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# Oscillation of first order delay differential equations with variable coefficients (**)

## 1 - Introduction

Consider the first order delay differential equation

$$\dot{x}(t) + p(t)\, x(t-\tau) = 0 \tag{1}$$

where $p(t) \geqslant 0$ is a continuous function and $\tau$ is a positive constant, or the more general one

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t)\, x(t-\tau_i) = 0 \tag{2}$$

where $p_i(t) \geqslant 0$ are continuous functions and $\tau_i$ are positive constants, $i = 1, 2, \ldots, n$.

By a solutions of equation (1) (or (2)) we mean a function $x \in C([t_0 - \varrho, \infty), \Re)$ for some $t_0 \geqslant 0$, where $\varrho = \tau$ (or $\varrho = \max_{1 \leqslant i \leqslant n} \{\tau_i\}$) satisfy equation (1) (or (2)) for all $t \geqslant t_0$. As it is customary, a solution of equation (1) (or (2)) is said to oscillate if it has an unbounded set of zeros for arbitrarily large $t$. Otherwise is called nonoscillatory. The equation will be called oscillatory if every solution defined on some ray is oscillatory.

Ladas [**1**] and Koplatadze and Chanturia [**2**] obtained the well-known oscilla-

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tion criterion for eq. (1)

$$\liminf_{t\to\infty}\int_{t-\tau}^{t} p(s)\,\mathrm{d}s > \frac{1}{e}\,. \tag{3a}$$

Ladas and Stavroulakis [**3**], Arino, Györi and other authors, (see [**4**], [**6**]) established different sufficient conditions for oscillation of eq. (2) which are in some sense extensions of (3$a$) in the case of several delays such as,

$$\liminf_{t\to\infty}\int_{t-\tau_{\max}}^{t} \sum_{i=1}^{n} p_i(s)\,\mathrm{d}s > \frac{1}{e}\,, \tag{3b}$$

$$\liminf_{t\to\infty}\left(\prod_{i=1}^{n} p_i(t)\right)^{1/n}\sum_{i=1}^{n}\tau_i > \frac{1}{e}\,, \tag{3c}$$

$$\limsup_{t\to\infty}\int_{t-\tau_{\min}}^{t} \sum_{i=1}^{n} p_i(s)\,\mathrm{d}s > 1\,, \tag{3d}$$

and

$$\liminf_{t\to\infty}\sum_{i=1}^{n}\tau_i p_i(t) > \frac{1}{e}\,. \tag{3e}$$

Recently, Li [**7**] obtained a sharper sufficient condition by improving condition (3$a$). Also he extended this result in his paper [**8**] for eq. (1) and eq. (2) as the following:

Theorem A. Let $\tau_n=\max\{\tau_1, \tau_2, \dots, \tau_n\}$. Suppose that $\sum_{i=1}^{n}\int_{t}^{t+\tau_i} p_i(s)\,\mathrm{d}s>0$ for $t\geqslant t_0$ for some $t_0>0$ and that

$$\limsup_{t\to\infty}\int_{t}^{t+\tau_n} p_n(s)\,\mathrm{d}s > 0\,. \tag{4a}$$

If, in addition,

$$\int_{t_0}^{\infty}\left(\sum_{i=1}^{n} p_i(t)\right)\ln\left(e\sum_{i=1}^{n}\int_{t}^{t+\tau_i} p_i(s)\,\mathrm{d}s\right)\mathrm{d}t = \infty\,, \tag{4b}$$

then every solution of eq. (2) oscillates.

Corollary. If

$$\liminf_{t\to\infty} \sum_{i=1}^{n} \int_{t}^{t+\tau_i} p_i(s)\, ds > \frac{1}{e} \tag{5}$$

then every solution of eq. (2) oscillates.

Finally, Tang and Shen in [**9**] obtained an other sharper sufficient condition for the oscillation of all solutions of eq. (1) which improves previously known results.

