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Complex Casson invariants (**)

1 - Introduction

The methods and the techniques of gauge theory are widely recognized as one of the main achievements in the theory of 4-manifolds over the last two decades, yielding a vast range of fundamental results, which have shed a new light on topology and geometry in dimension 3 and 4: just recall Donaldson theory (see e.g. [**8**]) and Seiberg-Witten theory (see [**9**] for an excellent overview).

The extension to higher dimension appears as a quite demanding challenge running through the research by many mathematicians in the quest for new invariants.

The goal of the present paper is to echo, discuss, and explain some of the attempts to define, via gauge theory, a Casson-like invariant for complex manifolds (see [**12**], [**6**], [**11**]).

One of the basic settings is the description of moduli spaces of bundle complex structures stemming from [**5**].

Note that we dare to insist (cf. [**4**]) with an «irregular» terminology: «complex» instead «almost-complex», «holomorphic» instead of «complex», «Hermitian» instead of «almost-Hermitian» etc...

2 - Real Casson invariants

We begin by quickly reviewing the geometric construction of Casson Invariants (see [**2**], [**1**]).

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Let M be a compact oriented 3-manifold; it is well known that M is parallelizable and so, for any Riemannian metric g on M, the principal $SO(3)$-bundle of oriented orthonormal frames on M, $SO_g(M)$, is trivial and so we can work with its $SU(2)$-covering P.

To simplify the situation, assume M has the homology of S^3. Roughly speaking, the Casson invariant counts the conjugacy classes of irreducible representations of $\pi_1(M)$ into $SU(2)$.

Let $f \in C^\infty(M, [0, 3])$ be a self-indexing Morse-Smale function; consider the corresponding Heegaard splitting:

$$M_- := f^{-1}\left(\left[0, \frac{3}{2}\right]\right), \qquad M_+ := f^{-1}\left(\left[\frac{3}{2}, 3\right]\right), \qquad \Sigma := f^{-1}\left(\frac{3}{2}\right)$$

therefore, Σ is a genus γ surface and $M_- \approx M_+$ is a standard γ-handlebody; set:

$$M_1 := f^{-1}\left(\left[0, \frac{7}{4}\right]\right), \qquad M_2 := f^{-1}\left(\left[\frac{5}{4}, 3\right]\right), \qquad M_0 := M_1 \cap M_2 = \Sigma \times \left[\frac{5}{4}, \frac{7}{4}\right]$$

for $k = 0, 1, 2$, let:

$$\begin{aligned} \mathfrak{m}_k &:= \{\textit{conjugacy classes of repr. of } \pi_1(M_k) \textit{ into } SU(2)\} \\ &= Hom\,(\pi_1(M_k), SU(2))/Ad(SU(2)) \end{aligned}$$

$$\widetilde{\mathfrak{m}}_k := \{\phi \in \mathfrak{m}_k \,|\, \phi \textit{ is irreducible}\}$$

now:

1) $\pi_1(M_1) = \pi_1(M_2)$ is a free group with γ generators; therefore

$$\mathfrak{m}_1 = \mathfrak{m}_2 = \prod_\gamma SU(2)/Ad(SU(2))$$

2) $\pi_1(M_0) = \pi_1(\Sigma) = \{[a_1, b_1, \ldots, a_\gamma, b_\gamma] \,|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1} = 1\}$ and so:

$$\mathfrak{m}_0 = \theta^{-1}(1)/Ad(SU(2))$$

with

$$\theta \colon \prod_{2\gamma} SU(2) \to SU(2), \; \theta(a_1, b_1, \ldots, a_\gamma, b_\gamma) = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1}$$

consider on the $\widetilde{\mathfrak{m}}_k$'s the induced structure; we have:

$$dim_{\mathbb{R}} \widetilde{\mathfrak{m}}_1 = dim_{\mathbb{R}} \widetilde{\mathfrak{m}}_2 = 3\gamma - 3 , \qquad dim_{\mathbb{R}} \widetilde{\mathfrak{m}}_0 = 6\gamma - 6$$

for $k = 1, 2$, the inclusion $j_k \colon M_0 \to M_k$ induces an embedding $j_k^* \colon \widetilde{\mathfrak{m}}_k \to \widetilde{\mathfrak{m}}_0$; the $\widetilde{\mathfrak{m}}_k$'s are all equipped with an orientation; in fact:

a) given an orientation on Σ, $H^1(\Sigma, \mathbb{R})$ has an intrinsic symplectic structure: if $[\alpha], [\beta] \in H^1(\Sigma, \mathbb{R})$, then:

$$\eta([\alpha], [\beta]) := \int_{\Sigma} \alpha \wedge \beta$$

is a symplectic form (1 is the middle dimension and $\eta([\alpha], [* \alpha]) > 0$!), so an orientation is induced on $H^1(\Sigma, \mathbb{R})$;

b) since, for $k = 1, 2$, M_k is an handlebody, we have:

$$0 \to H^1(M_k, \mathbb{R}) \xrightarrow{j_k^*} H^1(\Sigma, \mathbb{R})$$

from Stokes' theorem it follows that: $j_k^*(H^1(M_k, \mathbb{R}))$ is a Lagrangian subspace of $H^1(\Sigma, \mathbb{R})$ and

$$\langle [\alpha], [\beta] \rangle := \int_{\Sigma} j_k^*(\alpha) \wedge j_k^*(\beta)$$

is a duality between $H^1(M_1, \mathbb{R})$ and $H^1(M_2, \mathbb{R})$; therefore, the choice of an orientation on $H^1(M_1, \mathbb{R})$ produces an orientation on $H^2(M_1, \mathbb{R})$, giving back the orientation on $H^1(\Sigma, \mathbb{R})$;

c) a choice of a basis for $H^1(M_1, \mathbb{R})$ (consistent with the chosen orientation) identifies $Hom\,(\pi_1(M_1), SU(2))$ with $\prod_{\gamma} SU(2)$; this fact and a fixed orientation on $SU(2)$ determine an orientation on $\widetilde{\mathfrak{m}}_1$ (and on $\widetilde{\mathfrak{m}}_2$);

d) let $\Sigma_0 := \Sigma - pt$; a choice of a basis for $H^1(\Sigma, \mathbb{R})$ identifies $Hom\,(\pi_1(\Sigma_0), SU(2))$ with $\prod_{2\gamma} SU(2)$: this orients $\widetilde{\mathfrak{m}}_0$.

