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# Electromagnetic waves in stratified media (**)

## 1 - Introduction

Electromagnetic wave propagation in inhomogeneous media is investigated both in the frequency domain and in the time domain. Recent approaches to the time domain involve imbedding, Green's functions methods and wave splitting techniques [**1**]-[**4**]. The frequency domain is decisively more familiar, especially in dispersive media [**5**]. The present investigation is framed within the frequency domain and shows that new properties and results can be derived on the basis of an appropriate decomposition of the matrix governing wave propagation through a first-order system.

The material properties are modelled by incorporating both instantaneous response and memory effects in a linear way. Since we let the pertinent fields be time-harmonic, the material properties result in complex-valued coefficients between independent and dependent vector fields. Recent investigations [**6**]-[**7**] of thermodynamic character allow us to specify the sign of the imaginary parts of the permeability and of the effective permittivity.

The constitutive parameters and functions are taken to depend on Cartesian coordinates $x$, $y$, $z$ and the Maxwell's equations are ultimately written in the form

$$\partial_z w = Aw$$

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where $w$ is the set of $x$- and $y$-components of the electric and the magnetic field while $A$ is a $4 \times 4$ matrix operator. This form may be viewed in the spirit of Stroh formulation [8] and is applied [9] to determine the reflection matrix for stratified anisotropic media. The system is then shown to decouple in two $2 \times 2$ systems if the trasverse inhomogeneity occurs through one coordinate only.

Next wave propagation is investigated by letting $A$ depend only on $z$ and the fields be independent of one coordinate ($y$). The Fourier transform of the pertinent fields with respect to $x$ is considered; the remaining dependence on $z$ is the subject of our analysis. By a proper choice of the eigenvectors of the (Fourier transform of the) matrix $A$, the evolution equation results in a peculiar form. Application to the Cauchy problem, relative to a surface $z =$ constant, shows that finer approximations to the solution are obtained. In general, the standard method of successive approximations may be applied to determine a single scalar unknown which in turn produces the solution to the Cauchy problem. Also, the recourse to the thermodynamic restrictions provides a deeper understanding of the results in that the solution proves to be given by suitable integrals of wave-like functions which decay as they propagate. As a check of consistency, Fresnel's formulae for reflection and transmission are shown to hold in the thin-layer limit for the slab.

The crucial step in the present approach is that, by means of the Stroh-like formulation, a proper choice of the unknown vector function and of the eigenvectors results in a peculiar form of the evolution equation. Hence a simpler and more convenient expression follows of the integral equation for the initial value problem. Also, the connection with the limit case of thin layers is more immediate.

## 2 - Time-harmonic waves

Consider an electromagnetic isotropic solid and let $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{J}$ be the electric field, the electric displacement, the magnetic field, the magnetic induction, and the electric current density. Hence, in MKSA units, we write Maxwell's equations as

$$\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \nabla \times \boldsymbol{H} = \dot{\boldsymbol{D}} + \boldsymbol{J}, \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \cdot \boldsymbol{D} = \varrho . \tag{2}$$

The free-charge density $\varrho$ and the electric current density $\boldsymbol{J}$ satisfy the continuity equation

$$\nabla\cdot\boldsymbol{J}+\dot{\varrho}=0\,.$$

The material is taken to exhibit memory effects. We let $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{J}$ and $\boldsymbol{H}$, $\boldsymbol{E}$ be related by linear memory functionals in the form

$$\boldsymbol{B}(t)=\mu^0\boldsymbol{H}(t)+\int_0^\infty\breve{\mu}(\xi)\,\boldsymbol{H}(t-\xi)\,\mathrm{d}\xi,\qquad \boldsymbol{D}(t)=\varepsilon^0\boldsymbol{E}(t)+\int_0^\infty\breve{\varepsilon}(\xi)\,\boldsymbol{E}(t-\xi)\,\mathrm{d}\xi\,,$$

$$\boldsymbol{J}(t)=\sigma^0\boldsymbol{E}(t)+\int_0^\infty\breve{\sigma}(\xi)\,\boldsymbol{E}(t-\xi)\,\mathrm{d}\xi\,,$$

where $\mu^0$, $\varepsilon^0$, $\sigma^0$ are the positive-valued instantaneous permeability, permittivity and conductivity. The functions $\breve{\mu}$, $\breve{\varepsilon}$, $\breve{\sigma}$ on $\mathbb{R}^+=[0,\ \infty)$ are required to be integrable so that the response $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{J}$ to constant histories is bounded.

