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An elliptic problem with a layer (**)

1 - Introduction

Let Ω be an open set in R^N . We denote by Σ the set $\Omega \cap \{x_N = 0\}$ and we assume $\Sigma \neq \emptyset$. Moreover we denote $x = (x', x_N) = (x_1, x_2, \dots, x_{N-1}, x_N)$. We consider the problem

$$(1.1) \quad \int Du Dv m(dx) + \int D' u D' v \sigma(dx) = 0$$

$$u \in H_{loc}^1(\Omega) \quad \text{with trace in } H_{loc}^1(\Sigma)$$

$$\forall v \in H_{loc}^1(\Omega) \quad \text{with trace in } H_{loc}^1(\Sigma), \quad \text{with } \text{supp}(v) \subseteq \Omega$$

where m denotes the Lebesgue measure on R^N and σ denotes the Lebesgue measure on R^{N-1} ; moreover we denote by D the gradient in R^N and by D' the tangential gradient on R_x^{N-1} . If u verifies (1.1) we say that u is a *solution* of (1.1). If we replace in (1.1) the equality by the inequality \leq (\geq) and we consider only positive test functions v we say that u is a *subsolution* (*supersolution*) of (1.1) in Ω .

The aim of this paper is to study the local regularity for a solution of (1.1).

If we consider a ball that does not intersect Σ the problem of the regularity of u is reduced to the problem of the regularity of an harmonic function; then in par-

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ticular Harnack inequality for nonnegative u and Hölder continuity for u hold. Problems arise in the case of balls having a non empty intersection with Σ (due to the different rescaling by the usual dilation of the two terms in (1.1)).

We also observe that the bilinear form in (1.1) defines a strongly local regular Dirichlet form on $L^2(\Omega, m + \delta_\Sigma)$, [3], but the measure $m + \delta_\Sigma$ does not verify a doubling property, then the regularity theory in [1] [2] does not apply.

To study the local regularity of u in $B(x_0, r)$ with $x_0 \in R_x^{N-1}$ we modify the definition of a ball defining $B(x_0, r) = \{x: |x' - (x_0)'|^4 + |x_N - (x_0)_N|^2 < r^4\}$ and we write $S(x_0, r) = B(x_0, r) \cap \{x_N = 0\}$. We define a cut-off function between $B(x_0, tr)$ and $B(x_0, sr)$, $s, t \in \left[\frac{1}{2}, 1\right)$ $s < t$, as $\eta(x) = \phi(d(x - x_0))$ where $d(x) = (|x'|^4 + x_N^2)^{\frac{1}{4}}$, $\phi(\varrho) = 1$ for $\varrho \leq sr$, $\phi(\varrho) = 0$ for $\varrho \geq tr$, $0 \leq \phi \leq 1$ and $|\phi'| \leq \frac{C}{(t-s)r}$. Then

$$|D'\eta| \leq \frac{C}{(t-s)r} \quad \text{on } S(x_0, r), \quad |D\eta| \leq \frac{C}{(t-s)r^2} \quad \text{on } B(x_0, r).$$

With such a modification we obtain:

Theorem 1.1. *Let u be a nonnegative solution of (1.1) in $B(x_0, 4r)$, $x_0 \in R_x^{N-1}$; then*

$$\sup_{B(x_0, r)} u \leq C \inf_{B(x_0, r)} u$$

where C is a constant depending only on N .

Theorem 1.1. *Let u be a solution of (1.1) in Ω ; then u is locally Hölder continuous in Ω .*

In Section 2 we prove suitable Poincaré and Sobolev type inequalities, that play a fundamental role in proving, by a Moser type iteration method, local L^∞ estimates for solutions (or subsolutions) of (1.1), as we give in Section 3. In Section 4 we prove Theorems 1.1 and 1.2. Theorem 1.2 is an easy consequence of Theorem 1.1; the proof Theorem 1.1 uses an iteration method introduced by Moser, [6], that allow us to consider estimates only on concentric balls $B(x_0, r)$; this last opportunity is useful on account of the different forms of the balls in the case $x_0 \in \Sigma$ or $x_0 \notin \Sigma$.

2 - Poincaré and Sobolev type inequalities

It is well known that a fundamental role in the local regularity theory of harmonic functions relative to an uniformly elliptic operator is played by the usual Poincaré and Sobolev inequalities.