Set:

$$\kappa := j_1^*(\mathfrak{m}_1) \cap j_2^*(\mathfrak{m}_2) \approx Hom\,(\pi_1(M), SU(2))/Ad(SU(2))$$

and

$$\tilde{\kappa} := j_1^*(\widetilde{\mathfrak{m}}_1) \cap j_2^*(\widetilde{\mathfrak{m}}_2) \tag{1}$$

finally define:

$$\lambda(M) := \frac{1}{2} \sum_{p \in \tilde{\kappa}} (-1)^{index\ p}$$

according with a choice of orientation as in $a \div d$.

We have:

Theorem 2.1. *$\lambda(M)$ (which is called the Casson invariant of M) depends only on M and its orientation.*

We can immediately establish a bridge with the holonomy theory of connections.

Let $\widehat{\mathcal{F}}(\Sigma)$ be the moduli space of simple flat connections on $P_{|\Sigma}$ (see next section); therefore $\widehat{\mathcal{F}}(\Sigma)$ is a finite dimensional manifold with an intrinsic symplectic structure:

$$\eta(\alpha, \beta) := \int_\Sigma tr(\alpha \wedge \beta)$$

set:

$$L^{\pm} := \{[\omega] \in \widehat{\mathcal{F}}(\Sigma) \,|\, \omega \ \textit{extends as flat connection on } M_{\pm}\}$$

then $L^{\pm}$ are Lagrangian submanifolds of $\widehat{\mathcal{F}}(\Sigma)$,

$$L^+ \cap L^- \approx \tilde{\kappa} \tag{2}$$

is the moduli space of simple flat connections on M and $\lambda(M)$ is the intersection index of L_+ and L_-.

Recall that, for a general principal G-bundle $\pi \colon P \to M$, if $\widetilde{M}$ denotes the universal covering of M, we have:

$$\{\varrho \colon \pi_1(M) \to G \,|\, P = \widetilde{M} \times_\varrho G\} \leftrightarrow \{\textit{flat connections on } P\}$$

in fact:

$\to$: just extend to P the natural flat connection on $\widetilde{M}$

$\leftarrow$: the parallel displacement gives a representation $\varrho\colon \pi_1(M) \to G$ whose image is the holonomy group of P and $P = \widetilde{M} \times_{\varrho} G$.

In our special case, the assumption $P = \widetilde{M} \times_{\varrho} SU(2)$ is automatically satisfied.

As we shall see in the next section, the relations between Casson invariants and gauge theory are much deeper.

3 - Moduli spaces of connections

We briefly recall some general facts (see [**3**] for more).

Let $\pi\colon P \to M$ be a principal G-bundle over a compact Riemannian manifold (M, g) and let γ be a Riemannian structure on its adjoint bundle $\mathcal{A} := P \times_{ad} \mathfrak{g}$; let $\mathcal{G}(P) := \{\phi \in Diff(P) \,|\, \pi \circ \phi = \phi,\ \phi \textit{ is G-equivariant}\}$ be the gauge group of P; let V be an $\mathbb{R}$-vector space and let $\sigma\colon G \to Aut(V)$ be any representation: recall that

$$\mathcal{T}^p(P, V, \sigma) := \{\alpha \in \wedge^p(P, V) \,|\, \alpha \textit{ is G-equivariant and horizontal}\}$$

let $\mathfrak{T}^p(P, V, \sigma)$ be the bundle whose sections are the elements of $\mathcal{T}^p(P, V, \sigma)$; $\mathcal{C}(P)$ denotes the space of connection forms on P: it is an infinite dimensional affine space having $\mathcal{T}^1(P, \mathfrak{g}, ad)$ as space of translations.

We recall also that the *Yang-Mills functional*

$$YM\colon \mathcal{C}(P) \to \mathbb{R}^+$$

is defined as:

$$YM(\omega) := \frac{1}{2} \int_M |\Omega_\omega|^2 \, \mathrm{d}\mu(g)$$

we have that:

$$YM'[\omega](\alpha) = \int_M \langle D_\omega \alpha, \Omega_\omega \rangle \, \mathrm{d}\mu(g)$$

and so the Euler-Lagrange equation associated to YM is $D_\omega^* \Omega_\omega = 0$ and its solutions are called *YM-connections*. In the case $dim_{\mathbb{R}} = 4$ there are special *YM*-connections: the so called *istantons*, i.e. the elements $\omega \in \mathcal{C}(P)$ satisfying $\Omega_\omega = \pm * \Omega_\omega$; they represent absolute minima for YM;

set:

$$\mathcal{G}_s(P) := L_s^2\text{-}completion\ \ of\ \ \mathcal{G}(P)$$

$$\mathcal{T}_s^p(P,\ \mathfrak{g}) := L_s^2\text{-completion of } \mathcal{T}^p(P,\ \mathfrak{g},\ ad)\text{(with respect to } g\otimes\gamma);$$

fixing $\omega_0 \in \mathcal{C}(P)$, define

$$\mathcal{C}_s(P) := \omega_0 + \mathcal{T}_s^1(P,\ \mathfrak{g})$$

therefore, if Ω is the curvature map, we have:

$$\Omega\colon \mathcal{C}_s(P) \to \mathcal{T}_{s-1}(P,\ \mathfrak{g}) \ \text{ and } \ \Omega_*[\omega]\colon \alpha \mapsto D_\omega \alpha$$