Theorem B. Let $p(t) \in C([t_0, \infty), \Re^+)$ and let $\tau$ be a positive constant. Define the following sequences of functions:

$$p_1(t) = \int_{t-\tau}^{t} p(s)\, ds\,, \qquad t \geq t_0 + \tau\,,$$

$$p_{k+1}(t) = \int_{t-\tau}^{t} p(s)\, p_k(s)\, ds\,, \qquad t \geq t_0 + (k+1)\,\tau\,,$$

$$\overline{p}_1(t) = \int_{t}^{t+\tau} p(s)\, ds\,, \qquad t \geq t_0$$

$$\overline{p}_{k+1}(t) = \int_{t}^{t+\tau} p(s)\, p_k(s)\, ds\,, \qquad t \geq t_0,\ k = 1, 2, 3, \ldots.$$

Suppose that there exist a $t_1 > t_0 + \tau$ and a positive integer $n$ such that

$$p_n(t) \geq \frac{1}{e^n}\,, \qquad \overline{p}_n(t) \geq \frac{1}{e^n}\,, \qquad t > t_1,$$

and

$$\int_{t_0+n\tau}^{\infty} p(t)\left[\exp\left(e^{n-1} p_n(t) - \frac{1}{e}\right) - 1\right] dt = \infty\,. \tag{6}$$

Then every solution of eq. (1) oscillates [**9**].

In this paper we obtain a sufficient condition for the oscillation of all solutions of eq. (2) which is more general than eq. (1), by extending the technique in [**9**] to be suitable for delay equations with several delays.

## 2 - Main results

Consider eq. (2) and define the following sequences of functions

$$p_{k,0}(t)=1\,,$$

(7)
$$p_{k,m}(t)=\int_{t-\tau_k}^{t}\Big(\sum_{i=1}^{n}p_i(s)\,p_{i,m-1}(s)\Big)\,\mathrm{d}s,\quad t\geqslant t_0+m\tau_j,\quad \forall j=1,2,\dots,n,\ m=1,2,\dots$$

and

$$\overline{P}_0(t)=1\,,$$

(8)
$$\overline{P}_m(t)=\sum_{k=1}^{n}\int_{t}^{t+\tau_k}p_k(s)\,\overline{P}_{m-1}(s)\,\mathrm{d}s\,,\qquad t\geqslant t_0,\qquad m=1,2,\dots .$$

The main result is the following theorem.

Theorem 1. *Let* $p_i(t)\in C([t_0,\infty),\Re^+)$, $i=1,2,\dots,n$ *and* $\tau_n=\max\{\tau_1,\tau_2,\dots,\tau_n\}$. *Suppose that there exist a* $t_1>t_0+\tau_n$ *and a positive integer* $m$ *such that*

(9)
$$p_{k,m}(t)\geqslant\frac{1}{e^m}\,,\qquad \overline{P}_m(t)\geqslant\frac{1}{e^m}\,,\qquad t>t_1,\qquad k=1,2,\dots,n,\ \ m=1,2,\dots$$

*and*

(10)
$$\int_{t_0+m\tau_n}^{\infty}\sum_{k=1}^{n}p_k(t)\Big\{\exp\Big[e^{m-1}p_{k,m}(t)-\frac{1}{e}\Big]-1\Big\}\,\mathrm{d}t=\infty\,,$$

*where* $p_{k,m}(t)$ *and* $\overline{P}_m(t)$ *are defined by* (7) *and* (8) *respectively. Then every solution of eq.* (2) *oscillates.*

Proof. Rearrange the terms of eq. (2) such that $\tau_1\leqslant\tau_2\leqslant\dots\leqslant\tau_n$. Assume the contrary. Then eq. (2) may have an eventually positive solution $x(t)$. Then there exists a $t_2\geqslant t_1$ such that $\dot{x}(t)\leqslant 0$, $x(t)>0$ for all $t\geqslant t_2$. Dividing eq. (2) by