The goal of this section is to prove suitably adapted Poincaré and Sobolev inequalities relative to the problem in consideration.

Proposition 2.1. *Let u be a function in $H^1(B(x_0, r))$, $x_0 \in \mathbb{R}_x^{N-1}$, with a trace in $H^1(S(x_0, r))$ and u_r be the average of u on $S(x_0, r)$ relative to the measure σ ; then*

$$\begin{aligned} & \int_{B(x_0, r)} |u - u_r|^2 m(dx) + \int_{S(x_0, r)} |u - u_r|^2 \sigma(dx) \\ & \leq C \left[r^4 \int_{B(x_0, r)} |D_N u|^2 m(dx) + r^2 \int_{S(x_0, r)} |D' u|^2 \sigma(dx) \right] \end{aligned}$$

where $\int_{B(x_0, r)} m(dx) \left(\int_{S(x_0, r)} \sigma(dx) \right)$ denotes the average on the set $B(x_0, r)$ ($S(x_0, r)$) relative to the measure m (σ).

The result in Proposition 2.1 is a consequence of the following Sobolev type inequality:

Proposition 2.2. *Let u be a function in $H^1(B(x_0, r))$, $x_0 \in \mathbb{R}_x^{N-1}$, with a trace in $H^1(S(x_0, r))$.*

(a) *Let $N > 3$; there exists $q > 2$ such that*

$$\begin{aligned} & \left[\int_{B(x_0, r)} |u - u_r|^q m(dx) + \int_{S(x_0, r)} |u - u_r|^q \sigma(dx) \right]^{\frac{2}{q}} \\ & \leq C \left[r^4 \int_{B(x_0, r)} |D_N u|^2 m(dx) + r^2 \int_{S(x_0, r)} |D' u|^2 \sigma(dx) \right]. \end{aligned}$$

(b) Let $N = 3$; for every $q > 2$ we have

$$\begin{aligned} & \left[\int_{B(x_0, r)} |u - u_r|^q m(dx) + \int_{S(x_0, r)} |u - u_r|^q \sigma(dx) \right]^{\frac{2}{q}} \\ & \leq C \left[r^4 \int_{B(x_0, r)} |D_N u|^2 m(dx) + r^2 \int_{S(x_0, r)} |D' u|^2 \sigma(dx) \right]. \end{aligned}$$

(c) Let $N = 2$; then

$$\operatorname{osc}_{B(x_0, r)} u \leq C \left[r^4 \int_{B(x_0, r)} |D_N u|^2 m(dx) + r^2 \int_{S(x_0, r)} |D' u|^2 \sigma(dx) \right]^{\frac{1}{2}}.$$

Proof. We prove the result for the case (a); the proof in the cases (b) and (c) is analogous.

It is enough to prove the result in the case $r = 1$ and we write $B(x_0, r) = B$, $S(x_0, r) = S$.

Let $s = \frac{2N-2}{N-3}$; we have

$$\left[\int_S |u - u_1|^s \sigma(dx) \right]^{\frac{2}{s}} \leq C \int_S |D' u|^2 \sigma(dx),$$

where we denote by C possibly different constants depending only on N .

Let $q = \frac{2+s}{2} = \frac{2N-4}{N-3}$; by easy computations we obtain

$$\begin{aligned} & \sup_{((x_0)_{N-1}, (x_0)_{N+1})_{B \cap \{x_N=t\}}} \int |u - u_1|^q \sigma(dx) \\ & \leq C \left[\left(\int_B |D_N u|^2 m(dx) \right)^{\frac{q}{2}} + \left(\int_S |u - u_1|^s \sigma(dx) \right)^{\frac{q}{s}} \right] \\ & \leq C \left[\left(\int_B |D_N u|^2 m(dx) \right)^{\frac{q}{2}} + \left(\int_S |D' u|^2 \sigma(dx) \right)^{\frac{q}{2}} \right] \end{aligned}$$

and the result follows.