$\mathcal{G}_{s+1}$ acts smoothly on the right on $\mathcal{C}_s(P)$; this action is not effective, in fact:

$$f\colon \omega \mapsto \omega \ \text{ for every } \ \omega \in \mathcal{C}_s(P) \ \Leftrightarrow f \in C(G) \subset \mathcal{G}_{s+1}(P)$$

($C(G)$ being the center of G); therefore, if we set $\mathcal{G}_{s+1}^*(P) := \mathcal{G}_{s+1}(P)/C(G)$, then $\mathcal{G}_{s+1}^*(P)$ acts effectively on $\mathcal{C}_s(P)$; we have:

Proposition 3.1. *Let $\omega \in \mathcal{C}_s(P)$ then the following facts are equivalent*:

a) $D_\omega\colon \mathcal{T}_{s+1}^0(P,\ \mathfrak{g}) \to \mathcal{T}_s^1(P,\ \mathfrak{g})$ *has a non trivial kernel*;

b) ω *is a fixed point for some* $f \in \mathcal{G}_{s+1}^*(P)$.

Definition 3.2. $\omega \in \mathcal{C}_s(P)$ *is said to be* simple *if* a (*or* b) *of previous proposition does not hold.*

Set

$$\widehat{\mathcal{C}}_s(P) := \{\omega \in \mathcal{C}_s(P)\,|\,\omega \ \ is\ \ simple\}$$

thus $\mathcal{G}_{s+1}^*(P)$ acts freely on $\widehat{\mathcal{C}}_s(P)$; note that, for any $\omega \in \widehat{\mathcal{C}}_s(P)$, we have:

$$T_\omega(\mathcal{G}_{s+1}^*(P)\omega) = D_\omega \mathcal{T}_{s+1}^0(P,\ \mathfrak{g}).$$

We have the following

Proposition 3.3 (*slice theorem*). *Let $\omega_0 \in \widehat{\mathcal{C}}_s(P)$; then, there exists a nbd V of ω_0 in $\omega_0 + Ker D_{\omega_0}^*$ such that $U := \mathcal{G}_{s+1}^*(P)\,V$ is a nbd of ω_0 in $\widehat{\mathcal{C}}_s(P)$ diffeomorphic to $V \times \mathcal{G}_{s+1}^*(P)$; more precisely, there exists a smooth map $\sigma\colon U \to \mathcal{G}_{s+1}^*(P)$ such that*:

a) *for every* $\omega \in U$, $\sigma(\omega)\omega \in V$;

b) $\Sigma\colon U \to V \times \mathcal{G}^*_{s+1}(P)$ *given by*

$$\Sigma(\omega) := (\sigma(\omega)\,\omega\,,\ \sigma(\omega))$$

is a $\mathcal{G}^*_{s+1}(P)$*-equivariant diffeomorphism (where* $\mathcal{G}^*_{s+1}(P)$ *acts on the right on* $V \times \mathcal{G}^*_{s+1}(P)$ *as* $g(\omega, f) := (\omega, g^{-1} f)$*).*

Therefore, we have:

Theorem 3.4.

$$\widetilde{\mathfrak{M}}_s(P) := \widehat{\mathcal{C}}_s(P) / \mathcal{G}^*_{s+1}(P)$$

is a Hilbert manifold and $\widehat{\mathcal{C}}_s(P) \to \widetilde{\mathfrak{M}}_s(P)$ *is a principal* $\mathcal{G}^*_{s+1}(P)$*-bundle; if* $\omega \in \widehat{\mathcal{C}}_s(P)$*, then* $T_{[\omega]}\widetilde{\mathfrak{M}}_s(P)$ *can be identified with* $\{\alpha \in \mathcal{T}^1_s(P, \mathfrak{g}) \,|\, D^*_\omega \alpha = 0\}$.

From now on, to simplify our notations, we shall drop the subscript s.

Coming back to the 3-dimensional case, consider the *Chern-Simons* functional $CS : \mathcal{C}(P) \to \mathbb{R}/\mathbb{Z}$

$$CS(\omega) := \int_{\sigma(M)} tr\left(\omega \wedge \Omega_\omega - \frac{1}{6}\,\omega \wedge [\omega, \omega]\right)$$

(where σ is any section of $P = SO_g(M)$); we have: $CS\colon \widetilde{\mathfrak{M}}(P) \to \mathbb{R}/\mathbb{Z}$ i.e. it is defined on a $\mathbb{Z}$-covering of $\widetilde{\mathfrak{M}}(P)$; moreover:

$$CS'[\omega](\alpha) = \int_M tr(\alpha \wedge \Omega_\omega)$$

clearly CS' defines a closed 1-form γ on $\mathfrak{M}(P)$ and the zeroes of γ correspond to the equivalence classes of flat connections on P.

Now

$$CS'[\omega](\alpha) = \int_M \langle \alpha, * \,\Omega_\omega \rangle \,\mathrm{d}\mu(g)$$

and $*\,\Omega_\omega$ descends to a vector field ξ on $\widetilde{\mathfrak{M}}(P)$.

ξ can be thought as the gradient vector field of CS.