Consider time-harmonic fields with time-dependence factorized by $\exp(-i\omega t)$. Hence eqs. (1) become

$$\nabla\times\boldsymbol{E}=i\omega\boldsymbol{B},\qquad \nabla\times\boldsymbol{H}=-\,i\omega\boldsymbol{D}+\boldsymbol{J}\,. \tag{3}$$

Taking the divergence and using the continuity equation yields (2). We then restrict attention only to eqs (3). Meanwhile, the constitutive equations become

$$\boldsymbol{B}=\mu\boldsymbol{H},\qquad \boldsymbol{D}=\varepsilon\boldsymbol{E},\qquad \boldsymbol{J}=\sigma\boldsymbol{E}$$

where

$$\mu=\mu^0+\int_0^\infty\breve{\mu}(\xi)\exp(i\omega\xi)\,\mathrm{d}\xi,\qquad \varepsilon=\varepsilon^0+\int_0^\infty\breve{\varepsilon}(\xi)\exp(i\omega\xi)\,\mathrm{d}\xi\,,$$

$$\sigma=\sigma^0+\int_0^\infty\breve{\sigma}(\xi)\exp(i\omega\xi)\,\mathrm{d}\xi\,.$$

Substitution in (3) yields

$$\nabla\times\boldsymbol{E}=i\omega\mu\boldsymbol{H},\qquad \nabla\times\boldsymbol{H}=-\,i\omega\bar{\varepsilon}\boldsymbol{E} \tag{4}$$

where

$$\bar{\varepsilon} = \varepsilon + i\sigma/\omega$$

is the effective permeability.

Compatibility with the second law of thermodynamics [**6**] requires that

$$\breve{\mu}_s(\omega) > 0, \qquad \breve{\varepsilon}_s(\omega) + \omega^{-1}(\sigma^0 + \breve{\sigma}_c(\omega)) > 0, \qquad \forall \omega \in \mathbb{R}^{++},$$

where the subscripts $s$ and $c$ denote (half-range) sine and cosine transform, e.g.

$$\breve{\mu}_s(\omega) = \int_0^\infty \breve{\mu}(\xi) \sin \omega\xi \, \mathrm{d}\xi, \qquad \breve{\sigma}_c(\omega) = \int_0^\infty \breve{\sigma}(\xi) \cos \omega\xi \, \mathrm{d}\xi .$$

Denote by $\Im$ and $\Re$ the imaginary and real parts of a complex number. Since $\Im\mu = \breve{\mu}_s$ and $\Im\bar{\varepsilon} = \breve{\varepsilon}_s(\omega) + \omega^{-1}(\sigma^0 + \breve{\sigma}_c(\omega))$, the thermodynamic requirements become

$$\Im\mu > 0, \qquad \Im\bar{\varepsilon} > 0, \qquad \forall \omega \in \mathbb{R}^{++}. \tag{5}$$

No thermodynamic restriction is placed upon $\breve{\mu}_c$, $\breve{\varepsilon}_c$ and $\breve{\sigma}_s$. However, $\mu^0$ and $\varepsilon^0$ are likely to be larger than the frequency-dependent terms $\breve{\mu}_c$ and $\breve{\varepsilon}_c + \breve{\sigma}_s/\omega$. Hence, since $\mu^0$, $\varepsilon^0 > 0$ it is reasonable to assume that

$$\Re\mu = \mu^0 + \breve{\mu}_c > 0, \qquad \Re\bar{\varepsilon} = \varepsilon^0 + \breve{\varepsilon}_c + \breve{\sigma}_s/\omega > 0 . \tag{6}$$

Because of the inhomogeneity, the permittivity $\mu$, and the effective permeability $\bar{\varepsilon}$ are $C^1$ functions of the position $\boldsymbol{x}$. In Cartesian components, eqs. (4) read

$$H_{y,z} - H_{z,y} = i\omega\bar{\varepsilon}E_x, \qquad H_{x,z} - H_{z,x} = -i\omega\bar{\varepsilon}E_y, \qquad H_{x,y} - H_{y,x} = i\omega\bar{\varepsilon}E_z,$$

$$E_{y,z} - E_{z,y} = -i\omega\mu H_x, \qquad E_{x,z} - E_{z,x} = i\omega\mu H_y, \qquad E_{x,y} - E_{y,x} = -i\omega\mu H_z$$

where the subscripts, $x$, $y$ and $z$ stand for the partial derivatives $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, and $\partial_z = \partial/\partial z$. Hence we have

$$E_z = \frac{i}{\omega\bar{\varepsilon}}(H_{y,x} - H_{x,y}), \qquad H_z = -\frac{i}{\omega\mu}(E_{y,x} - E_{x,y}).$$