$x(t)$ and integrating from $t-\tau_k$ to $t$, $t \geqslant t_2+\tau_n$, we get

$$\frac{x(t-\tau_k)}{x(t)} = \exp\left[\int_{t-\tau_k}^{t}\left(\sum_{i=1}^{n} p_i(s)\,\frac{x(s-\tau_i)}{x(s)}\right)\mathrm{d}s\right]. \tag{11}$$

Putting

$$w_k(t) = \frac{x(t-\tau_k)}{x(t)}, \qquad k=1,2,\dots,n\,. \tag{12}$$

Since $\dot{x}(t) \leqslant 0$, $x(t)>0$. Then $w_k(t) \geqslant 1$ for all $k=1,2,\dots n$ and for all $t \geqslant t_2+\tau_n$. From (11) and (12) one can write

$$w_k(t) = \exp\left[\int_{t-\tau_k}^{t}\left(\sum_{i=1}^{n} p_i(s)\,w_i(s)\right)\mathrm{d}s\right] \tag{13}$$

and consequently,

$$w_k(t) \geqslant e\left[\int_{t-\tau_k}^{t}\left(\sum_{i=1}^{n} p_i(s)\,w_i(s)\right)\mathrm{d}s\right], \qquad k=1,2,\dots,n\,. \tag{14}$$

Set

$$w_{k,0}(t) = w_k(t)$$

$$w_{k,1}(t) = \int_{t-\tau_k}^{t}\sum_{i=1}^{n} p_i(s)\,w_i(s)\,\mathrm{d}s\,, \qquad t \geqslant t_2+\tau_n,$$

$$w_{k,2}(t) = \int_{t-\tau_k}^{t}\sum_{i=1}^{n} p_i(s)\,w_{i,1}(s)\,\mathrm{d}s\,, \qquad t \geqslant t_2+2\tau_n,$$

$$\vdots$$

$$w_{k,m}(t) = \int_{t-\tau_k}^{t}\sum_{i=1}^{n} p_i(s)\,w_{i,m-1}(s)\,\mathrm{d}s\,, \qquad t \geqslant t_2+m\tau_n, \tag{15}$$

and

$$v_k(t) = w_k(t) - 1\,, \qquad t \geqslant t_1, \tag{16}$$

$$v_{k,1}(t) = \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_i(s)\, \mathrm{d}s\,, \qquad t \geqslant t_2 + \tau_n,$$

$$v_{k,2}(t) = \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,1}(s)\, \mathrm{d}s\,, \qquad t \geqslant t_2 + 2\tau_n,$$

$$\vdots$$

$$v_{k,m}(t) = \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,m-1}(s)\, \mathrm{d}s\,, \qquad t \geqslant t_2 + m\tau_n. \tag{17}$$

Since $w_k(t) \geqslant 1$ for all $k = 1, 2, \ldots, n$, it follows that

$$v_k(t) \geqslant 0\,, \quad v_{k,m}(t) \geqslant 0\,, \quad t \geqslant t_2 + m\tau_n, \quad k = 1, 2, \ldots, n, \; m = 1, 2, \ldots. \tag{18}$$

From (14) and (15), we get

$$w_k(t) \geqslant e^{m-1} w_{k,m-1}(t)\,, \qquad t \geqslant t_2 + (m-1)\,\tau_n \tag{19}$$

and consequently, from (13) and (19) we have

$$w_k(t) \geqslant \exp\left[e^{m-1} \int_{t-\tau_k}^{t} \left(\sum_{i=1}^{n} p_i(s)\, w_{i,m-1}(s)\right) \mathrm{d}s\right], \qquad t \geqslant t_2 + m\tau_n. \tag{20}$$

