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Rings satisfying the generalized polynomial identity $(x - x^n)([x, y]_k - [x, y]_k^m) = 0$ (**)

0 - Introduction

Throughout, R will represent an associative ring with center C and Jacobson radical $J(R)$. If $(x_i)_{i \in \mathbb{N}}$ is a sequence of elements of R and k is a positive integer we define $[x_1, \ldots, x_{k+1}]$ inductively as follows:

$$[x_1, x_2] = x_1 x_2 - x_2 x_1$$

$$[x_1, \ldots, x_k, x_{k+1}] = [[x_1, \ldots, x_k], x_{k+1}].$$

If $x_1 = x$ and $x_2 = \ldots = x_{k+1} = y$, we write $[x_1, \ldots, x_{k+1}] = [x, y]_k$. Also for $k = 0$ we define $[x, y]_k = x$.

By a ring R with torsion-free commutators, we mean that $m[x, y] = 0$ implies $[x, y] = 0$ for all $m \geq 1$, $x, y \in R$.

A ring R is called left (resp. right) s-unital [8] if for each $x \in R$ we have $x \in Rx$(*resp*. $x \in xR$). A ring R is called s-unital if for each x in R, $x \in xR \cap Rx$. If R is an s-unital ring , then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ for all $x \in F$ (see [8]). Such an element e will be called a pseudo-identity of F.

In [3] Hirano and Yaqub studied the rings satisfying $(x - x^n)(y - y^n) = 0$. Later in [7], Komatsu and Tominaga extended Theorem 3 of [3] as follows: If R is a ring satisfying $(x - x^n)(y - y^n) = 0$ $(n > 1)$ and if for each $x, y \in R$, either

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$(xy)^n - (yx)^n \in C$, or $x^n y^n - y^n x^n \in C$ or $(xy)^n - y^n x^n \in C$ then R is commutative. (Note that, here $(x - x^n)(y - y^n) = 0$, implies the Chacron's condition). On the other hand Giambruno, Goncalves and Mandel [**5**] have investigated the commutativity of rings satisfying $[x, y]_k^n = [x, y]_k$. Now our objective is to investigate the commutativity of rings satisfying any of the following conditions:

(**P1**) For each x, y in R there exist positive integers $n = n(x, y) > 1$, $m = m(x, y) > 1$ and $k = k(x, y) \geqslant 1$, such that $(x - x^n)([x, y]_k - [x, y]_k^m) = 0$.

(**P2**) For each x, y in R there exist positive integers $n = n(x, y) > 1$, $m = m(x, y) > 1$ and $k = k(x, y) \geqslant 1$ such that $(1 - x^n)([x, y]_k - [x, y]_k^m) = 0$ (here 1 is formal).

In section 1, we prove the following theorems:

Theorem 1. *Let R be a division ring which satisfies* (**P1**), *then R is commutative.*

Although this result can not be extended to primitive rings, we show that:

Theorem 2. *If R is a ring which satisfies* (**P2**), then the commutator ideal of R is nil.

In section **2**, we generalize the result of Komatsu and Tominaga [**6**] by proving:

Theorem 3. *Let R be an s-unital ring, and $n > 1$ a fixed positive integer. Suppose that for any $x, y \in R$ there exist $r = r(x, y) \geqslant 1$ and $m = m(x, y) > 1$ such that, either $(xy)^r - (yx)^r \in C$, or $x^r y^r - y^r x^r \in C$, or $(xy)^r - y^r x^r \in C$; and*

(I) $(x - x^n)([x, y] - [x, y]^m) = 0$,

(II) *The commutator ideal of R is $n!$-torsion free, then R is commutative.*

(Note that, here $(x - x^n)([x, y] - [x, y]^m) = 0$ does not imply the Chacron's condition).

1 - Commutativity results

In preparation for the proof of the main theorems we start with the following lemmas. Proof of Lemma 1 can be found in [**5**] and Lemma 2 is obvious.

Lemma 1. *Let R be a division ring. If for each x and y in R there exist positive integers $n = n(x, y) > 1$ and $k = k(x, y) \geqslant 1$ such that $[x, y]_k = [x, y]_k^n$. Then R is commutative.*

Lemma 2. *Let $b \in R$ and $a \in J(R)$. If $ba = b$ then $b = 0$.*

With the above lemmas established, we are able to complete the proof of Theorem 1 and 2.

Proof of Theorem 1. By Lemma 1 it is enough to show that if there exist x and y such that

$$(1.1) \qquad [x, y] \neq [x, y]^m \quad \text{for all } m \geqslant 1$$

then $[x, y]_{k+1} = [x, y]_{k+1}^m$ for some positive integer k and $m > 1$. In order to show this we replace x by $[x, y]$ in (**P1**), thereby obtaining

$$(1.2) \qquad ([x, y] - [x, y]^n)([x, y]_{k+1} - [x, y]_{k+1}^m) = 0$$

for some $n > 1$, $m > 1$ and $k \geqslant 1$. Comparing (1.1) with (1.2) now yields $[x, y]_{k+1} = [x, y]_{k+1}^m$, as desired.

