-1}(y) + eG^\star_{n-k}(y) = G^\star_n(y).$$

That is, Lemma 2 holds for every n, if $1 \leqslant n \leqslant k$.

Now, we shall deal with the case $n > k$. If $n = k+1$ then

$$f_{k+1}(y) = \begin{vmatrix} y-\alpha & -\varepsilon^3\alpha_{2-k} & -\varepsilon^4\alpha_{3-k} & \dots & -\varepsilon^{k+1}\alpha_0 & 0 \\ -\varepsilon^3 & y & 0 & \dots & 0 & -\varepsilon^{k+1} \\ 0 & -\varepsilon^3 & y & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -\varepsilon^3 & y \end{vmatrix}$$

$$= yf_k(y) + \varepsilon^3 \begin{vmatrix} y-\alpha & -\varepsilon^3\alpha_{2-k} & -\varepsilon^4\alpha_{3-k} & \dots & -\varepsilon^k\alpha_{-1} & 0 \\ -\varepsilon^3 & y & 0 & \dots & 0 & -\varepsilon^{k+1} \\ 0 & -\varepsilon^3 & y & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -\varepsilon^3 & 0 \end{vmatrix}.$$

Developing successively the resulting determinants with respect to their last row, we have

$$f_{k+1}(y) = yf_k(y) + (\varepsilon^3)^{k-1}\begin{vmatrix} y-\alpha & 0 \\ -\varepsilon^3 & -\varepsilon^{k+1} \end{vmatrix}$$

$$= yG_k^\star(y) - \varepsilon^{3k-3}\varepsilon^{k+1}(y-\alpha) = yG_k^\star(y) + eG_1^\star(y) = G_{k+1}^\star(y).$$

Let us suppose that $f_{n-j}(y) = G_{n-j}^\star(y)$ $(1 \leqslant j \leqslant k)$ holds for an integer $n \geqslant k+2$. In this case, by (8),

$$f_n(y) = \begin{vmatrix} y-\alpha & -\varepsilon^3\alpha_{2-k} & \dots & -\varepsilon^{k+1}\alpha_0 & 0 & 0 & \dots & 0 & 0 \\ -\varepsilon^3 & y & \dots & 0 & -\varepsilon^{k+1} & 0 & \dots & 0 & 0 \\ 0 & -\varepsilon^3 & \dots & 0 & 0 & -\varepsilon^{k+1} & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\varepsilon^3 & y \end{vmatrix}$$

$$= yf_{n-1}(y) + \varepsilon^3 \begin{vmatrix} y-\alpha & -\varepsilon^3\alpha_{2-k} & \dots & -\varepsilon^{k+1}\alpha_0 & 0 & \dots & 0 & 0 \\ -\varepsilon^3 & y & \dots & 0 & -\varepsilon^{k+1} & \dots & 0 & 0 \\ 0 & -\varepsilon^3 & \dots & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & 0 & \dots & -\varepsilon^3 & 0 \end{vmatrix}.$$

Now, develop successively the resulting determinants with respect to their last row. Then one can get the following equalities:

$$f_n(y) = yf_{n-1}(y) + (\varepsilon^3)^{k-1}(-\varepsilon^{k+1})\, f_{n-k}(y)$$

$$= yG_{n-1}^\star(y) - \varepsilon^2 G_{n-k}(y) = yG_{n-1}^\star(y) + eG_{n-k}(y) = G_n^\star(y).$$

This completes the proof of Lemma 2.

3 - Proof of the Theorem

Using our lemmas the Theorem can already be proved. According to (5), (6) and (7)

$$G_n(px+q,\, 0,\, 0,\, \dots,\, 0,\, e,\, a_{2-k},\, a_{3-k},\, \dots,\, a_0,\, rx+s)$$

$$= \frac{r}{p} G_n^\star(y,\, 0,\, 0,\, \dots,\, 0,\, e,\, \alpha_{2-k},\, \alpha_{3-k},\, \dots,\, \alpha_0,\, y-\alpha)$$

holds for every $n \geqslant 2-k$. Since, by Lemma 2, $G_n^\star(y)$ is the characteristic polynomial of matrix $\boldsymbol{A}_n$, therefore the zeros of polynomial $G_n^\star$ are equal to the eigenvalues of matrix $\boldsymbol{A}_n$. Applying the Gershgorin's theorem, we have that these eigenvalues can be found in the set $C_1 \cup C_2$, where

$$C_1 = \left\{\omega\colon \omega \in \boldsymbol{C},\ |\omega - \alpha| \leqslant \sum_{j=2-k}^{0} |\alpha_j|\right\}$$

and

$$C_2 = \{\omega\colon \omega \in \boldsymbol{C},\ |\omega| \leqslant 2\}.$$

These sets C_1 and C_2 are called Gershgorin circles (It is sufficient to consider only these two Gershgorin circles, because the other ones are parts of the set C_1 or C_2.) Thus, if a complex number $y = \varrho$ is a zero of the polynomial $G_n^{\star}(y)$ with some $n \geqslant 1$, then

$$|\varrho| \leqslant \max\left(|\alpha| + \sum_{j=2-k}^{0} |\alpha_j|, 2\right). \tag{9}$$

Applying (7), we have

$$|\varrho| \leqslant \max\left(\frac{|ps - rq|}{|r|} + \sum_{j=2-k}^{0} \frac{|pa_j|}{|r|}, 2\right) \tag{10}$$

and hence, by (4), the following inequality can be obtained for any zero $x = z(z = (\varrho - q)/p)$ of the polynomial $G_n(x)$.

$$|z| \leqslant \frac{|\varrho| + |q|}{|p|} \leqslant \frac{\max\left(\frac{|ps - rq|}{|r|} + \sum_{j=2-k}^{0} \frac{|pa_j|}{|r|}, 2\right) + |q|}{|p|} =$$

$$\frac{1}{|pr|}\left(\max\left(|ps - rq| + |p|\sum_{j=2-k}^{0} |a_j|, 2|r|\right) + |rq|\right).$$

The proof of the Theorem is complete.

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Abstract

Let $k \geqslant 2$ *be an integer, while let* $G_j(x) = a_j \in \boldsymbol{C}$ $(2-k \leqslant j \leqslant 0)$ *and* $px+q$, $G_1(x) = rx+s$ *be given polynomials of x with complex coefficients, where* $pr \neq 0$*. For* $n \geqslant 2$ *the sequence* $\{G_n(x)\}$ *is defined by the following recursion of order k.*

$$G_n(x) = (px+q)G_{n-1}(x) + eG_{n-k}(x), \quad \text{where } e = 1 \text{ or } e = -1.$$

We prove that the absolute values of the zeros of polynomials $G_n(x)$ $(n \geqslant 1)$ *have a common upper bound, which depends only on* a_j $(2-k \leqslant j \leqslant 0)$, p, q, r *and s. Namely, if* $G_n(z) = 0$ *for a* $z \in \boldsymbol{C}$ *with some* $n \geqslant 1$ *then*

$$|z| \leqslant \frac{1}{|pr|}\left(\max\left(|ps-rq| + |p|\sum_{j=2-k}^{0}|a_j|,\, 2|r|\right) + |rq|\right).$$

This result extends and generalizes some earlier results presented in [5], [6] *and* [7] *for the case* $k=2$.

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