Upon evaluation of $E_{z,x}$, $E_{z,y}$, $H_{z,x}$, $H_{z,y}$ and substitution we obtain a system of four equations in the four unknowns $H_x$, $H_y$, $E_x$, $E_y$ in the form

$$\partial_z \begin{bmatrix} H_x \\ H_y \\ E_x \\ E_y \end{bmatrix} = i \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ E_x \\ E_y \end{bmatrix}$$

where 0 denotes the $2 \times 2$ zero matrix while $M$ and $N$ are given by

$$M = \begin{bmatrix} (1/\omega\mu)_{,\,x}\,\partial_y + (1/\omega\mu)\,\partial_{xy} & \omega\,\overline{\varepsilon} - (1/\omega\mu)_{,\,x}\,\partial_x - (1/\omega\mu)\,\partial_{xx} \\ \omega\,\overline{\varepsilon} + (1/\omega\mu)_{,\,y}\,\partial_y + (1/\omega\mu)\,\partial_{yy} & -(1/\omega\mu)_{,\,y}\,\partial_x - (1/\omega\mu)\,\partial_{xy} \end{bmatrix},$$

$$N = \begin{bmatrix} -(1/\omega\overline{\varepsilon})_{,\,x}\,\partial_y - (1/\omega\overline{\varepsilon})\,\partial_{xy} & \omega\mu + (1/\omega\overline{\varepsilon})_{,\,x}\,\partial_x + (1/\omega\overline{\varepsilon})\,\partial_{xx} \\ -\omega\mu - (1/\omega\overline{\varepsilon})_{,\,y}\,\partial_y - (1/\omega\overline{\varepsilon})\,\partial_{yy} & (1/\omega\overline{\varepsilon})_{,\,y}\,\partial_x + (1/\omega\overline{\varepsilon})\,\partial_{xy} \end{bmatrix}.$$

The entries off the secondary diagonal involve mixed derivatives $\partial_{xy}$ or products such as $(1/\omega\mu)_{,\,x}\,\partial_y$. Hence, if the transverse dependence is only through one coordinate, $x$ or $y$, the off-diagonal entries vanish. The system then decouples in two $2 \times 2$ systems, one for $H_x$ and $E_y$, the other for $H_y$ and $E_x$. If, further, the dependence of the material properties is only through $z$ and $H_x$, $H_y$, $E_x$, $E_y$ are independent of $y$ we have

$$\partial_z \begin{bmatrix} H_x \\ E_y \end{bmatrix} = -\,i \begin{bmatrix} 0 & \omega\,\overline{\varepsilon} + (1/\omega\mu)\,\partial_{xx} \\ \omega\mu & 0 \end{bmatrix} \begin{bmatrix} H_x \\ E_y \end{bmatrix}, \tag{7}$$

$$\partial_z \begin{bmatrix} H_y \\ E_x \end{bmatrix} = i \begin{bmatrix} 0 & \omega\,\overline{\varepsilon} \\ \omega\mu + (1/\omega\,\overline{\varepsilon})\,\partial_{xx} & 0 \end{bmatrix} \begin{bmatrix} H_y \\ E_x \end{bmatrix}. \tag{8}$$

### 3 - Property of $A$ in the Fourier-transform domain

The form of the systems (7), (8) suggests that we apply the Fourier transform with respect to $x$. Letting

$$\widehat{f}(k_x) = \int_{-\infty}^{\infty} \exp(-ik_x x)\, f(x)\, \mathrm{d}x$$

we obtain

$$w' = Aw$$

where the prime denotes differentiation with respect to $z$ and

$$w = \begin{bmatrix} \widehat{H}_x \\ \widehat{E}_y \end{bmatrix}, \qquad A = -i\begin{bmatrix} 0 & \omega\bar{\varepsilon} - (1/\omega\mu)\, k_x^2 \\ \omega\mu & 0 \end{bmatrix} \tag{9}$$

or

$$w = \begin{bmatrix} \widehat{H}_y \\ \widehat{E}_x \end{bmatrix}, \qquad A = i\begin{bmatrix} 0 & \omega\bar{\varepsilon} \\ \omega\mu - (1/\omega\bar{\varepsilon})\, k_x^2 & 0 \end{bmatrix}. \tag{10}$$

The eigenvalues $\lambda_1, \lambda_2$ of the $2 \times 2$ matrices $A$ in (9) and (10) satisfy

$$\begin{aligned} \lambda_{1,2}^2 &= k_x^2 - \omega^2 \mu \bar{\varepsilon} \\ &= -\omega^2\{(\Re\mu)(\Re\bar{\varepsilon}) - (\Im\mu)(\Im\bar{\varepsilon}) - k_x^2/\omega^2 + i[(\Im\mu)(\Re\bar{\varepsilon}) + (\Im\bar{\varepsilon})(\Re\mu)]\}. \end{aligned} \tag{11}$$