In the view of (7), (16) and (17) we obtain

$$w_{k,m-1}(t) = v_{k,m-1}(t) + p_{k,m-1}(t)\,,$$

and then substituting in (20), we get

$$w_k(t) \geqslant \exp\left[e^{m-1} \int_{t-\tau_k}^{t} \left(\sum_{i=1}^{n} p_i(s)[v_{i,m-1}(s) + p_{i,m-1}(s)]\right) \mathrm{d}s\right]$$

$$= \exp\left[e^{m-1} \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,m-1}(s)\, \mathrm{d}s + e^{m-1} \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, p_{i,m-1}(s)\, \mathrm{d}s\right]$$

$$= \exp\left[e^{m-1} \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,m-1}(s)\, \mathrm{d}s + \frac{1}{e} + e^{m-1} p_{k,m} - \frac{1}{e}\right], \qquad t \geqslant t_2 + m\tau_n,$$

and consequently

$$(21)\quad w_k(t) \geqslant \left[e^m \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,m-1}(s)\, \mathrm{d}s + 1\right] + \exp\left[e^{m-1} p_{k,m}(t) - \frac{1}{e}\right], \quad t \geqslant t_2 + m\tau_n.$$

From (21) we have

$$p_k(t)\left\{w_k(t) - \left[e^m \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,m-1}(s)\, \mathrm{d}s + 1\right]\right\}$$

$$(22)\quad \geqslant p_k(t)\left[e^m \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,m-1}(s)\, \mathrm{d}s + 1\right]\left\{\exp\left[e^{m-1} p_{k,m}(t) - \frac{1}{e}\right] - 1\right\}$$

$$\geqslant p_k(t)\left\{\exp\left[e^{m-1} p_{k,m}(t) - \frac{1}{e}\right] - 1\right\}, \qquad t \geqslant t_2 + m\tau_n.$$

In view of (16) and (17), then (22) becomes

$$p_k(t)[v_k(t) - e^m v_{k,m}(t)] \geqslant p_k(t)\left\{\exp\left[e^{m-1} p_{k,m}(t) - \frac{1}{e}\right] - 1\right\}, \qquad t \geqslant t_2 + m\tau_n.$$

i.e.

$$\sum_{k=1}^{n} p_k(t)[v_k(t) - e^m v_{k,m}(t)] \geqslant \sum_{k=1}^{n} p_k(t)\left\{\exp\left[e^{m-1} p_{k,m}(t) - \frac{1}{e}\right] - 1\right\}, \qquad t \geqslant t_2 + m\tau_n.$$

Integrating the above inequality from $t_3 = t_2 + m\tau_n$ to $T > t_3 + m\tau_n$ we get

$$(23)\qquad \int_{t_3}^{T} \sum_{k=1}^{n} p_k(t)[v_k(t) - e^m v_{k,m}(t)]\, \mathrm{d}t \geqslant \int_{t_3}^{T} \sum_{k=1}^{n} p_k(t)\left\{\exp\left[e^{m-1} p_{k,m}(t) - \frac{1}{e}\right] - 1\right\} \mathrm{d}t.$$

From (10) and (23), we obtain

$$(24)\qquad \lim_{T\to\infty} \int_{t_3}^{T} \sum_{k=1}^{n} p_k(t)[v_k(t) - e^m v_{k,m}(t)]\, \mathrm{d}t = \infty\ .$$

Since

$$\int_{t_3}^{T} \sum_{k=1}^{n} p_k(t)\, v_{k,\,m}(t)\, \mathrm{d}t = \int_{t_3}^{T} \sum_{k=1}^{n} p_k(t) \left[ \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, v_{i,\,m-1}(s)\, \mathrm{d}s \right] \mathrm{d}t$$

$$\geqslant \sum_{k=1}^{n} \int_{t_3}^{T-\tau_k} \sum_{i=1}^{n} p_i(s)\, v_{i,\,m-1}(s) \left[ \int_{s}^{s+\tau_k} p_k(t)\, \mathrm{d}t \right] \mathrm{d}s$$

$$\geqslant \int_{t_3}^{T-\tau_n} \sum_{i=1}^{n} p_i(s)\, v_{i,\,m-1}(s)\, \overline{P}_1(s)\, \mathrm{d}s$$

$$= \int_{t_3}^{T-\tau_n} \overline{P}_1(s) \sum_{i=1}^{n} p_i(s) \left[ \int_{s-\tau_i}^{s} \sum_{k=1}^{n} p_k(\xi)\, v_{k,\,m-2}(\xi)\, \mathrm{d}\xi \right] \mathrm{d}s$$