We are now in position to mimic the finite dimensional theory; Recall that, if N is a compact differentiable manifold and ξ is a vector field, then we can assume that, possibly after a perturbation, ξ has non degenerate zeroes, i.e., as section of TN, it intersects the zero section transversally; the zero section defines horizontal subspaces along itself and thus a covariant differentiation ∇; we have:

$$\chi(N) = \sum_{\xi(p)=0} sign\ det\,(\nabla\xi)(p)\,.$$

For every $\omega \in \widehat{\mathcal{C}}(P)$ define:

- $L[\omega] \in End_S(T_\omega \widehat{\mathcal{C}}(P))$ as $L[\omega](\alpha) := * D_\omega \alpha$
- $(\nabla\xi)[\omega] \in End_S(T_{[\omega]}\widetilde{\mathfrak{M}}(P))$ as $(\nabla\xi)[\omega](\alpha) := * D_\omega \alpha - D_\omega t(\alpha)$ where $t(\alpha) \in \mathcal{T}^0(P, \mathfrak{su}(2), ad)$ satisfies $\Delta_\omega t(\alpha) = *(\Omega_\omega \wedge \alpha)$.

We have the following

Theorem 3.5. *Let $\omega \in \widehat{\mathcal{C}}(P)$ be a smooth, simple connection; then $(\nabla\xi)[\omega]$ defines a closed, essentially selfadjoint Fredholm operator; its eigenvectors form a complete orthonormal basis of $T_{[\omega]}\widetilde{\mathfrak{M}}(P)$ and the eigenvalues form a discrete subset of the real line, with no accumulation points and unbounded in both directions; each eigenvalue has finite multiplicity, and so, in particular, $Ker(\nabla\xi)[\omega]$ is finite dimensional.*

Definition 3.6. *A* nondegenerate *zero of ξ is an equivalence class of flat connections $[\omega]$ such that $0 \notin sp\,((\nabla\xi)[\omega])$.*

We have:

Lemma 3.7. *A nondegenerate zero of ξ is isolated*

and

Theorem 3.8 (*C. H. Taubes* [**10**]).

$$\chi(\xi) := \frac{1}{2} \sum_{\xi([\omega])=0} sign\,(det\,(\nabla\xi)[\omega])$$

is well defined and depends only on M; $\chi(\xi)$ can be thought as the Euler characteristic of $\widetilde{\mathfrak{M}}(P)$, thus call it $\chi(\widetilde{\mathfrak{M}}(P))$. Moreover we have:

$$\chi(\widetilde{\mathfrak{M}}(P)) = \lambda(M)\,.$$

The proof of thm. (3.8) evokes the so called «standard elliptic theory» and it is based on the key point that the complexes which are involved are elliptic and on the possibility to build up an efficient deformation/perturbation procedure.

Recall that the integral curves of the gradient vector field ξ of *CS* connecting different critical points give a chain complex: the famous *Floer's complex* (see [7]). The resulting homology groups are independent of the metric g on M and are formally the homology groups of $\widetilde{\mathfrak{M}}(P)$ in «semi-infinite dimensions».

Consider a curve $\omega: S^1 \to \mathcal{C}(P)$ and setting $\widetilde{P} := p_1^*(P)$, where $p_1: M \times S^1 \to M$ is the natural projection, interpret ω as $\widetilde{\omega} \in \mathcal{C}(\widetilde{P})$ simply defining $\widetilde{\omega}[(u, t)] := \omega(t)[u]$; consider in $M \times S^1$ the product metric $g + dt \otimes dt$ and the Hodge operator $\widetilde{*}$; then we have:

$$\Omega_{\widetilde{\omega}} = \Omega_{\omega} + \frac{\mathrm{d}}{\mathrm{d}t} \omega(t)\, \mathrm{d}t$$

and so

$$\Omega_{\widetilde{\omega}} = \pm \widetilde{*}\, \Omega_{\widetilde{\omega}} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t} \omega(t) = \pm * \Omega_{\omega(t)}$$

i.e. the gradient flow equation for *CS* coincides with the *YM*-instanton equation in $M \times S^1$.

4 - Moduli spaces of bundle complex structures

The first step in order to extend the previous constructions to the complex case is to study the moduli space of bundle complex structures; the theory of stability as developed in [**5**] enables us to describe it as a suitable moduli space of connections, making therefore the tools of gauge theory fully available.

Let (M, J_M) be a n-dimensional complex manifold.

Definition 4.1. A complex vector bundle $(E, \widehat{J})$ *of (complex) rank* r *over* M *is a real vector bundle* E *of rank* $2r$ *over* M *equipped with a section* $\widehat{J}$ *of* $End(E)$ *such that* $\widehat{J}^2 = -id_E$.

Given a complex vector bundle E of *rank* r, we can consider the principal $GL(r, \mathbb{C})$-bundle $C(E)$ of complex linear frames on E; thus:

$$E = C(E) \times_{GL(r, \mathbb{C})} \mathbb{R}^{2r}$$

where $GL(r, \mathbb{C})$ acts on $\mathbb{R}^{2r}$ via $\varrho: GL(r, \mathbb{C}) \to GL(2r, \mathbb{R})$, the standard real representation.

Definition 4.2. *Let* $(E, \widehat{J})$ *be a complex vector bundle of rank* r *over the complex manifold* (M, J_M);

1) *a* bundle complex structure (bucs) *on* $C(E)$ *is a complex structure* J *on* $C(E)$ *such that*:

(a) *the bundle projection* $\pi : C(E) \to M$ *is* (J, J_M)*-holomorphic*;

(b) *J induces the standard integrable complex structure J_S on the fibres*;

(c) *$GL(r, \mathbb{C})$ acts J-holomorphically on $C(E)$.*

$\mathcal{B}(C(E))$ will denote the set of bucs on $C(E)$;

2) *let $J \in \mathcal{B}(C(E))$; then J is said to be simple if any J-holomorphic endomorphism of E is of the form λid_E, with $\lambda \in C^\infty(M)$, satisfying $\overline{\partial}_M \lambda = 0$.*

$\widehat{\mathcal{B}}(C(E))$ will denote the set of simple bucs on $C(E)$.