It follows from (5) and (6) that

$$\Im\lambda_{1,2}^2 < 0, \qquad \omega > 0. \tag{12}$$

Also, if $k_x \neq 0$, $\Re\lambda_{1,2}^2$ is positive for small values of $\omega$ and negative for large values of $\omega$, if $\Re\mu > \Im\mu$ and $\Re\bar{\varepsilon} > \Im\bar{\varepsilon}$. Let $\lambda_1$ be the value with the minimal argument. Irrespective of the value of $k_x$ and $\omega > 0$, it follows from (12) that

$$\Im\lambda_1 > 0, \qquad \Re\lambda_1 < 0, \tag{13}$$

namely $\arg\lambda_1 \in (\pi/2, \pi)$.

As a consequence of (11) and (13) we have $\lambda_1 = -\lambda_2 \neq 0$ and hence the matrices $A$ are simple, which means that the eigenvectors $w_{(1)}, w_{(2)}$ are linearly independent.

Let $Q$ be the matrix whose columns are the eigenvectors of $A$. Hence $u = Q^{-1}w$ satisfies the differential equation

$$u' = \Lambda u - Q^{-1}Q'u \tag{14}$$

where $\Lambda = \mathrm{diag}(\lambda_1, -\lambda_1) = Q^{-1}AQ$. An equation of the form (14) for the evolution with $z$ is considered by Kennett [**10**] within seismic wave propagation and by Karlsson [**3**] in electromagnetic one-dimensional media. The form (14) occurs also in a paper by Keller & Keller [**11**] on systems of linear differential equations. In electromagnetism, $Q^{-1}Q'$ can be given a peculiar form - cf. (16) and (17) - which allows for simpler procedures to obtain the solution to related problems.

Look at the matrix $A$ of (9). To find the expression of $Q^{-1}Q'$ we choose the eigenvectors $w_{(1,2)}$ as

$$w_{(1)} = \begin{bmatrix} i\lambda_1/\omega\mu \\ 1 \end{bmatrix}, \qquad w_{(2)} = \begin{bmatrix} i\lambda_2/\omega\mu \\ 1 \end{bmatrix}. \tag{15}$$

On observing that $Q = [w_{(1)}, w_{(2)}]$ we obtain

$$Q^{-1}Q' = \frac{\phi'}{2\phi}\,\Xi \tag{16}$$

where

$$\Xi = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$\phi = \lambda_1/\omega\mu\,.$$

Also,

$$w = Qu = \left[\frac{i\lambda_1}{\omega\mu}(u_1 - u_2),\; u_1 + u_2\right]^T.$$

Concerning the matrix $A$ of (10), the eigenvalues turn out to be the same as for (9) while

$$w_{(1)} = \begin{bmatrix} 1 \\ -i\lambda_1/\omega\bar{\varepsilon} \end{bmatrix}, \qquad w_{(2)} = \begin{bmatrix} 1 \\ -i\lambda_2/\omega\bar{\varepsilon} \end{bmatrix}.$$

Hence we find that

$$Q^{-1}Q' = \frac{\psi'}{2\psi}\,\Xi \tag{17}$$

where

$$\psi = \lambda_1/\omega\bar{\varepsilon}\,.$$

The result that, in both cases, $Q^{-1}Q'$ is a scalar function times the constant symmetric matrix $\Xi$, is crucially related to the choice of the eigenvectors. That is presumably why, so far, such a property has not been exhibited. By (16) — or similarly by (17) — we can write

$$u' = \Lambda u - \frac{\phi'}{2\phi} g \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{18}$$

where $g = (\Xi u)_1 = u_1 - u_2$. Upon an integration, $u$ is then determined in terms of the single scalar variable $g = u_1 - u_2$. Indeed, summation of the two components gives

$$(u_1 + u_2)' = \lambda_1 g$$

and hence $u_1 + u_2$ is determined by $g$.

A single scalar unknown occurs also in other approaches but then the pertinent problem is governed by a second-order differential equation. The next section examines how the function $g(z)$, and hence $u$ and $w$, may be determined by solving an integral equation.

The peculiar properties of the model under investigation is that a $2 \times 2$ matrix $A$ occurs with zero diagonal terms, and hence with equal and opposite eigenvalues $\lambda_2 = -\lambda_1$ such that $\Im \lambda_{1,2}^2 < 0$ as $\omega > 0$. Moreover, by a choice of the eigenvectors the pertinent first-order system takes the form (18). The same aspects pertain to the system for a transmission line where

$$w = \begin{bmatrix} V \\ I \end{bmatrix}, \quad A = i \begin{bmatrix} 0 & \omega(L + iR/\omega) \\ \omega(C + iG/\omega) & 0 \end{bmatrix},$$

where $V$ is the voltage, $I$ is the current and $L$, $R$, $C$, $G$ are the inductance, the resistance, the capacitance and the shunt conductance per unit length.