3 - The local L^∞ estimate for subsolutions

We prove at first an L^∞ estimate for nonnegative subsolution of (1.1) and finally we prove the general L^∞ estimate for nonnegative solutions of (1.1)

Proposition 3.1. *Let u be a function in $H^1(B(x_0, r))$, $x_0 \in \mathbb{R}_x^{N-1}$, with a trace in $H^1(S(x_0, r))$. Assume that u is a nonnegative subsolution in a neighbourhood of $B(x_0, r)$; then there exists constants d and C such that for $\alpha \in \left[\frac{1}{2}, 1\right)$ and $p \geq 2$ we have*

$$\left(\sup_{B(x_0, \alpha r)} u \right)^p \leq \frac{C}{(1-\alpha)^d} \left[\int_{B(x_0, r)} u^p m(dx) + \int_{S(x_0, r)} u^p \sigma(dx) \right]^{\frac{1}{p}}.$$

Proof. Let $\beta \geq 1$ and $0 < M < +\infty$; we define

$$H_M(t) = t^\beta \quad \text{for } t \in [0, M]$$

$$H_M(t) = M^\beta + \beta M^{\beta-1}(t - M) \quad \text{for } t > M.$$

The function $H_M(t)$ is Lipschitz-continuous for every fixed M .

We assume that u is Lipschitz continuous (if it is not the case we use an approximation of u in $H^1(B(x_0, r))$ and in $H^1(S(x_0, r))$ by a sequence $\{u_k\}$ of nonnegative Lipschitz-continuous functions).

For a fixed M we define

$$\phi(x) = \eta(x)^2 \int_0^{u(x)} H'_M(t)^2 dt$$

where η is a Lipschitz continuous function with support in $B(x_0, r)$ to be chosen. We observe that ϕ are nonnegative Lipschitz continuous functions defined in $B(x_0, r)$ and

$$(3.1) \quad D_i \phi = \eta^2 H'_M(u)^2 D_i u + 2\eta D_i \eta \int_0^{u(x)} H'_M(t)^2 dt$$

for $i = 1, 2, \dots, N$. Since u is a subsolution we have

$$(3.2) \quad \int Du \eta^2 H'_M(u)^2 Du m(dx) + \int Du 2\eta D\eta \left(\int_0^{u_k} H'_M(t)^2 dt \right) m(dx) \\ + \int D'u \eta^2 H'_M(u)^2 D'u \sigma(dx) + \int D'u 2\eta D'\eta \left(\int_0^u H'_M(t)^2 dt \right) \sigma(dx) \leq 0.$$

We observe that

$$\left| D_i u 2\eta D_i \eta \left(\int_0^u H'_M(t)^2 dt \right) \right| \leq \frac{1}{2} D_i u \eta^2 H'_M(u)^2 D_i u \\ + 2 |D_i \eta|^2 \left(\frac{1}{H'_M(u)} \int_0^u H'_M(t)^2 dt \right)^2 \\ \leq \frac{1}{2} D_i u \eta^2 H'_M(u)^2 D_i u + 2 |D_i \eta|^2 (u H'_M(u))^2.$$

From (3.2) it follows

$$(3.3) \quad \frac{1}{2} \int |D(H_M(u))|^2 \eta^2 m(dx) + \frac{1}{2} \int |D'(H_M(u))|^2 \eta^2 \sigma(dx) \\ \leq 2 \int |D\eta|^2 (u H'_M(u))^2 m(dx) + 2 \int |D'\eta|^2 (u H'_M(u))^2 \sigma(dx).$$

We choose now η as the cut-off function between $B(x_0, sr)$ and $B(x_0, tr)$. From (3.3) we obtain

$$(3.4) \quad \int_{B(x_0, sr)} |D(H_M(u))|^2 m(dx) + \int_{S(x_0, sr)} |D'(H_M(u))|^2 \sigma(dx) \\ \leq \frac{c_2}{(t-s)^2 r^4} \int_{B(x_0, tr)} (u H'_M(u))^2 m(dx) + \frac{c_2}{(t-s)^2 r^2} \int_{S(x_0, tr)} (u H'_M(u))^2 \sigma(dx).$$

Using the Sobolev inequality we obtain

$$\begin{aligned} & \left[\int_{B(x_0, sr)} |H_M(u) - (H_M(u))_{sr}|^q m(dx) + \int_{S(x_0, sr)} |H_M(u) - (H_M(u))_{sr}|^q \sigma(dx) \right]^{\frac{1}{q}} \\ & \leq c_3 \frac{s}{t-s} \left[\int_{B(x_0, tr)} (uH'_M(u))^2 m(dx) + \int_{S(x_0, tr)} (uH'_M(u))^2 \sigma(dx) \right]^{\frac{1}{2}} \end{aligned}$$

where q is as Proposition 2.2 for $N \geq 3$ and is an arbitrarily fixed real number greater than 2 if $N = 2$.