$$\geqslant \sum_{i=1}^{n} \int_{t_3}^{T-\tau_n-\tau_i} \sum_{k=1}^{n} p_k(\xi)\, v_{k,\,m-2}(\xi) \left[ \int_{\xi}^{\xi+\tau_i} p_i(s)\, \overline{P}_1(s)\, \mathrm{d}s \right] \mathrm{d}\xi$$

$$\geqslant \int_{t_3}^{T-2\tau_n} \sum_{k=1}^{n} p_k(\xi)\, v_{k,\,m-2}(\xi) \left[ \sum_{i=1}^{n} \int_{\xi}^{\xi+\tau_i} p_i(s)\, \overline{P}_1(s)\, \mathrm{d}s \right] \mathrm{d}\xi$$

$$= \int_{t_3}^{T-2\tau_n} \overline{P}_2(\xi) \sum_{k=1}^{n} p_k(\xi)\, v_{k,\,m-2}(\xi)\, \mathrm{d}\xi\,.$$

So we have,

$$e^m \int_{t_3}^{T} \sum_{k=1}^{n} p_k(t)\, v_{k,\,m}(t)\, \mathrm{d}t \geqslant e^m \int_{t_3}^{T-m\tau_n} \overline{P}_m(t) \left[ \sum_{k=1}^{n} p_k(t)\, v_k(t) \right] \mathrm{d}t$$

using (9) the above inequality becomes

$$e^m \int_{t_3}^{T} \sum_{k=1}^{n} p_k(t)\, v_{k,\,m}(t)\, \mathrm{d}t \geqslant \int_{t_3}^{T-m\tau_n} \sum_{k=1}^{n} p_k(t)\, v_k(t)\, \mathrm{d}t\,. \tag{25}$$

Thus,

$$\int_{t_3}^{T}\sum_{k=1}^{n} p_k(t)[v_k(t)-e^m v_{k,m}(t)]\,\mathrm{d}t \leqslant \int_{t_3}^{T}\sum_{k=1}^{n} p_k(t)\, v_k(t)\,\mathrm{d}t - \int_{t_3}^{T-m\tau_n}\sum_{k=1}^{n} p_k(t)\, v_k(t)\,\mathrm{d}t$$

$$= \int_{T-m\tau_n}^{T}\sum_{k=1}^{n} p_k(t)\, v_k(t)\,\mathrm{d}t\,.$$

In view of (24) we have,

$$\lim_{T\to\infty}\int_{T-m\tau_n}^{T}\sum_{k=1}^{n} p_k(t)\, v_k(t)\,\mathrm{d}t = \infty\,. \tag{26}$$

This shows that either

$$\lim_{T\to\infty}\int_{T-m\tau_n}^{T}\sum_{k=1}^{n} p_k(t)\,\mathrm{d}t = \infty \tag{27}$$

or, there exists $k^* \in \{1, 2, \dots, n\}$ such that

$$\limsup_{t\to\infty} v_{k^*}(t) = \infty\,. \tag{28}$$

If (27) holds, then

$$\limsup_{t\to\infty}\int_{t-\tau_1}^{t}\sum_{k=1}^{n} p_k(s)\,\mathrm{d}s = \infty\,.$$

By a knowning result in [**5**], [**6**], every solution of eq. (2) oscillates.

If (28) holds, then

$$\limsup_{t\to\infty} w_{k^*}(t) = \infty\,. \tag{29}$$

Since

$$w_k = \frac{x(t-\tau_k)}{x(t)} \geqslant 1\,, \qquad k = 1, 2, \dots, n$$

then

$$x(t-\tau_1) \geqslant x(t-\tau_k) \geqslant x(t)\,,$$

and consequently

(30) $$w_1 \geqslant w_k \geqslant 1\,, \qquad k = 1, 2, \ldots, n\,.$$

On the other hand, integrating eq. (2) from $t - \tau_k$ to $t$, we obtain

$$x(t) - x(t - \tau_k) + \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, x(s - \tau_i)\, \mathrm{d}s = 0\,.$$

i.e.,

(31) $$x(t - \tau_k) > \int_{t-\tau_k}^{t} \sum_{i=1}^{n} p_i(s)\, x(s - \tau_i)\, \mathrm{d}s\,, \qquad k = 1, 2, \ldots, n\,.$$