The same properties pertain also to horizontally-polarized mechanical waves in isotropic solids where

$$w = \begin{bmatrix} u_y \\ t_y \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1/\mu \\ \mu k_x^2 - \varrho\omega^2 & 0 \end{bmatrix},$$

where $u_y$ is the (horizontal) displacement, $t_y$ is the traction and $\mu$ is the shear elasticity, $\varrho$ is the density. Such different contexts provide a further motivation for the investigation of the system (18).

**4 - The solution in a layer**

We consider the differential equation (18) in $\mathbb{C}$ together with the initial condition on $u(0)$. Here $u$ is a pair of complex quantities while $Q^{-1}Q' = (\phi'/2\phi)\,\Xi$. Since $\mu$ and $\bar{\varepsilon}$ are complex valued, both $\lambda_1$ and $\phi$ are complex valued. The dependence of $\mu$ and $\bar{\varepsilon}$ on the space variable $z$ is taken to be of class $C^1$ and hence so is $\phi$. Also, $\lambda_1^2 \neq 0$ everywhere in $\mathbb{R}$ and so does $\phi$. With a view to later use, we let $\phi' \equiv 0$ in $(-\infty, 0] \cup [d, \infty)$, $d > 0$.

Though the matrix $\Lambda$ is dependent on $z$, the diagonal form allows us to write

$$\frac{\mathrm{d}}{\mathrm{d}z} \exp\left[\int_0^z \Lambda(\xi)\,\mathrm{d}\xi\right] = \exp\left[\int_0^z \Lambda(\xi)\,\mathrm{d}\xi\right] \Lambda(z).$$

Hence integration of (18) yields

$$u(z) = \exp\left[\int_0^z \Lambda(\xi)\,\mathrm{d}\xi\right] u(0) - \int_0^z \frac{\phi'}{2\phi}(\xi) \exp\left[\int_\xi^z \Lambda(\eta)\,\mathrm{d}\eta\right] g(\xi)\,\mathrm{d}\xi \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{19}$$

namely a linear Volterra equation for $u$.

The occurrence of $g$ in (19) suggests that we look for an equation in the single unknown $g$. Let $G$ be the first component of $\Xi \exp\left[\int_0^z \Lambda(\xi)\,\mathrm{d}\xi\right] u(0)$ namely

$$G(z) = \exp\left[\int_0^z \lambda_1(\xi)\,\mathrm{d}\xi\right] u_1(0) - \exp\left[-\int_0^z \lambda_1(\xi)\,\mathrm{d}\xi\right] u_1(0).$$

Application of $\Xi$ to (19) yields the integral equation

$$g(z) = G(z) - \int_0^z \frac{\phi'}{2\phi}(\xi) \left[\exp\left(\int_\xi^z \lambda_1(\eta)\,\mathrm{d}\eta\right) + \exp\left(-\int_\xi^z \lambda_1(\eta)\,\mathrm{d}\eta\right)\right] g(\xi)\,\mathrm{d}\xi\,. \tag{20}$$

Differentiation of (20) with respect to $z$ yields

$$g'(z) = G'(z) - \frac{\phi'}{\phi}(z)\, g(z)$$

$$-\lambda_1(z)\int_0^z \frac{\phi'}{2\phi}(\xi)\left[\exp\left(\int_\xi^z \lambda_1(\eta)\,\mathrm{d}\eta\right)-\exp\left(-\int_\xi^z \lambda_1(\eta)\,\mathrm{d}\eta\right)\right] g(\xi)\,\mathrm{d}\xi\,.$$

Multiplication by $\phi(z)$ and integration result in

$$(\phi g)(z)-(\phi g)(0)=\int_0^z(\phi G')(\xi)\,\mathrm{d}\xi$$

(21)

$$-\int_0^z \phi(\xi)\,\lambda_1(\xi)\int_0^\xi \frac{\phi'}{2\phi}(s)\left[\exp\left(\int_s^\xi \lambda_1(\eta)\,\mathrm{d}\eta\right)-\exp\left(-\int_s^\xi \lambda_1(\eta)\,\mathrm{d}\eta\right)\right] g(s)\,\mathrm{d}s\,.$$