Remark 1. Theorem 1 can not be extended to primitive rings because a trivial computation (by computer) shows that the noncommutative ring of 2×2 matrix over $GF(2)$ with $m = n = 4$ and $k = 1$, satisfies the condition (**P1**).

Proof of Theorem 2. We prove Theorem 2 by dividing its proof into several steps.

Step 1. Clearly Theorem 2 is true for any division ring (by Theorem 1).

Step 2. Theorem 2 is true for any left primitive ring R.

In this case either $R \approx D$ for some division ring D-in which case we would deduce that R is commutative by use of step 1-or for some $k > 1$ D_k is a homomorphic image of a subring of R. We wish to show that this latter possibility does not arise. If it did, D_k as a homomorphic image of a subring of R would inherit the property (**P2**). This is seen to be patently false by considering the elements $x = E_{21}$ and $y = E_{22}$, for these satisfy $(1 - x^n)([x, y]_k - [x, y]_k^m) = E_{21} \neq 0$, for all $n > 1$, $m > 1$ and $k \geqslant 1$. Thus , if R is primitive it must be commutative.

Step 3. Theorem 2 is true for any semiprimitive ring R.

For, we have a subdirect product representation $R \to \prod_{i \in I} R_i$, where the $R'_i s$ are left primitive rings. Each R_i satisfies (**P2**), and is therefore commutative. Hence R is also commutative.

Step 4. Theorem 2 is true for any ring R.

Let $x, y \in R$, then by (**P2**) we have

$$(1 - x^n)[x, y]_k = (1 - x^n)[x, y]_k^m$$

$$= (1 - x^n)[x, y]_k [x, y]_k^{m-1}$$

for some positive integers $k \geq 1$, $m > 1$, $n > 1$. But Step 3 shows that $R/J(R)$ is commutative, hence $[x, y]_k^{m-1} \in J(R)$ and therefore by Lemma 2 $(1 - x^n)[x, y]_k = 0$. Replacing x by $[x, y]$ in the recent equality, we get $(1 - [x, y]^n)[x, y]_{k+1} = 0$, for some $n > 1, k \geq 1$. Since $[x, y]^n \in J(R)$, we would deduce that $[x, y]_{k+1} = 0$ by Lemma 2. Therefore the commutator ideal is nil, by [**4**].

The following corollary is an immediate consequence of [8].

Corollary 1. *If for each $x, y \in R$ there exist positive integers $n = n(x, y) > 1$, $m = m(x, y) > 1$, $k = k(x, y) \geq 1$ and a fixed integer $r \geq 1$ such that*

(i) $(1 - x^n)([x, y]_k - [x, y]_k^m) = 0$,

(ii) If $r[x, y] = 0$ then $[x, y] = 0$,

(iii) $[x^r, y^r] = 0$.

Then R is commutative.

2 - Extensions

In preparation for the proof of the Theorem 3, we start with the following lemmas. Proof of Lemma 3 can be found in [**6**], [**1**] and [**2**], Lemma 4 is obvious and the proof of Lemma 5 can be found in [8].

Lemma 3. *Suppose that R is a semiprimitive ring which satisfies any of the following conditions*:

(i) *For all x, y in R there exists a positive integer $r = r(x, y) \geq 1$ such that $(xy)^r - (yx)^r \in C$.*

(ii) *For all x, y in R there exists a positive integer $r = r(x, y) \geq 1$ such that $x^r y^r - y^r x^r \in C$.*

(iii) *For all* x, y *in* R *there exists a positive integer* $r = r(x, y) \geqslant 1$ *such that* $(xy)^r - y^r x^r \in C$.

Then R is commutative.

Lemma 4. *If* $[x, y]$ *commutes with* x, *then* $[x^r, y] = rx^{r-1}[x, y]$ *for all positive integer* r.

Lemma 5. *Let* R *be an s-unital ring and* e *a pseudo-identity of* $\{x, z\} \subseteq R$. *If* $x^m z = (x+e)^m z$ *for some positive integer* m, *then* $z = 0$.

With the above lemmas established, we are able to complete the proof of Theorem 3.

