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# Tristimulus-space transformations and luminance-coefficient computation in *RGB* colour mixing (**)

*In memory of the friend and colleague Gian Lorenzo Braglia*

## 1 - Luminance and tristimulus space [2, 4, 8]

*Transduction* and *signal processing* are the phenomena that generate a colour sensation in correspondence to any visible radiation entering into the eye [**2**]. The correspondence between electromagnetic radiations and colour sensations is defined by the *activation* of the photoreceptors of the retina of the eye. The Rushton's *univariance* principle [**5**] states that the activation is a step of the transduction and is proportional to the power absorbed by the photoreceptors. The photoreceptors of the colour vision, named *cones* for their shape, are of three types (fig. 1.):

1) the L cones, whose spectral sensitivity $\bar{p}(\lambda)$ has maximum at $\lambda = 560$ nm in the Long wavelength region;

2) the M cones, whose spectral sensitivity $\bar{d}(\lambda)$ has maximum at $\lambda = 545$ nm in the Medium wavelength region;

3) the S cones whose spectral sensitivity $\bar{t}(\lambda)$ has maximum at $\lambda = 445$ nm in the Short wavelength region.

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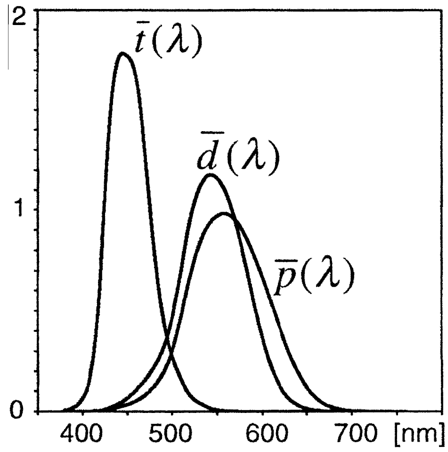


Fig. 1. – Colour-matching functions in the fundamental reference frame, in which represent the spectral sensitivities of the three kinds of cones.

A spectral radiance $L_{e,\lambda}$ generates the following cone activations

$$(1.1) \qquad P = \int_{380}^{780} L_{e,\lambda}\bar{p}(\lambda)\,\mathrm{d}\lambda\,, \qquad D = \int_{380}^{780} L_{e,\lambda}\bar{d}(\lambda)\,\mathrm{d}\lambda\,, \qquad T = \int_{380}^{780} L_{e,\lambda}\bar{t}(\lambda)\,\mathrm{d}\lambda$$

where $\lambda$ is the wavelength and the integration domain is between 380 and 780 nm. It exists an infinite set of different spectral radiances $L_{e,\lambda}$ that can generate the same set of three positive numbers $(P, D, T)$, i.e the correspondence between electromagnetic radiations and colour sensations in a defined viewing situation is many to one. The monochromatic radiations are excluded from such a correspondence because in this case the correspondence is one to one. Moreover, the correspondence between cone activations and colour sensations is one to one and then the $(P, D, T)$ are good to specify the colour sensations.

The set of the elements $(P, D, T)$ constitutes a linear vector space defined on the positive real numbers and the addition is the internal composition law. This space is named *tristimulus space*. Such a mathematical structure for describing the human colour vision was first defined by the *Grassman* laws (1853) [**1**]; then Krantz (1975) [**3**] redefined these laws in the present form.

The set of three functions $(\bar{p}(\lambda), \bar{d}(\lambda), \bar{t}(\lambda))$ represents the cone activations due to unit monochromatic radiances $L_{e,\lambda} = \delta(\lambda - \lambda')$, where $\delta(\lambda - \lambda')$ is the

Dirac $\delta$ function, and the corresponding tristimulus vector is denoted by

$$\boldsymbol{E}_{PDT}(\lambda) = \begin{pmatrix} \overline{p}(\lambda) \\ \overline{d}(\lambda) \\ \overline{t}(\lambda) \end{pmatrix}. \tag{1.2}$$

The directions of the tristimulus vectors is in a one to one correspondence with the intersections points of the line of these vectors with the plane $P + D + T = 1$; therefore, in practice, the *chromaticity* of a colour stimulus is defined by the co-ordinates of this intersection point

$$p = \frac{P}{P+D+T}, \qquad d = \frac{D}{P+D+T}, \qquad t = 1 - p - d = \frac{T}{P+D+T}. \tag{1.3}$$

This reference frame of the tristimulus-space is named *fundamental*. Infinite other reference frames can be obtained by linear transformations $\boldsymbol{M}$ from the fundamental one and, among these, the most used in colorimetry is the *XYZ* one, proposed by the *Commission Internationale de l'Éclairage* (CIE) in 1931 and known as CIE 1931. In this last reference frame, the $Y$ component is proportional to the luminance $L_\nu$ associated to the radiance $L_{e,\lambda}$. The practical role of the luminance makes this reference frame generally preferable.