Equation (20) can be given the form

$$g(z)=G(z)+\int_0^z k(z,\,\xi)\,g(\xi)\,\mathrm{d}\xi \tag{22}$$

where

$$k(z,\,\xi)=-\frac{\phi'}{2\phi}(\xi)\left[\exp\left(\int_\xi^z \lambda_1(\eta)\,\mathrm{d}\eta\right)+\exp\left(-\int_\xi^z \lambda_1(\eta)\,\mathrm{d}\eta\right)\right].$$

Upon the assumption that $k$ is bounded as $\xi\in[0,\,d]$ and $z\in[0,\,d]$, it follows that the solution $g$ to (22) exists and is unique in $L^2(0,\,d)$ (cf. [**12**]). The same conclusion follows for the unknown $\phi g$ by merely letting $k(z,\,\xi)\to K(z,\,\xi)=\phi(z)\,k(z,\,\xi)/\phi(\xi)$; in this regard it is essential that $\phi\neq 0$ as a consequence of (12).

To obtain the solution $g$ to (22) we apply the method of successive approximations through the integral equation

$$g(z)=G(z)+\sum_{m=1}^{n-1}\int_0^z k_m(z,\,\xi)\,G(\xi)\,\mathrm{d}\xi+\int_0^z k_n(z,\,\xi)\,g(\xi)\,\mathrm{d}\xi$$

where $k_m$, $m=2,3,\ldots$, is generated by the recursive relation

$$k_{m+1}(z,\,\xi)=\int_\xi^z k_m(z,\,\eta)\,k(\eta,\,\xi)\,\mathrm{d}\eta$$

and $n$ is chosen such that

$$(K^n g)(z) := \int_0^z k_n(z, \xi)\, g(\xi)\, \mathrm{d}\xi$$

is a contraction, i.e. $\|K^n\| < 1$.

We now make use of a generalization [**13**] of Gronwall's lemma which improves a standard inequality (cf. [**14**]). Let $h, p, q$ be continuous functions on $[0, d]$ and $pq \geqslant 0$. If $f$ has the property that

$$f(z) \leqslant h(z) + p(z) \int_0^z q(y)\, f(y)\, \mathrm{d}y, \qquad z \in [0, d],$$

then

$$f(z) \leqslant h(z) + p(z) \int_0^z (qh)(y) \exp\left[\int_y^z (qp)(\eta)\, \mathrm{d}\eta\right] \mathrm{d}y, \qquad z \in [0, d]. \tag{23}$$

Application of (23) to

$$|g(z)| \leqslant |G(z)| + \int_0^z |k(z, \xi)|\, |g(\xi)|\, \mathrm{d}\xi$$

and the identifications

$$f = |g|, \qquad h = |G|, \qquad p = 1, \qquad q = |k|,$$

shows that $|g|$ is bounded on $[0, d]$. Hence, on the assumption that $\lambda_1$, $\phi$, $\phi'$ be bounded it follows from (21) that

$$(\phi g)(z) - (\phi g)(0) = O(z). \tag{24}$$

Accordingly, the function $g$ satisfies

$$g(z) = \frac{(\phi g)(0)}{\phi(z)} + O(z).$$

The conclusion remains valid as $d \to 0$, in which case $\phi'$ may become unbounded, provided only that $|\phi'|d$ is kept bounded. This means that, $g(z)$ is closer and closer to $(\phi g)(0)/\phi(z)$ both when $z$ becomes small while $\phi$ is kept fixed and when $d \to 0$ while $\phi(0)$ and $\phi(d)$ are fixed.

**5 - Wave propagation**

With (13) in mind we now investigate the meaning of (21). The last term provides the higher-order corrections, in the sense of successive approximations, to the solution determined by the initial value $u(0)$. Examine the first two terms.

Let $u(0) = [u_0^+, u_0^-]$. Observe that $G'$ takes the form

$$G'(z) = \lambda_1(z)\left[u_0^+ \exp\left(\int_0^z \lambda_1(\xi)\,\mathrm{d}\xi\right) + u_0^- \exp\left(-\int_0^z \lambda_1(\xi)\,\mathrm{d}\xi\right)\right];$$

the initial values $u_0^+, u_0^-$ enter the solution through $G'(z)$.