From (31) by successively substituting $(m-2)$ times and using the decreasing nature of $x(t)$, it follows that

(32) $$x(t - \tau_k) > x(t - \tau_1)\, p_{k,\, m-1}(t)\,.$$

In view of (9) for any $t \geqslant t_1 + \tau_1$ there exists $\xi \in (t - \tau_1, t)$ such that

(33) $$\int_{\xi}^{t} \sum_{k=1}^{n} p_k(s)\, p_{k,\, m-1}(s)\, \mathrm{d}s \geqslant \frac{1}{2e^m}\,, \qquad \int_{t}^{\xi+\tau_1} \sum_{k=1}^{n} p_k(s)\, p_{k,\, m-1}(s)\, \mathrm{d}s \geqslant \frac{1}{2e^m}\,.$$

Integrating eq. (2) over $[\xi, t]$ and $[t, \xi + \tau_1]$, we have

(34) $$x(t) - x(\xi) + \int_{\xi}^{t} \sum_{k=1}^{n} p_k(s)\, x(s - \tau_k)\, \mathrm{d}s = 0\,, \qquad t \geqslant t_2 + (m-1)\,\tau_1,$$

and

(35) $$x(\xi + \tau_1) - x(t) + \int_{t}^{\xi+\tau_1} \sum_{k=1}^{n} p_k(s)\, x(s - \tau_k)\, \mathrm{d}s = 0\,, \qquad t \geqslant t_2 + (m-1)\,\tau_1.$$

Omitting the first terms in (34) and (35) and substituting (32) in resulting inequalities, we get

$$(36)\qquad \begin{aligned} x(\xi) &> \int_{\xi}^{t} \sum_{k=1}^{n} p_k(s)\, p_{k,\, m-1}(s)\, x(s-\tau_1)\, \mathrm{d}s \\ &> x(t-\tau_1) \int_{\xi}^{t} \sum_{k=1}^{n} p_k(s)\, p_{k,\, m-1}(s)\, \mathrm{d}s \geqslant \frac{1}{2e^{m}}\, x(t-\tau_1) \end{aligned}$$

and

$$(37)\qquad \begin{aligned} x(t) &> \int_{t}^{\xi+\tau_1} \sum_{k=1}^{n} p_k(s)\, p_{k,\, m-1}(s)\, x(s-\tau_1)\, \mathrm{d}s \\ &> x(\xi) \int_{t}^{\xi+\tau_1} \sum_{k=1}^{n} p_k(s)\, p_{k,\, m-1}(s)\, \mathrm{d}s \geqslant \frac{1}{2e^{m}}\, x(\xi)\,. \end{aligned}$$

From (36) and (37), we obtain

$$x(t) > \frac{1}{4e^{2m}}\, x(t-\tau_1)$$

and consequently,

$$(38)\qquad w_1(t) < 4e^{2m}, \qquad t \geqslant t_2 + (m-1)\,\tau_1 .$$

But from (30) we get

$$1 \leqslant w_{k^*}(t) \leqslant w_1(t) < 4e^{2m}, \qquad t \geqslant t_2 + (m-1)\,\tau_1 .$$

This contradicts (29) and completes the proof.