We use the inequality $H(t) \leq tH'(t)$ and we obtain

$$\begin{aligned} & \left[\int_{B(x_0, sr)} |H_M(u)|^q m(dx) + \int_{S(x_0, sr)} |H_M(u)|^q \sigma(dx) \right]^{\frac{1}{q}} \\ & \leq c_5 \left(\frac{s}{t-s} + 1 \right) \left[\int_{B(x_0, tr)} (uH'_M(u))^2 m(dx) + \int_{S(x_0, tr)} (uH'_M(u))^2 \sigma(dx) \right]^{\frac{1}{2}}. \end{aligned}$$

We observe that $\left(\frac{s}{t-s} + 1 \right) \leq 2 \frac{s}{t-s}$. We take into account the definition of H_M and we let $M \rightarrow +\infty$; then

$$\begin{aligned} & \left[\int_{B(x_0, sr)} u^{\beta q} m(dx) + \int_{S(x_0, sr)} u^{\beta q} \sigma(dx) \right]^{\frac{1}{q}} \\ & \leq c_6 \beta \frac{s}{t-s} \left[\int_{B(x_0, tr)} u^{2\beta} m(dx) + \int_{S(x_0, tr)} u^{2\beta} \sigma(dx) \right]^{\frac{1}{2}}. \end{aligned}$$

We write $2\beta = \nu$, $q = 2\tau$ ($\tau > 1$) and we obtain

$$\begin{aligned} & \left[\int_{B(x_0, sr)} u^{\tau\nu} m(dx) + \int_{S(x_0, sr)} u^{\tau\nu} \sigma(dx) \right]^{\frac{1}{\tau\nu}} \\ (3.5) \quad & \leq \left(c_6 \nu \frac{s}{t-s} \right)^{\frac{2}{\nu}} \left[\int_{B(x_0, tr)} u^{\nu} m(dx) + \int_{S(x_0, tr)} u^{\nu} \sigma(dx) \right]^{\frac{1}{\nu}}. \end{aligned}$$

From (3.5) an iteration method of Moser's type (see for example [3]) gives the result.

Proposition 3.2. *Let u be a local nonnegative solution of our problem (1.1) in $B(x_0, 2r)$, $x_0 \in \mathbb{R}_x^{N-1}$. Then there exists constants d , $\tau > 1$ and C such that for $\alpha \in \left[\frac{1}{2}, 1\right)$ and every real number p we have*

$$\left(\sup_{B(x_0, \alpha r)} u\right)^p \leq \frac{C}{(1-\alpha)^d} (1+|p|)^{2\tau/(\tau-1)} \left[\int_{B(x_0, r)} u^p m(dx) + \int_{S(x_0, r)} u^p \sigma(dx) \right]^{\frac{1}{p}}.$$

Proof. It is enough to prove the result in the case $-\infty < p < 2$ and $u \geq \varepsilon > 0$.

By Proposition 3.1 u is bounded in $B(x_0, r)$; we define $\phi = \eta^2 u^\beta$, with $\beta \leq 1$ and we can prove that ϕ is in $H^1(B(x_0, r))$ and its trace is in $H^1(S(x_0, r))$.

We recall that

$$\begin{aligned} D_i \phi &= \eta^2 \beta u^{\beta-1} D_i u + 2\eta D_i \eta u^\beta \\ D_i \left(u^{\frac{\beta+1}{2}}\right) &= \frac{\beta+1}{2} u^{\frac{\beta-1}{2}} D_i u. \end{aligned}$$

Then for $\beta \neq -1$ we obtain

$$\begin{aligned} (3.6) \quad & \left| \frac{\beta}{\beta+1} \right|^2 \int_{B(x_0, r)} |D(u^{\frac{\beta+1}{2}})|^2 \eta^2 m(dx) + \left| \frac{\beta}{\beta+1} \right| \int_{S(x_0, r)} |D'(u^{\frac{\beta+1}{2}})|^2 \eta^2 \sigma(dx) \\ & \leq \int_{B(x_0, r)} |D(u^{\frac{\beta+1}{2}}) D \eta| u^{\frac{\beta+1}{2}} \eta m(dx) + \int_{S(x_0, r)} |D'(u^{\frac{\beta+1}{2}}) D' \eta| u^{\frac{\beta+1}{2}} \eta \sigma(dx). \end{aligned}$$