$\mathcal{G}(C(E))$ acts in a natural way on the right on $\mathcal{B}(C(E))$ and $\widehat{\mathcal{B}}(C(E))$: given $\gamma \in \mathcal{G}(C(E))$ and $J \in \mathcal{B}(C(E))$, we define

$$\gamma^{\&}(J) := \gamma_*^{-1} \circ J \circ \gamma_* .$$

The object we are interested in is therefore:

$$\widetilde{\mathcal{K}}(E) := \widehat{\mathcal{B}}(C(E)) / \mathcal{G}^*(C(E)) .$$

Again, we have to make a massive use of connections (see [**5**] for details and proofs).

Lemma 4.3. *Let $J \in \mathcal{B}(C(E))$ and let $\omega \in \mathcal{C}(C(E))$; then*

$$\omega^{(0,1)} \in \mathcal{T}^{0,1}(C(E), \mathfrak{gl}(r, \mathbb{C}), ad)$$

and consequently:

$$\omega^{(1,0)} \in \mathcal{C}(C(E)) .$$

Proposition 4.4. *Given $\omega \in \mathcal{C}(C(E))$, there exists a unique $J \in \mathcal{B}(C(E))$ for which ω is of type $(1, 0)$; this J is given by the formula*:

$$J[u](X) = ((\pi^{-1})_* \circ J_M \circ \pi_*)[u](X^{(h)}) + J_S[u](X^{(v)})$$

i.e. J is obtained considering the standard structure on the fibre (vertical component) and J_M on the ω-horizontal component.

Therefore, we have just constructed a surjective map $\chi : \mathcal{C}(C(E)) \to \mathcal{B}(C(E))$; it is easy to check that χ is $\mathcal{G}(C(E))$-equivariant.

Definition 4.5. *Given $J \in \mathcal{B}(C(E))$, we set:*

$$\mathcal{C}_J^{1,0}(C(E)) := \chi^{-1}(J)$$

i.e. $\mathcal{C}_J^{1,0}(C(E))$ *is the set of all connection* 1*-forms in* $C(E)$ *that are of type* $(1, 0)$ *with respect to* J.

We have now the following

Proposition 4.6. *Let* $J \in \mathcal{B}(C(E))$ *and let* $\omega \in \mathcal{C}_J^{1,0}(C(E))$; *then*:

$$(D_\omega)^{0,1} = \bar{\partial}_J$$

and consequently

$$D_\omega \colon \mathcal{T}^0(C(E)) \to \mathcal{T}^1(C(E))$$

splits as

$$D_\omega = \partial_\omega + \bar{\partial}_J$$

where $\partial_\omega := (D_\omega)^{1,0}$; *more in general, we have that*

$$D_\omega \colon \mathcal{T}^{p,q}(C(E)) \to \mathcal{T}^{p+q+1}(C(E))$$

decomposes as

$$D_\omega = A_J + \partial_\omega + \bar{\partial}_J + \overline{A}_J.$$

Proposition 4.7. *Let* $J \in \mathcal{B}(C(E))$ *and let* $\omega \in \mathcal{C}_J^{1,0}(C(E))$; *then:*

$$N_J = \lambda_\omega \circ N_{J_M} \circ \pi_* + 4(\Omega_\omega^{0,2})^*$$

(*where* λ_ω *is the horizontal lift with respect to* ω).

Let $J \in \mathcal{B}(C(E))$, assume a Hermitian structure h is assigned on E and let $U_h(E)$ be the principal $U(r)$-bundle of h-unitary frames on E; we have the following basic result:

Proposition 4.8. *There exists a unique connection* $\omega_h \in \mathcal{C}(U_h(E))$ *such that its extension to* $\mathcal{C}(C(E))$ *is of type* $(1, 0)$ (*in other words* $\mathcal{C}_J^{1,0}(C(E)) \cap \mathcal{C}(U_h(E))$ *consists of a single element*); *we have*:

$$\omega_h = \hat{h}^{-1} \partial_J \hat{h}$$

where $\widehat{h}\colon C(E) \to GL(r, \mathbb{C})$ *is defined as*

$$\widehat{h}(u) := \overline{\varrho}^{-1}(u^*(h))$$

ω_h *is called the* canonical Hermitian *connection.*

Note that the uniqueness stems from the relation $\mathfrak{u}(r) \cap i\mathfrak{u}(r) = 0$; we have also that the projections

$$p_1\colon \mathcal{T}(U_h(E), \mathfrak{u}(r), ad) \to \mathcal{T}^{1,0}(C(E), \mathfrak{gl}(r, \mathbb{C}), ad)$$

and

$$p_2\colon \mathcal{T}(U_h(E), \mathfrak{u}(r), ad) \to \mathcal{T}^{0,1}(C(E), \mathfrak{gl}(r, \mathbb{C}), ad)$$

are both injective and, given $\alpha \in \mathcal{T}(U_h(E), \mathfrak{u}(r), ad)$, we have:

$$\alpha = \frac{1}{2}(\alpha^{1,0} - (\alpha^{1,0})^{\#}) = \frac{1}{2}(\alpha^{0,1} - (\alpha^{0,1})^{\#})$$

where, $\#\colon \mathcal{T}^0(C(E), \mathfrak{gl}(r, \mathbb{C}), ad) \to \mathcal{T}^0(C(E), \mathfrak{gl}(r, \mathbb{C}), ad)$ is defined by the relation

$$s^{\#}(u) := \widehat{h}^{-1}(u)\,{}^t\overline{s(u)}\,\widehat{h}(u)$$

and extends naturally to positive degree forms.

Therefore we have:

Corollary 4.9. *There is a one-to-one correspondence between the set* $\mathcal{B}(C(E))$ *of bucs on* $C(E)$ *and the affine space* $\mathcal{C}(U_h(E))$ *of connections on* $U_h(E)$.