The transformations between the fundamental reference frame to any other one transform the cone-spectral sensitivities $(\overline{p}(\lambda), \overline{d}(\lambda), \overline{t}(\lambda))$ into sets of three functions named *colour-matching functions*.

The transformations $\boldsymbol{M}$ can be represented by matrices and the sets $(P, D, T)$ by column vectors. Particularly, the matrix for the transformation $(P, D, T) \rightarrow (X, Y, Z)$ holds

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boldsymbol{M}_{PDT \rightarrow XYZ} \begin{pmatrix} P \\ D \\ T \end{pmatrix} \tag{1.4}$$

with the usual normalisation constraint

$$\begin{pmatrix} X = 1 \\ Y = 1 \\ Z = 1 \end{pmatrix} = \boldsymbol{M}_{PDT \rightarrow XYZ} \begin{pmatrix} P = 1 \\ D = 1 \\ T = 1 \end{pmatrix} \tag{1.5}$$

and the colour-matching functions are

$$\boldsymbol{E}_{XYZ}(\lambda) = \begin{pmatrix} \bar{x}(\lambda) \\ \bar{y}(\lambda) \\ \bar{z}(\lambda) \end{pmatrix} = \boldsymbol{M}_{PDT \to XYZ} \begin{pmatrix} \bar{p}(\lambda) \\ \bar{d}(\lambda) \\ \bar{t}(\lambda) \end{pmatrix} = \boldsymbol{M}_{PDT \to XYZ} \boldsymbol{E}_{PDT}(\lambda)\,. \tag{1.6}$$

The chromaticity co-ordinates associated to the vector $(X, Y, Z)$ are

$$x = \frac{X}{X+Y+Z}\,, \qquad y = \frac{Y}{X+Y+Z}\,, \qquad z = 1 - x - y = \frac{Z}{X+Y+Z}\,. \tag{1.7}$$

The CIE 1931 reference frame is possible because properties analogous to the Grassman laws, known as **Abney** law, hold for the luminance, too. The luminance is defined by the integral

$$L_v = K_m \int_{380}^{780} L_{e,\,\lambda} V(\lambda)\, \mathrm{d}\lambda\,, \tag{1.8}$$

analogous to the integrals (1.1). The luminance is measured by $\mathrm{cd/m^2}$ and $K_m = 683\ \mathrm{lm/W}$ if $L_{e,\,\lambda}$ is measured by $\mathrm{W/(sr\,m^2)}$ and $V(\lambda)$ is the *relative spectral luminous efficiency function* (fig. 2). The comparison between the integrals (1.1) and (1.8) gives that $Y$ represents the luminance up to the factor $K_m$ if $\bar{y}(\lambda) = V(\lambda)$.

The analogy between the integrals (1.1) and (1.8) could induce us to suppose

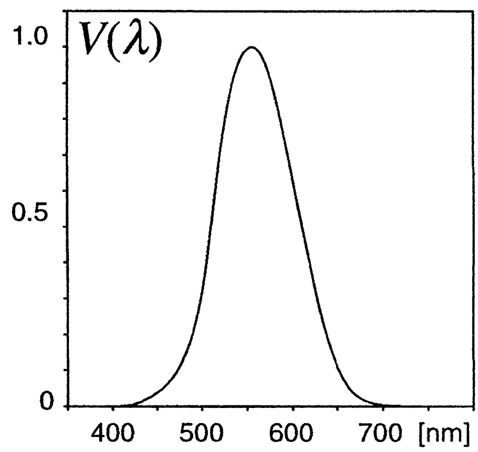


Fig. 2. – Relative photopic luminous efficiency function defined by CIE in 1924. This function can be written as linear function of $\bar{p}(\lambda)$ and $\bar{d}(\lambda)$, given in the figure 1.

the existence of a fourth kind of photoreceptors, but this is not necessary because the function $V(\lambda)$ can be written as linear function of the three spectral cone sensitivities

$$V(\lambda) = L_P \overline{p}(\lambda) + L_D \overline{d}(\lambda) + L_T \overline{t}(\lambda) = \widetilde{\Lambda}_{PDT} \boldsymbol{E}_{PDT}(\lambda) \tag{1.9}$$

(particularly, $L_t = 0$ because the S cones give no contribution to the luminance); therefore

$$L_v = K_m (L_P P + L_D D + L_T T). \tag{1.10}$$

The coefficients $L_P$, $L_D$ and $L_T$, named *luminance coefficients*, are the weights by which the three kind of cones contribute to the luminance.