Replace $(\phi g)(s)$ with $(\phi g)(0) + O(s)$. Since the time dependence is through the factor $\exp(-i\omega t)$, in the space-time variables $z, t$ we obtain from (21) that

$$(\phi g)(z, t) - (\phi g)(0, t)$$

$$= \int_0^z \phi(\xi)\,\lambda_1(\xi)\left\{u_0^+ \exp\left(-\int_0^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)\exp\left[i\left(\int_0^\xi \Im\lambda_1(\eta)\,\mathrm{d}\eta - \omega t\right)\right]\right.$$

$$\left. + u_0^- \exp\left(\int_0^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)\exp\left[-i\left(\int_0^\xi \Im\lambda_1(\eta)\,\mathrm{d}\eta + \omega t\right)\right]\right\}\mathrm{d}\xi$$

$$-(\phi g)(0)\int_0^z \phi(\xi)\,\lambda_1(\xi)\int_0^\xi \frac{\phi'}{2\phi^2}(s)\,\lambda_1(\xi)\left\{\exp\left(-\int_s^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)\exp\left[i\left(\int_s^\xi \Im\lambda_1(\eta)\,\mathrm{d}\eta - \omega t\right)\right]\right.$$

$$\left. - \exp\left(\int_s^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)\exp\left[-i\left(\int_s^\xi \Im\lambda_1(\eta)\,\mathrm{d}\eta + \omega t\right)\right]\right\}\mathrm{d}s\,\mathrm{d}\xi - \dots$$

the dots indicating the higher-order corrections due to $O(s)$. The result shows that the wave at a point $z$ consists of a superposition of elementary terms of the form

$$\exp\left(-\int_0^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)\exp\left[i\left(\int_0^\xi \Im\lambda_1(\eta)\,\mathrm{d}\eta - \omega t\right)\right], \qquad \xi \in [0, z], \tag{25}$$

$$\exp\left(\int_0^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)\exp\left[-i\left(\int_0^\xi \Im\lambda_1(\eta)\,\mathrm{d}\eta+\omega t\right)\right],\qquad \xi\in[0,z]. \tag{26}$$

The function (25) represents a wave propagating in the positive $z$-direction with mean wavenumber $\xi^{-1}\int_0^\xi \Im\lambda_1(\eta)\,\mathrm{d}\eta$. The wave decays, as it propagates, through the factor $\exp\left(-\int_0^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)$, which means that $\xi^{-1}\int_0^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta$ represents the mean attenuation rate. The function (26) represents a wave propagating in the negative $z$-direction. The amplitude decays as $\xi$ decreases because of the factor $\exp\left(\int_0^\xi |\Re\lambda_1|(\eta)\,\mathrm{d}\eta\right)$.

## 6 - Reflection and transmission from a layer

A layer of thickness $d$, $z\in[0,d]$, is placed among two homogeneous half spaces. Let $\lambda_0$ and $\lambda_d$ be the value of the eigenvalue $\lambda_1$ as $z\in(-\infty,0]$ and $z\in[d,\infty)$. Since $\Im(\mu\bar{\varepsilon})>0$ it follows that

$$\Im\lambda_0>0,\qquad \Re\lambda_0<0$$

$$\Im\lambda_d>0,\qquad \Re\lambda_d<0$$

for any values of $\omega\neq 0$ and $k_x$. By solving $u'=\Lambda u$ in $z\leqslant 0$ and $z\geqslant d$ and letting

$$u(0)=\begin{bmatrix}u_0^+\\ u_0^-\end{bmatrix},\qquad u(d)=\begin{bmatrix}u_d^+\\ u_d^-\end{bmatrix},$$

we have

$$u(z)=\begin{bmatrix}u_0^+\exp(-|\Re\lambda_0|z)\exp(i\Im\lambda_0 z)\\ u_0^-\exp(|\Re\lambda_0|z)\exp(-i\Im\lambda_0 z)\end{bmatrix},\qquad z\leqslant 0,$$

$$u(z)=\begin{bmatrix}u_d^+\exp(-|\Re\lambda_d|(z-d))\exp(i\Im\lambda_d(z-d))\\ u_d^-\exp(|\Re\lambda_d|(z-d))\exp(-i\Im\lambda_d(z-d))\end{bmatrix},\qquad z\geqslant d.$$

We may view the terms with $u_0^+$, $u_d^-$ as incoming waves, those with $u_0^-$, $u_d^+$ as outgoing waves.

A reflection-transmission problem is modelled by considering an incoming wave and two outgoing waves. Let the incident (incoming) wave come from the negative side $z \leqslant 0$, namely $u_0^+ \exp(-|\Re\lambda_0|z)\exp(i\Im\lambda_0 z)$. The two outgoing waves are $u_0^- \exp(|\Re\lambda_d|z)\exp(-i\Im\lambda_d z)$ at $z \leqslant 0$ and $u_d^+ \exp(-|\Re\lambda_d| \cdot (z-d))\exp(i\Im\lambda_d(z-d))$ at $z \geqslant d$. The position $u_d^- = 0$ means that no wave is incoming from $z \geqslant d$.