The one-to-one correspondence between $\mathcal{B}(C(E))$ and $\mathcal{C}(U_h(E))$ induces a right action of $\mathcal{G}(C(E))$ on $\mathcal{C}(U_h(E))$; more precisely, we have the following

Proposition 4.10. $\mathcal{G}(C(E))$ *acts on the right on* $\mathcal{C}(U_h(E))$ *in the following way: for* $\gamma \in \mathcal{G}(C(E))$ *such that* $\gamma(u) = up(u)$ *and* $\omega \in \mathcal{C}(U_h(E))$, *we have*:

$$\gamma^{\&}(\omega) := \omega + p^{\#}\partial_\omega(p^{\#})^{-1} + p^{-1}\overline{\partial}_\omega p \tag{3}$$

this action corresponds to the natural action of $\mathcal{B}(C(E))$ *on* $\mathcal{B}(C(E))$ *via the bi-*

jection $\mathcal{B}(C(E)) \leftrightarrow \mathcal{C}(U_h(E))$; *moreover, if* $\gamma \in \mathcal{G}(U_h(E))$ *then*

$$\gamma^{\&} = \gamma^*$$

and so (3) *extends the standard action of* $\mathcal{G}(U_h(E))$ *and* $\mathcal{G}(C(E))$ *can be viewed as the complexification of* $\mathcal{G}(U_h(E))$.

Remark 4.11. a) *The action of* $\mathcal{G}(C(E))$ *on* $\mathcal{C}(U_h(E))$ *can be described also in the following way: let* $\gamma \in \mathcal{G}(C(E))$ *such that* $\gamma(u) = up(u)$ *and let* k *such that* $\widehat{k} = \widehat{h}(pp^{\#})^{-1}$; *therefore*

$$\gamma \colon U_h(E) \to U_k(E)$$

let $\omega \in \mathcal{C}(U_h(E))$ *and let* $J := \chi(\omega)$ *be the corresponding bucs* (*and so* $\omega = \widehat{h}^{-1}\partial_J \widehat{h}$); *let* $\omega_k = \widehat{k}^{-1}\partial_J \widehat{k}$; *therefore we have*:

$$\gamma^{\&}(\omega) = \gamma^*(\omega_k)$$

b) *it is easy to check that* $\widehat{\mathcal{B}}(C(E))$ *corresponds to* $\widehat{\mathcal{C}}(U_h(E))$ *and so*:

$$\widehat{\mathcal{K}}(E) = \widehat{\mathcal{C}}(U_h(E)) / \mathcal{G}^*(C(E))$$

this, of course, is not a moduli space of connections, so we need further investigations.

From now on, let (M, J_M, g) be a compact n-dimensional Hermitian manifold with Kähler form κ normalized in such a way that $d\kappa^{n-1} = 0$;
given $\omega \in \mathcal{C}(U_h(E))$, set:

$$K_\omega^{1,1} := \Lambda_\kappa \Omega_\omega^{1,1} \text{ (contraction with } \kappa)$$
$$\sigma_\omega := tr K_\omega^{1,1}$$
$$deg(E) := \int_M c_1(E) \wedge \kappa^{n-1} = \frac{1}{2\pi n}\int_M \sigma_\omega \kappa^n$$
$$H_\omega := K_\omega^{1,1} - \frac{2\pi i deg(E)}{rn!\, Vol_g(M)} I.$$

If $d\kappa^{n-2} = 0$, we have:

(4)
$$YM(\omega) = -\frac{n(n-1)}{12} Ch_2(E)\cdot[\kappa^{n-2}] + \frac{r}{2}(c_1(E)\cdot[\kappa^{n-1}])^2 + 2\int_M |\Omega_\omega^{0,2}|^2 \kappa^n + \frac{1}{2}\int_M |H_\omega|^2 \kappa^n.$$

First of all, we have a standard Hermitian structure on $\mathcal{T}_h^1 := \mathcal{T}^1(U_h(E), \mathfrak{u}(r), ad)$;

in fact: let $\alpha, \beta \in \mathcal{T}_h^1$, $\alpha = \pi^*(\mu) \otimes a$, $\beta = \pi^*(\nu) \otimes b$, for $\mu, \nu \in \bigwedge^1(M)$, $a, b \in \mathcal{T}_h^0$; set:

$$\langle \alpha, \beta \rangle := -\frac{1}{2} g(\mu, \nu)\, tr\, ab$$

and

$$(\alpha, \beta) := \frac{1}{n!} \int_M \langle \alpha, \beta \rangle\, \kappa^n$$

moreover, set:

$$\mathfrak{J}\alpha := \pi^*(J_M \mu) \otimes a$$

therefore:

$$\langle \mathfrak{J}\alpha, \beta \rangle = -\langle \alpha, \mathfrak{J}\beta \rangle$$

and, consequently:

$$(\mathfrak{J}\alpha, \beta) = -(\alpha, \mathfrak{J}\beta)$$

clearly $(\,,)$ and $\mathfrak{J}$ extend to $\mathcal{T}_h^1$ and so $(\mathcal{T}_h^1, \mathfrak{J}, (\,,))$ is a Hermitian vector space; note that, if $\alpha \in \mathcal{T}_h^1$, then

$$\mathfrak{J}\alpha = i\alpha^{1,0} - i\alpha^{0,1} \qquad \text{and} \qquad \mathfrak{J}\alpha[u](X) = \alpha[u](Y)$$

where $\pi_*[u](Y) = J_M \pi_*[u](X)$; moreover

$$Q(\alpha, \beta) := (\mathfrak{J}\alpha, \beta) = \int_M tr(\alpha \wedge \beta) \wedge \kappa^{n-1}.$$

Clearly, $\mathfrak{J}$ and Q extend to the Sobolev L_s^2-complection $V := \mathcal{T}_s^1(U_h(E), \mathfrak{u}(r), ad)$ of $\mathcal{T}_h^1$ and, in particular, Q represents a symplectic form on V.