Corollary 1. *Let* $p_i(t) \in C([t_0, \infty), \Re^+)$, $i = 1, 2, \ldots n$ *and* $\tau_n = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$. *Suppose that there exist a* $t_1 > t_0 + \tau_n$ *and a positive integer* $m$ *such that*

$$(39)\qquad \liminf_{t\to\infty} p_{k,m}(t) > \frac{1}{e^{m}} \;\; \textit{and} \;\; \liminf_{t\to\infty} \overline{P}_m(t) > \frac{1}{e^{m}}, \;\; t > t_1,\; k = 1, 2, \ldots, n,\; m = 1, 2, \ldots$$

*where* $p_{k,\, m}(t)$ *and* $\overline{P}_m(t)$ *are defined by* (7) *and* (8) *respectively. Then every solution of eq.* (2) *oscillates.*

### 3 - Example

Consider the delay differential equation

$$\dot{x}(t) + \frac{1}{3e}(1+\cos t)\, x(t-\pi) + \frac{1}{15e}(1+\sin t)\, x(t-2\pi) = 0\,, \qquad t \geqslant 0\,. \tag{40}$$

i.e., $p_1 = \frac{1}{3e}(1+\cos t),\ \tau_1 = \pi,$

$p_2(t) = \frac{1}{15e}(1+\sin t),\ \tau_2 = 2\pi.$

We have:

(1) $$\liminf_{t\to\infty}\left(\int_t^{t+\pi}\frac{1}{3e}(1+\cos\zeta)\,d\zeta + \int_t^{t+2\pi}\frac{1}{15e}(1+\sin\zeta)\,d\zeta\right) = \frac{1}{15e}(7\pi-10) < \frac{1}{e}\,.$$

This shows that Corollary of Li (5) do not hold.

(2) $$\liminf_{t\to\infty}\left(\left[\frac{1}{3e}(1+\cos t)\right]\left[\frac{1}{15e}(1+\sin t)\right]\right)^{1/2}(3\pi) < \frac{1}{e}\,.$$

This shows that (3*c*) do not hold.

(3) $$\limsup_{t\to\infty}\int_{t-\pi}^{t}\left\{\frac{1}{3e}(1+\cos\zeta) + \frac{1}{15e}(1+\sin\zeta)\right\}d\zeta = \frac{1}{15e}(6\pi+\sqrt{26}) < 1\,.$$

This shows that (3*d*) do not hold.

(4) $$\liminf_{t\to\infty}\left(\frac{\pi}{3e}(1+\cos t) + \frac{2\pi}{15e}(1+\sin t)\right) = \frac{\pi}{15e}(7-\sqrt{29}) < \frac{1}{e}\,.$$

This shows that (3*e*) do not hold. But according (7) and (8) one can write

$$p_{1,1}(t) = \int_{t-\pi}^{t}(p_1(\zeta)\,p_{1,0}(\zeta) + p_2(\zeta)\,p_{2,0}(\zeta))\,d\zeta = \int_{t-\pi}^{t}\left(\frac{1}{3e}(1+\cos\zeta) + \frac{1}{15e}(1+\sin\zeta)\right)d\zeta$$

$$= \frac{2\pi}{5e} + \frac{2}{3e}\sin t - \frac{2}{15e}\cos t\,,$$

$$P_{2,1}(t) = \int_{t-2\pi}^{t}(p_1(\zeta)\,p_{1,0}(\zeta) + p_2(\zeta)\,p_{2,0}(\zeta))\,d\zeta$$

$$= \int_{t-2\pi}^{t}\left(\frac{1}{3e}(1+\cos\zeta) + \frac{1}{15e}(1+\sin\zeta)\right)d\zeta = \frac{4\pi}{5e}\,.$$

Then

$$\liminf_{t\to\infty} p_{1,1}(t) = \frac{1}{15e}(6\pi - \sqrt{26}) < \frac{1}{e}\,.$$