From (3.6) we easily obtain for $\beta \neq 0, -1$

$$\begin{aligned} & \int_{B(x_0, r)} |D(u^{\frac{\beta+1}{2}})|^2 \eta^2 m(dx) + \int_{S(x_0, r)} |D'(u^{\frac{\beta+1}{2}})|^2 \eta^2 \sigma(dx) \\ & \leq \left(\frac{\beta+1}{\beta}\right)^2 \int_{B(x_0, r)} |D \eta|^2 u^{\beta+1} m(dx) + \left(\frac{\beta+1}{\beta}\right)^2 \int_{S(x_0, r)} |D' \eta|^2 u^{\beta+1} \sigma(dx). \end{aligned}$$

Then, taking again η as the cut-off function between $B(x_0, sr)$ and $B(x_0, tr)$,

$\frac{1}{2} \leq s < t < 1$, we have

$$\begin{aligned} & \int_{B(x_0, sr)} |D(u^{\frac{\beta+1}{2}})|^2 m(dx) + \int_{S(x_0, sr)} |D'(u^{\frac{\beta+1}{2}})|^2 \sigma(dx) \\ & \leq \left(\frac{\beta+1}{\beta}\right)^2 \frac{1}{(t-s)^2 r^4} \int_{B(x_0, tr)} u^{\beta+1} m(dx) + \left(\frac{\beta+1}{\beta}\right)^2 \frac{1}{(t-s)^2 r^2} \int_{S(x_0, tr)} u^{\beta+1} \sigma(dx). \end{aligned}$$

We use now the Sobolev inequality in Proposition 2.2; then by the same methods as in Proposition 3.1 we have

$$\begin{aligned} (3.7) \quad & \left[\int_{B(x_0, sr)} u^{\frac{\beta+1}{2}q} m(dx) + \int_{S(x_0, sr)} u^{\frac{\beta+1}{2}q} \sigma(dx) \right]^{\frac{1}{q}} \\ & \leq c \left| \frac{\beta+1}{\beta} \right| \left(\left| \frac{s}{t-s} \right| + 1 \right) \left[\int_{B(x_0, tr)} u^{\beta+1} m(dx) + \int_{S(x_0, tr)} u^{\beta+1} \sigma(dx) \right]^{\frac{1}{2}}. \end{aligned}$$

Setting $\beta+1 = \nu$ and $q = 2\tau$ we have for any $-\infty < \nu \leq 2$, $\nu \neq 0, -1$

$$\begin{aligned} (3.8) \quad & \left[\int_{B(x_0, sr)} u^{\tau\nu} m(dx) + \int_{S(x_0, sr)} u^{\tau\nu} \sigma(dx) \right]^{\frac{1}{\tau|\nu|}} \\ & \leq c \frac{2}{|\nu|} \left(\left| \frac{\nu}{\nu-1} \right| \left| \frac{s}{t-s} \right| + 1 \right)^{\frac{2}{|\nu|}} \left[\int_{B(x_0, tr)} u^{\nu} m(dx) + \int_{S(x_0, tr)} u^{\nu} \sigma(dx) \right]^{\frac{1}{|\nu|}}. \end{aligned}$$

From (3.8) the result follows by a Moser's type iteration argument (see for example [3]).

Proposition 3.3. *Let the assumptions of Proposition 3.2 hold and assume that $u \geq \varepsilon > 0$. For $\alpha \in \left[\frac{1}{2}, 1\right)$ let us define k by*

$$\log k = \int_{S(x_0, ar)} \log u \sigma(dx),$$

$x_0 \in \Sigma$; then for $\lambda > 0$ we have

$$m \left\{ x \in B(x_0, \alpha r); \left| \log \left(\frac{u(x)}{k} \right) \right| > \lambda \right\} \geq \frac{C}{\lambda(1-\alpha)} m(B(x_0, \alpha r))$$

$$\sigma \left\{ x \in S(x_0, \alpha r); \left| \log \left(\frac{u(x)}{k} \right) \right| > \lambda \right\} \geq \frac{C}{\lambda(1-\alpha)} \sigma(S(x_0, \alpha r))$$

where C is a constant that does not depend on ε .