Let $\{\omega_t\}_{-\varepsilon \leq t \leq \varepsilon}$ be a curve in $\mathcal{C}(U_h(E))$ such that $\omega_0 = \omega$ and $\left(\frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right)_{|t=0} = \alpha$; therefore

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} H_{\omega_t}\right)_{|t=0} = i(\bar{\partial}_\omega^* - \partial_\omega^*)\,\alpha = -(D_\omega^* \circ \mathfrak{J})\,\alpha\,.$$

Consider the map:

$$\mu\colon \omega \mapsto H_\omega$$

we have: for any $X \in \mathcal{T}_h^0$, $\omega \in \mathcal{C}(U_h(E))$, $\alpha \in T_\omega \mathcal{C}(U_h(E))$:

$$(X, \mathrm{d}\mu[\omega](\alpha)) = (X, -D_\omega^* \mathfrak{J}\alpha) = (\mathfrak{J}D_\omega X, \alpha) = Q(X^*(\omega), \alpha)$$

i.e. μ is a momentum map; of course μ induces a momentum map on $\widehat{\mathcal{C}}(U_h(E))$.

Let

$$\widehat{\mathcal{E}}(U_h(E)) := \{\omega \in \widehat{\mathcal{C}}(U_h(E)) \,|\, H_\omega = 0\}$$

we have the following:

Proposition 4.12.

$$\widetilde{\mathcal{E}}(E) := \widehat{\mathcal{E}}(U_h(E)) / \mathcal{G}^*(U_h(E))$$

is an infinite dimensional symplectic manifold; in particular, given $[\omega] \in \widetilde{\mathcal{E}}(U_h(E))$ *we have*:

$$T_{[\omega]}\widetilde{\mathcal{E}}(U_h(E)) = Ker\, D_\omega^* \cap \mathfrak{J} Ker\, D_\omega^* = \{\alpha \in \mathcal{T}_h^1 \,|\, \bar{\partial}_\omega^* \alpha^{0,1} = 0\}$$

and thus $T_{[\omega]}\widetilde{\mathcal{E}}(U_h(E))$ *is* $\mathfrak{J}$*-invariant and so* $(\widetilde{\mathcal{E}}(E), \mathfrak{J})$ *is an infinite dimensional complex manifold.*

Now

Proposition 4.13. *The natural map*

$$j: \widetilde{\mathcal{E}}(E) \to \widetilde{\mathcal{K}}(E)$$

is an embedding.

Proof. Let $\omega \in \widehat{\mathcal{E}}(U_h(E))$ and $\gamma \in \mathcal{G}^*(C(E))$ such that $\gamma^{\&} \in \widehat{\mathcal{E}}(U_h(E))$; thus, if $\gamma(u) = up(u)$ and $\hat{k} := \hat{h}(pp^{\#})^{-1}$, then h and k are Hermite-Einstein structures with respect to ω; by uniqueness, $h = k$ and so $\gamma \in \mathcal{G}^*(U_h(E))$. ■

Recall now the following

Theorem 4.14 (*PdB-G. Tian* [**5**]). *Let* (M, J, g) *be a compact* n*-dimensional Hermitian manifold whose Kähler form* η *satisfies* $\partial_M \bar{\partial}_M \eta^{n-1} = 0$; *let* $(E, \hat{J})$ *be a complex vector bundle of rank* r *over* M *and let* $J \in \mathcal{B}(C(E))$ *such that* E *is* J*-stable; then there exists a unique (up to homotheties) Hermitian structure* h *on* E *satisfying the Hermite-Einstein condition* $H_{\omega_h} = 0$.

Recall also that stability is a generic condition; therefore we have:

Proposition 4.15. *The image of j is dense in $\widetilde{\mathcal{K}}(E)$.*

Proof. As a consequence of (4.14), we have that $\mathcal{G}^*(C(E))\widehat{\mathcal{E}}(U_h(E))$ dense in $\widehat{\mathcal{C}}(U_h(E))$; moreover:

$$\mathcal{G}^*(C(E))\,\widehat{\mathcal{E}}(U_h(E))/\mathcal{G}^*(C(E)) = \widehat{\mathcal{E}}(U_h(E))/\mathcal{G}^*(U_h(E)) \qquad \blacksquare$$

we can set the following

Definition 4.16. *It is natural to call* $\tilde{\mathcal{E}}(E)$ *the* sound moduli space of bucs *on* E; *an element* $[\omega] \in \tilde{\mathcal{E}}(E)$ *is called an* (*equivalence class of*) E-holomorphic connection(s) *if it satisfies* $\Omega_\omega^{0,2} = 0$;
set:

$$\mathcal{W}(E) := \{[\omega] \in \tilde{\mathcal{E}}(E)\,|\,\Omega_\omega^{0,2} = 0\}.$$

It is clear that, if J_M is integrable, then E-holomorphic connections correspond to integrable bucs.

The complex counterpart of the topological triviality of the tangent bundle of oriented 3-manifolds is represented by *Calabi-Yau manifolds*, i.e. Kähler n-manifolds (M, κ) equipped with a holomorphic $(n, 0)$-form ε, satisfying $\varepsilon \wedge \bar{\varepsilon} = \dfrac{\kappa^n}{n!}$.