Proof of Theorem 3. Let $x, y \in R$, then by (I) there exist positive integers $n > 1$ and $m = m(x, y) > 1$ such that

$$(x - x^n)[x, y] = (x - x^n)[x, y]^m$$

$$= (x - x^n)[x, y][x, y]^{m-1}.$$

But, by Lemma 3, $[x, y]^{m-1} \in J(R)$, therefore

$$(x - x^n)[x, y] = 0 \tag{2.1}$$

by Lemma 2. Since R is an s-unital ring we can replace x by $x + e$ in (2.1), where e is the pseudo-identity of $\{x, y\}$, thereby obtaining:

$$\left(x^n + \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} + \ldots + \binom{n}{n-2} x^2 + \binom{n}{n-1} x + e^n - x - e\right)[x,y] = 0\,. \tag{2.2}$$

Note that $(e^n - e)[x, y] = 0$. Comparing (2.1) and (2.2) now yields

$$\left(\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} + \ldots \binom{n}{n-2} x^2 + \binom{n}{n-1} x\right)[x, y] = 0$$

and therefore

$$\binom{n}{1} x^{n-1}[x, y] = \left(-\binom{n}{2} x^{n-2} - \ldots - \binom{n}{n-2} x^2 - \binom{n}{n-1} x\right)[x, y]\,. \tag{2.3}$$

Again, if we replace x by $x+e$ in (2.3), we get

$$\binom{n}{1}\left(x^{n-1}+\binom{n-1}{1}x^{n-2}+\ldots+e^{n-1}\right)[x,\,y]$$

(2.4) $$=\left(-\binom{n}{2}\left(x^{n-2}+\binom{n-2}{1}x^{n-3}+\ldots+e^{n-2}\right)-\ldots-\binom{n}{n-2}(x^2+2x+e^2)-\binom{n}{n-1}(x+e)\right)[x,\,y].$$

Comparing (2.3) and (2.4) now yields

(2.5) $$\binom{n}{1}\binom{n-1}{1}x^{n-2}[x,y]=\left(\left(-\binom{n}{1}\binom{n-1}{2}-\binom{n}{2}\binom{n-2}{1}\right)x^{n-3}-\ldots-\left(\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}\right)\right)[x,y].$$

Continuing the above process, we reach that

(2.6) $$\binom{n}{1}\binom{n-1}{1}\ldots\binom{2}{1}x[x,\,y]=s[x,\,y]$$

where s is an integer.

Now replacing x by $x+e$ in (2.6) yields $n![x,\,y]=0$, and therefore $[x,\,y]=0$, by (II).

Remark 2. If n is an even integer in Theorem 3, then (II) can be replaced by:

(II)′ The commutator ideal of R is 2-torsion free.

Proof. Let $x,\,y\in R$, then by (2.1) there exist $n>1$ such that $(x-x^n)[x,\,y]=0$. Since n is even, replacing x by $-x$ in (2.1), we deduce that

(2.7) $$x[x,\,y]=-\,x^n[x,\,y].$$

Comparing (2.1) and (2.7) now yields $2x[x,\,y]=0$ and therefore $x[x,\,y]=0$, by (II)′. Since R is an s-unital ring we can replace x by $x+e$, thereby obtaining $x[x,\,y]+[x,\,y]=0$. Hence $[x,\,y]=0$, as desired.

Corollary 2. *Let R be an s-unital ring and $n>1$, $r>1$ fixed positive inte-*

gers. Suppose that for any $x, y \in R$ *there exist positive integer* $m = m(x, y)$ *such that*

(I) $(x - x^n)([x, y] - [x, y]^m) = 0$

(II) $[x^r, y^r] = 0$

(III) *The commutator ideal of* R *is* $r(2^n - 2)$*-torsion free. Then* R *is commutative.*

Proof. Let $x, y \in R$, then by (2.1), $(x - x^n)[x, y] = 0$. Hence

$$x[x, y] = x^n[x, y]$$

$$= x^{n-1}x[x, y].$$

Now if $x \in J(R)$, applying Lemma 2 we deduce that $x[x, y] = 0$ and therefore $(2^n - 2)[x, y] = 0$ by (2.4), and so $[x, y] = 0$ by (III). On the other hand in view of Lemma 3, (II) implies that $[x, y] \in J(R)$, hence $x[x, y] = [x, y]x$ and therefore

$$x^{r-1}[x, y^r] = 0 \tag{2.8}$$

by Lemma 4, (II) and (III). Now replacing x by $x + e$, where e is the pseudo-identity of $\{x, y\}$, we conclude that

$$(x + e)^{r-1}[x, y^r] = 0 = x^{r-1}[x, y^r]$$

and therefore $[x, y^r] = 0$ by Lemma 5. By repeating the above argument, we get $[x, y] = 0$.

Example 1. In Theorem 3 the ring R must be s-unital because the following noncommutative ring satisfies all of the other hypotheses.

$$A = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \text{where } a, b, c \text{ are any real numbers} \right\}$$

for $r = m = n = 3$.

Example 2. The hypothesis (II) of Theorem 3 is essential as the following example showes.

$$A=\left\{\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix} \mid | a,\, b,\, c,\, d\in GF(2)\right\}$$

with $n=m=2, r=1$.

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Abstract

The paper deals with the study of sufficient conditions for commutativity of a ring, namely with the partial generalizations of the Maclegan-Wedderburn theorem according to Jacobsons idea.

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