Proof. By easy computations we obtain

$$\begin{aligned} & \int |D(\log u)|^2 \eta^2 m(dx) + \int |D'(\log u)|^2 \eta^2 \sigma(dx) \\ & \leq 4 \int |D\eta|^2 m(dx) + \int |D'\eta|^2 \sigma(dx) \\ & \leq \frac{c}{(1-\alpha)^2} \left(\frac{m(B(x_0, r))}{r^4} + \frac{\sigma(S(x_0, r))}{r^2} \right) \end{aligned}$$

where η is the cut-off function between $B(x_0, \alpha r)$ and $B(x_0, r)$. By Proposition 2.1 we obtain

$$(3.9) \quad \int_{B(x_0, \alpha r)} |\log u - \log k|^2 m(dx) \leq \frac{c}{(1-\alpha)^2} \frac{m(B(x_0, r))}{r^4}$$

$$(3.10) \quad \int_{S(x_0, \alpha r)} |\log u - \log k|^2 \sigma(dx) \leq \frac{c}{(1-\alpha)^2} \frac{\sigma(S(x_0, r))}{r^2}.$$

From (3.10) and (3.11) the result easily follows.

4 - Proof of Theorems 1.1 and 1.2

We are now in position to prove Theorem 1.1.

Lemma 4.1. *Let $m, \mu, C, \theta \in \left[\frac{1}{2}, 1 \right)$ be positive constants and let $w > 0$ be*

a function in $H^1(B(x_0, r))$, $x_0 \in \mathbb{R}_x^{N-1}$, such that

$$(4.1) \quad \begin{aligned} \sup_{B(x_0, sr)} w^p &\leq \frac{C}{(t-s)^d} \frac{1}{m(B(x_0, r))} \int_{B(x_0, tr)} w^p m(dx) \\ &+ \frac{C}{(t-s)^d} \frac{1}{\sigma(S(x_0, r))} \int_{S(x_0, tr)} w^p \sigma(dx) \end{aligned}$$

for all $\frac{1}{2} \leq \theta \leq s < t \leq 1$, $0 < p < \mu^{-1}$. Moreover, let

$$(4.2) \quad m(x \in B(x_0, r); \log w \geq \lambda) \leq \frac{C\mu}{\lambda} m(B(x_0, r))$$

$$(4.3) \quad \sigma(x \in S(x_0, r); \log w \geq \lambda) \leq \frac{C\mu}{\lambda} \sigma(B(x_0, r))$$

for all $\lambda > 0$. Then there exists a constant $\gamma = \gamma(\theta, d, C)$ such that

$$\sup_{B(x_0, \theta r)} u \leq \gamma^\mu.$$

Proof. We assume, without loss of generality, $r = 1$. Replacing w by w^μ and λ by $\lambda\mu$ we reduce us to the case $\mu = 1$.

Define

$$\phi(s) = \sup_{B(x_0, s)} \log w, \quad \theta \leq s < 1.$$

We observe that $\phi(s)$ is a nondecreasing function.

We now prove that the following inequality holds:

$$(4.4) \quad \phi(s) \leq \frac{3}{4} \phi(t) + \frac{\gamma_1}{(t-s)^{2d}}$$

where $\theta \leq s < t \leq 1$ and γ_1 is a constant depending on θ, d, C .

We decompose $B(x_0, t)$ and $S(x_0, t)$ into the sets where $\log w > \frac{1}{2} \phi(t)$ and

where $\log w \leq \frac{1}{2} \phi(t)$; then taking into account (4.2) and (4.3) we obtain

$$\int_{B(x_0, t)} w^p m(dx) \leq \left(e^{p\phi(t)} \frac{2C}{\phi(t)} + e^{p\frac{\phi(t)}{2}} \right) m(B(x_0, 1))$$

$$\int_{S(x_0, t)} w^p \sigma(dx) \leq \left(e^{p\phi(t)} \frac{2C}{\phi(t)} + e^{p\frac{\phi(t)}{2}} \right) \sigma(S(x_0, 1)).$$