Let $\tilde{u} = \Xi u = g[1, \ -1]^T$. By (14) and (16) we have

$$u(d) = \exp\left(\int_0^d \Lambda(\xi)\, d\xi\right) u(0) - \int_0^d \frac{\phi'}{2\phi}(\xi) \exp\left(\int_\xi^d \Lambda(\eta)\, \mathrm{d}\eta\right) \tilde{u}(\xi)\, \mathrm{d}\xi\,.$$

Define the transmission coefficient $T$ and the reflection coefficient $R$ such that $u_d^+ = Tu_0^+$, $u_0^- = Ru_0^+$. The continuity of $u$ at $z = d$ implies that

$$\begin{aligned}\begin{bmatrix} T \\ 0 \end{bmatrix} &= \begin{bmatrix} \exp\left(\int_0^d \lambda_1(\xi)\, d\xi\right) \\ R\exp\left(-\int_0^d \lambda_1(\xi)\, d\xi\right) \end{bmatrix} \\ &+ \frac{1}{u_0^+}\begin{bmatrix} -\int_0^d [\phi'/2\phi](\xi)\exp\left(\int_\xi^d \lambda_1(\eta)\,\mathrm{d}\eta\right) g(\xi)\,\mathrm{d}\xi \\ \int_0^d [\phi'/2\phi](\xi)\exp\left(-\int_\xi^d \lambda_1(\eta)\,\mathrm{d}\eta\right) g(\xi)\,\mathrm{d}\xi \end{bmatrix}.\end{aligned} \tag{27}$$

Substitution of $g$, as given by the integral equation (20) or by (21), into (27) yields the transmission and reflection coefficients $T, R$.

It is of interest to evaluate $T$ and $R$ in the limit case when the thickness $d$ of the layer approaches zero while the values $\phi_0 = \phi(0)$ and $\phi_d = \phi(d)$ are kept fixed. In this regard it is essential to use the estimate

$$\phi(\xi)\,\frac{g(\xi)}{u_0^+} - \phi_0\left(1 - \frac{u_0^-}{u_0^+}\right) = O(\xi)$$

which follows from (24). Substitution in (27) yields

(28)
$$\begin{bmatrix} T \\ 0 \end{bmatrix} = \begin{bmatrix} \exp\left(\int_0^d \lambda_1(\xi)\,\mathrm{d}\xi\right) \\ R\exp\left(-\int_0^d \lambda_1(\xi)\,\mathrm{d}\xi\right) \end{bmatrix} + \phi_0(1-R)\begin{bmatrix} -\int_0^d [\phi'/2\phi^2](\xi)\exp\left(\int_\xi^d \lambda_1(\eta)\,\mathrm{d}\eta\right)[1+O(\xi)]\,\mathrm{d}\xi \\ \int_0^d [\phi'/2\phi^2](\xi)\exp\left(-\int_\xi^d \lambda_1(\eta)\,\mathrm{d}\eta\right)[1+O(\xi)]\,\mathrm{d}\xi \end{bmatrix}.$$

The limit as $d$ approaches zero yields

$$T = 1 + \frac{\phi_0(1-R)}{2}\left[\frac{1}{\phi_d} - \frac{1}{\phi_0}\right],$$

$$0 = R - \frac{\phi_0(1-R)}{2}\left[\frac{1}{\phi_d} - \frac{1}{\phi_0}\right].$$

Hence it follows that

(29)
$$R = \frac{\phi_0 - \phi_d}{\phi_0 + \phi_d}, \qquad T = \frac{2\phi_0}{\phi_0 + \phi_d}.$$

Since $\phi = \lambda_1/\omega\mu$ we have

$$\phi = \frac{i(\mu\bar{\varepsilon} - k_x^2/\omega)^{1/2}}{\mu}$$

where the square root is that with minimal argument. In the picture of plane waves, $\mu\bar{\varepsilon} - k_x^2/\omega^2$ may be viewed as the complex-valued normal component of the wave vector. Accordingly the limit relations (29) may be regarded as the generalization of Fresnel's formulae (cf. [**15**] and [**5**], p. 120).

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## Summary

*Time-harmonic wave propagation is considered in inhomogeneous, isotropic, electromagnetic solids. The particular case is then assumed that the material properties (permittivity and permeability) depend on a Cartesian coordinate only and the problem is shown to satisfy a system of linear, first-order, ordinary differential equations. A suitable change of unknown functions is performed by means of the eigenvectors of the coefficient matrix. The peculiar structure of the system of evolution equations results in a single equation for a suitable unknown variable thus providing finer approximations to the solution of a Cauchy problem with data at a surface. The recourse to properties of thermodynamic character allows a deeper understanding of the results. The coefficients of reflection and transmission are evaluated in closed form. As a check of consistency, in the thin-layer limit for a stratified slab, Fresnel's formulae for reflection and transmission are recovered.*

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