Equations (1.9) and (1.10) can be written as scalar product between pair of vectors belonging to row and column dual spaces, i.e.

$$V(\lambda) = (L_P \;\; L_D \;\; L_T) \begin{pmatrix} \overline{p}(\lambda) \\ \overline{d}(\lambda) \\ \overline{t}(\lambda) \end{pmatrix} = \widetilde{\Lambda}_{PDT} \begin{pmatrix} \overline{p}(\lambda) \\ \overline{d}(\lambda) \\ \overline{t}(\lambda) \end{pmatrix} \tag{1.11}$$

and

$$L_v = K_m (L_P \;\; L_D \;\; L_T) \begin{pmatrix} P \\ D \\ T \end{pmatrix} = K_m \widetilde{\Lambda}_{PDT} \boldsymbol{Q}_{PDT}, \tag{1.12}$$

where $\boldsymbol{Q}_{PDT}$ represents the column tristimulus vector associated to the cone activations $(P, D, T)$ and $\widetilde{\Lambda}_{PDT}$ is the row vector $(L_P, L_D, L_T)$. The luminance is an invariant and, hence, is independent of the reference frame in the tristimulus space, i.e. [**6**]

$$\begin{aligned} L_v &= K_m (L_P \;\; L_D \;\; L_T)\, \boldsymbol{M}^{-1}_{PDT \to XYZ} \boldsymbol{M}_{PDT \to XYZ} \begin{pmatrix} P \\ D \\ T \end{pmatrix} = \\ &= K_m (L_P \;\; L_D \;\; L_T) \begin{pmatrix} P \\ D \\ T \end{pmatrix} = K_m (L_X \;\; L_Y \;\; L_Z) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \end{aligned} \tag{1.13}$$

Equation (1.13) defines the rule for computing the luminance coefficients $\widetilde{\Lambda}$ in any reference frame. Particularly, in the $XYZ$ reference it holds

$$\widetilde{\Lambda}_{XYZ} = (L_P \ \ L_D \ \ L_T)\, \boldsymbol{M}^{-1}_{PDT\to XYZ} = (L_X = 0 \ \ L_Y = 1 \ \ L_Z = 0)\,, \tag{1.14}$$

Moreover, from equations (1.14) and (1.11), we obtain

$$V(\lambda) = (L_X \ \ L_Y \ \ L_Z) \begin{pmatrix} \overline{x}(\lambda) \\ \overline{y}(\lambda) \\ \overline{z}(\lambda) \end{pmatrix} = \overline{y}(\lambda) = \widetilde{\Lambda}_{XYZ}\, \boldsymbol{E}_{XYZ}(\lambda)\,. \tag{1.15}$$

## 2 - The $RGB$-reference frame [4, 7, 8]

Usually, the colorimetry of the displays, like those for the television systems, is defined for the standard observer CIE 1931. In practice, the most used reference frame in the tristimulus space is the $RGB$ one. Any reference frame of this kind is directly associated to a proper trichromatic display, because the set of three independent vectors chosen as reference represents the stimuli $\boldsymbol{R}$, $\boldsymbol{G}$ e $\boldsymbol{B}$ related to the red, green and blue colours, respectively, typical of the display. These stimuli represented in the $XYZ$ CIE 1931-reference frame are

$$\begin{aligned} \boldsymbol{R} &= (X_r,\, Y_r,\, Z_r) = c_r(x_r,\, y_r,\, z_r) \quad \text{with } c_r = X_r + Y_r + Z_r \\ \boldsymbol{G} &= (X_g,\, Y_g,\, Z_g) = c_g(x_g,\, y_g,\, z_g) \quad \text{with } c_g = X_g + Y_g + Z_g \\ \boldsymbol{B} &= (X_b,\, Y_b,\, Z_b) = c_b(x_b,\, y_b,\, z_b) \quad \text{with } c_b = X_b + Y_b + Z_b \end{aligned} \tag{2.1}$$

whose lengths are chosen in order to obtain the neutral or achromatic stimulus as their sum

$$\boldsymbol{W} = \boldsymbol{R} + \boldsymbol{G} + \boldsymbol{B} = (X_n,\, Y_n = 100,\, Z_n) \tag{2.2}$$