Summing up the two inequalities we have

$$\frac{1}{m(B(x_0, 1))} \int_{B(x_0, t)} u^p m(dx) + \frac{1}{\sigma(S(x_0, 1))} \int_{S(x_0, t)} u^p \sigma(dx)$$

$$\leq 2 \left(e^{p\phi(t)} \frac{2C}{\phi(t)} + e^{p\frac{\phi(t)}{2}} \right).$$

We choose now p such that the two terms in the right-hand side are equal:

$$p = \frac{2}{\phi(t)} \log \left(\frac{\phi(t)}{2C} \right)$$

provided the term in the right-hand side is less than $\mu^{-1} = 1$; this last inequality requires

$$(4.5) \quad \phi(t) > c_1$$

where c_1 depends only on C .

In that case we have

$$\frac{1}{m(B(x_0, 1))} \int_{B(x_0, t)} u^p m(dx) + \frac{1}{\sigma(S(x_0, 1))} \int_{S(x_0, t)} u^p \sigma(dx) \leq 4e^{p\frac{\phi(t)}{2}}.$$

Hence by (4.1) we obtain

$$\phi(s) \leq \frac{1}{p} \log \left(\frac{4C}{(t-s)^d} e^{p\frac{\phi(t)}{2}} \right) = \frac{1}{p} \log \left(\frac{4C}{(t-s)^d} \right) + \frac{1}{2} \phi(t).$$

Then, taking into account the fixed value of p , the above inequality becomes

$$\phi(s) \leq \frac{1}{2} \phi(t) \left\{ \frac{\log \left(\frac{4C}{(t-s)^d} \right)}{\log \left(\frac{\phi(t)}{2C} \right)} + 1 \right\}.$$

If

$$(4.6) \quad \phi(t) \geq \frac{32C^3}{(t-s)^{2d}}$$

we obtain

$$\phi(s) \leq \frac{3}{4} \phi(t)$$

then (4.4) holds.

If (4.5) or (4.6) does not hold then

$$\phi(s) \leq \phi(t) \leq \frac{\gamma_1}{(t-s)^{2d}}$$

$\theta \leq s < t \leq 1$, where γ_1 is a constant depending on C , c_1 , d , θ ; so (4.4) holds again.

We have so proved the inequality (4.4); the result now follows by iteration as in Lemma 3 in [6].

We are now in position to prove the result of Theorem 1.1. We assume, without loss of generality, $u \geq \varepsilon > 0$. We use the result in Proposition 4.1 for $\frac{u}{k}$ and $\frac{k}{u}$, where $\log k = \int_{S(x_0, r)} \log u \sigma(dx)$. The assumptions of proposition 4.1 hold in $B(x_0, 2r)$ by Proposition 3.2 and 3.3; then

$$\sup_{B(x_0, r)} \frac{u}{k} \leq C, \quad \sup_{B(x_0, r)} \frac{k}{u} \leq C.$$

Then the result follows.

Theorem 1.2 follows from Theorem 1.1 by standard methods (see [5]).

References

- [1] M. BIROLI and U. MOSCO, *Formes de Dirichlet et estimations structurelles dans les milieux discontinus*, C. R. Acad. Sci. Paris **313** (1991), 593-598.
- [2] M. BIROLI and U. MOSCO, *A Saint-Venant principle for Dirichlet forms on discontinuous media*, Ann. Mat. Pura Appl. **CLXIX** (1995), 125-181.
- [3] S. CHANILLO and R. L. WHEEDEN, *Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations*, Comm. in P.D.E. **11** (1986), 1111-1134.
- [4] M. FUKUSHIMA, *Dirichlet forms and Markov processes*, North-Holland, Amsterdam 1980.
- [5] D. GILBARG and N. S. TRUDINGER, *Second order elliptic equations*, Grundlehren der Math. Wiss. 224, Springer Verlag, Berlin-Heidelberg-New York 1977.
- [6] J. MOSER, *On a pointwise estimate for parabolic differential equations*, Comm. Pure Appl. Math. **XXIV** (1971), 727-740.

Abstract

We consider the problem of local regularity for a local solution of an elliptic problem with a layer. The difficulty comes from the different homogeneity degree between the measure of a ball and the measure of its intersection with the layer. The main tool is a modification of the balls centered on the layer.