So, we find $p_{1,2}(t)$ and $p_{2,2}(t)$ as following,

$$\begin{aligned} p_{1,2} &= \int_{t-\pi}^{t} (p_1(\zeta)\,p_{1,1}(\zeta) + p_2(\zeta)\,p_{2,1}(\zeta))\,\mathrm{d}\zeta \\ &= \int_{t-\pi}^{t} \left\{ \frac{1}{3e}(1+\cos\zeta)\left(\frac{2\pi}{5e} + \frac{2}{3e}\sin\zeta - \frac{2}{15e}\cos\zeta\right) + \frac{4\pi}{75e^2}(1+\sin\zeta)\right\}\mathrm{d}\zeta \\ &= \frac{\pi}{225e^2}(42\pi - 5) + \frac{4(3\pi-1)}{45e^2}\sin t - \frac{4(25+6\pi)}{225e^2}\cos t\,, \end{aligned}$$

$$\liminf_{t\to\infty} p_{1,2}(t) = \frac{\pi}{225e^2} - \frac{4}{225e^2}\sqrt{25(3\pi-1)^2 + (25+6\pi)^2} > \frac{1}{e^2}$$

and

$$\begin{aligned} p_{2,2} &= \int_{t-2\pi}^{t} (p_1(\zeta)\,p_{1,1}(\zeta) + p_2(\zeta)\,p_{2,1}(\zeta))\,\mathrm{d}\zeta \\ &= \int_{t-2\pi}^{t} \left\{ \frac{1}{3e}(1+\cos\zeta)\left(\frac{2\pi}{5e} + \frac{2}{3e}\sin\zeta - \frac{2}{15e}\cos\zeta\right) + \frac{4\pi}{75e^2}(1+\sin\zeta)\right\}\mathrm{d}\zeta \\ &= \frac{2\pi}{225e^2}(30\pi + 7) > \frac{1}{e^2}\,. \end{aligned}$$

Also,

$$\begin{aligned} \overline{p}_1(t) &= \int_{t}^{t+\pi} p_1(\zeta)\,p_0(\zeta)\,\mathrm{d}\zeta + \int_{t}^{t+2\pi} p_2(\zeta)\,p_0(\zeta)\,\mathrm{d}\zeta \\ &= \int_{t}^{t+\pi} \frac{1}{3e}(1+\cos\zeta)\,\mathrm{d}\zeta + \int_{t}^{t+2\pi} \frac{1}{15e}(1+\sin\zeta)\,\mathrm{d}\zeta = \frac{1}{15e}(7\pi - 10\ \sin t), \end{aligned}$$

$$\liminf_{t\to\infty} \overline{p}_1(t) = \frac{1}{15e}(7\pi - 10) < \frac{1}{e}\,.$$

But,

$$\overline{p}_2(t) = \int_t^{t+\pi} p_1(\zeta)\,\overline{p}_1(\zeta)\,\mathrm{d}\zeta + \int_t^{t+2\pi} p_2(\zeta)\,\overline{p}_1(\zeta)\,\mathrm{d}\zeta$$

$$= \int_t^{t+\pi} \frac{1}{45e^2}(1+\cos\zeta)(7\pi - 10\sin\zeta)\,\mathrm{d}\zeta + \int_t^{t+2\pi} \frac{1}{225e^2}(1+\sin\zeta)(7\pi-10\sin\zeta)\,\mathrm{d}\zeta$$

$$= \frac{1}{225e^2}\left[39\pi^2 - 10\pi - 2(10\cos t + 7\pi\sin t)\right].$$

Consequently,

$$\liminf_{t\to\infty} \overline{p}_2(t) = \frac{1}{225e^2}\left[39\pi^2 - 10\pi - 2\sqrt{100+49\pi^2}\right] > \frac{1}{e^2}$$

Then, by Corollary 1, every solution of (40) oscillates.

Remark. Equation (40) is also oscillatory by (3*b*), where

$$\liminf_{t\to\infty} \int_{t-2\pi}^{t} \left\{ \frac{1}{3e}(1+\cos\zeta) + \frac{1}{15e}(1+\sin\zeta)\right\}\mathrm{d}\zeta = \frac{4\pi}{5e} > \frac{1}{e}\,.$$

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## Abstract

*A sufficient condition for the oscillation of all solutions of*

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t)\, x(t - \tau_i) = 0$$

*where* $p_i(t) \geqslant 0$ *are continuous functions and* $\tau_i$ *are positive constants, is obtained, which is an extension for the previously known results.*

* * *