(some time $Y_n = 1$ instead of $Y_n = 100$). The following transformation holds between the $XYZ$ and $RGB$ reference frame

$$\begin{pmatrix} R \\ G \\ B \end{pmatrix} = \boldsymbol{M}_{XYZ\to RGB} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boldsymbol{M}^{-1}_{XYZ\to RGB} \begin{pmatrix} R \\ G \\ B \end{pmatrix} \tag{2.3}$$

with

$$\boldsymbol{M}_{XYZ\to RGB} = \frac{1}{\Delta}\begin{pmatrix} r_x/c_r & r_y/c_r & r_z/c_r \\ g_x/c_g & g_y/c_g & g_z/c_g \\ b_x/c_b & b_y/c_b & b_z/c_b \end{pmatrix} \tag{2.4}$$

and

$$\boldsymbol{M}^{-1}_{XYZ\to RGB} = \begin{pmatrix} c_r x_r & c_g x_g & c_b x_b \\ c_r y_r & c_g y_g & c_b y_b \\ c_r z_r & c_g z_g & c_b z_b \end{pmatrix} \tag{2.5}$$

where

$$\begin{aligned}
&c_r = [X_n(y_g z_b - y_b z_g) - Y_n(x_g z_b - x_b z_g) + Z_n(x_g y_b - x_b y_g)]/\Delta \\
&c_g = [-X_n(y_r z_b - y_b z_r) + Y_n(x_r z_g - x_b z_g) - Z_n(x_r y_b - x_b y_r)]/\Delta \\
&c_b = [X_n(y_r z_g - y_g z_r) - Y_n(x_r z_g - x_g z_r) + Z_n(x_r y_g - x_g y_r)]/\Delta \\
&r_x = (y_g z_b - y_b z_g), \qquad r_y = (x_b z_g - x_g z_b), \qquad r_z = (x_g y_b - x_b y_g) \\
&g_x = (y_b z_r - y_r z_b), \qquad g_y = (x_r z_b - x_b z_r), \qquad g_z = (x_b y_r - x_r y_b), \\
&b_x = (y_r z_g - y_g z_r), \qquad b_y = (x_g z_r - x_r z_g), \qquad b_z = (x_r y_g - x_g y_r), \\
&\Delta = x_r(y_g z_b - y_b z_g) + x_g(y_b z_r - y_r z_b) + x_b(y_r z_g - y_g z_r).
\end{aligned} \tag{2.6}$$

In this reference frame, the vector $(R, G, B)$ is defined by the integrals

$$R = \int_{380}^{780} L_{e,\lambda}\,\bar{r}(\lambda)\,\mathrm{d}\lambda\,, \qquad G = \int_{380}^{780} L_{e,\lambda}\,\bar{g}(\lambda)\,\mathrm{d}\lambda\,, \qquad B = \int_{380}^{780} L_{e,\lambda}\,\bar{b}(\lambda)\,\mathrm{d}\lambda \tag{2.7}$$

where

$$\begin{pmatrix} \bar{r}(\lambda) \\ \bar{g}(\lambda) \\ \bar{b}(\lambda) \end{pmatrix} = \boldsymbol{E}_{RGB}(\lambda) = \boldsymbol{M}_{XYZ\to RGB}\begin{pmatrix} \bar{x}(\lambda) \\ \bar{y}(\lambda) \\ \bar{z}(\lambda) \end{pmatrix} = \boldsymbol{M}_{XYZ\to RGB}\,\boldsymbol{E}_{XYZ}(\lambda)\,, \tag{2.8}$$

the constraint (1.5) is substituted by

$$\begin{pmatrix} R = 100 \\ G = 100 \\ B = 100 \end{pmatrix} = \boldsymbol{M}_{XYZ \to RGB} \begin{pmatrix} X_n \\ Y_n = 100 \\ Z_n \end{pmatrix} \tag{2.9}$$

and the luminance coefficients are

$$(L_R = c_r y_r \;\; L_G = c_g y_g \;\; L_B = c_b y_b) = (L_X = 0 \;\; L_Y = 1 \;\; L_Z = 0)\, \boldsymbol{M}^{-1}_{XYZ \to RGB}. \tag{2.10}$$

The chromaticity of the stimulus $\boldsymbol{Q}_{RGB} = (R, G, B)$ is defined as

$$r = \frac{R}{R+G+B}, \qquad g = \frac{G}{R+G+B}, \qquad b = 1 - r - g = \frac{B}{R+G+B}. \tag{2.11}$$

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## Abstract

*Luminace is a quantity that contributes to the colour specification, either directly or indirectly. Its simplest definition is given in the XYZ system, proposed as a standard by the Commission Internationale de l'Éclairage (CIE). However usually, in the informatics systems with cathodic-ray-tube, plasma or liquid-crystal displays, the colour is specified in the RGB reference frame. This reference frame is obtained by proper linear transformations, that are summarised in this paper with the aim to compute the luminance by means of a simple operation. Here, the CIE 1931 Standard Observer is considered, but this treatment holds for CIE 1964 Standard Observer and the Judd-Vos one, too.*

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