In any CY n-manifold (M, κ, ε), we can define

$$*_\varepsilon \colon \wedge^{0,p} \to \wedge^{0,n-p}$$

by means of the relation $\alpha \wedge *_\varepsilon \alpha = |\alpha|^2 \bar{\varepsilon}$ and so $*_\varepsilon \alpha = \bar{*}(\alpha \wedge \varepsilon)$; note that

$$(*_\varepsilon \alpha, \beta) = \overline{(\alpha, *_\varepsilon \beta)}$$

if $n = 4$, then $\alpha \in \wedge^2(M)$ is said to be *complex asd* (resp. *sd*) w. r. to ε if $*_\varepsilon \alpha = -\alpha$ (resp. $*_\varepsilon \alpha = \alpha$); set

$$\wedge_\pm^{0,2}(\varepsilon) := \{\alpha \in \wedge^{0,2}(M)\,|\,*_\varepsilon \alpha = \pm\alpha\}$$

since the condition $\varepsilon \wedge \bar{\varepsilon} = \dfrac{\kappa^4}{4!}$ defines ε up to $\lambda \in U(1)$, complex asd and complex sd interchange simply passing from ε to $-\varepsilon$: the special elements in $\mathcal{C}(U_h(E))$ are therefore the eigenvectors of $*_\varepsilon$: given such a connection, ε can be always normalized in such a way the corresponding eigenvalue is 1.

Consider a Calabi-Yau 3-fold (M, κ, ε) and set, with some abuse of notation:

$$CCS([\omega]) := \int_M tr\left(\Omega_\omega \wedge \omega - \frac{1}{6}\,\omega \wedge [\omega, \omega]\right) \wedge \varepsilon$$

this is a multivalued function defined on a covering space of $\tilde{\mathcal{E}}(E)$ with covering group at most $H^3(M, \mathbb{Z})$; we have

$$\partial_{\mathfrak{J}} CCS = 0$$

and

$$\begin{aligned} \bar{\partial}_{\mathfrak{J}} CCS[\![\omega]\!](\alpha) &= \int_M tr(\Omega_\omega^{0,2} \wedge \alpha^{0,1}) \wedge \varepsilon \\ &= \int_M tr\,(\alpha^{0,1} \wedge \bar{*}\,(\bar{*}(\Omega_\omega^{0,2} \wedge \varepsilon))) = \int_M \langle \alpha^{0,1}, *_\varepsilon \Omega_\omega^{0,2}\rangle \, \mathrm{d}\mu(g)\,. \end{aligned} \tag{5}$$

The complex gradient vector field of CCS at $[\omega]$ is given by $*_\varepsilon \Omega_\omega^{0,2}$ and the zeroes of $*_\varepsilon \Omega\omega^{0,2}$ (i.e. the elements of $\mathcal{W}(E)$) are the equivalence classes of holomorphic bundles of a fixed topological type over M and «counting» them will conjecturally yield an invariant which represents the complex counterpart of the Casson invariant; the ellipticity still holds as in the real case, but, in order to produce a rigorous definition, we must handle with some care the perturbation/deformation theory and this is not yet entirely available.

If $[\omega] \in \tilde{\mathcal{E}}(E)$ satisfies $\Omega_\omega^{0,2} = 0$ then

$$\|\Omega_\omega\|^2 = -\,Ch_2(E)\cdot[\kappa]^{n-2} + \frac{r}{2}\,(c_1(E)\cdot[\kappa]^{n-1})^2$$

(cf. also in the following) and this ensures the compactness of the moduli space.

Consider now a map $\omega : \mathbb{T}^2 \to \hat{\mathcal{E}}(U_h(E))$; again setting $\tilde{P} := p_1^*(P)$, where $p_1 \colon M \times \mathbb{T}^2 \to M$ is the natural projection, interpret ω as $\tilde{\omega} \in \mathcal{C}(\tilde{P})$ simply defining $\tilde{\omega}[(u, z)] := \omega(z)[u]$; consider, in $M \times \mathbb{T}^2$, $\tilde{\varepsilon} = \varepsilon \wedge \mathrm{d}z$; we have:

$$\Omega_{\tilde{\omega}} = \Omega_\omega + \frac{\partial \omega}{\partial z}\,\mathrm{d}z + \frac{\partial \omega}{\partial \bar{z}}\,\mathrm{d}\bar{z}$$

and so

$$\left(\frac{\partial\omega}{\partial\bar{z}}\right)^{0,1} = \pm *_{\varepsilon}\Omega_{\omega}^{0,2} \Leftrightarrow \begin{cases} \Omega_{\tilde{\omega}}^{0,2} = \pm \tilde{*}_{\tilde{\varepsilon}}\Omega_{\tilde{\omega}}^{0,2}. \\ H_{\tilde{\omega}} = 0\,. \end{cases} \tag{6}$$

These equations make sense in any CY 4-fold and they are called $SU(4)$-*istanton equations*.

We have:

$$SU(4) \approx Spin(6) \hookrightarrow Spin(7)$$

and the $Spin(7)$-representation splits as

$$\wedge^2(M) = A \oplus B$$

restricting to $SU(4)$ we have:

$$B = \mathbb{R}\kappa \oplus \wedge_{+}^{0,2}(\varepsilon)$$

and so the $SU(4)$-instanton equations correspond to the vanishing of the B-component for $Spin(7)$-connections and so they make sense for any $Spin(7)$-structure, i.e. for any Riemannian 8-manifold with holonomy $\subset Spin(7)$.

$SU(4)$-istantons fit, as a special case, in a much more general theory: η-antiselfduality (η-asd) ([**11**]), which will be the object of next section.

We want to show first two things:

a) we have the following picture:

1) In the Real Case: Geometric Definition of Casson invariant $\leftrightarrow$ Definition via Gauge Theory

2) In the Complex Case: Definition via Gauge Theory $\rightarrow$ Conjectural Geometric Definition ([**6**]).

Let us explain a little bit more about the last arrow; a possible way to reconstruct, in the complex case, the Heegaard splitting situation is the following: let Σ be a Calabi-Yau surface; let $M^{\pm}$ be two Calabi-Yau 3-folds such that Σ is embedded in $M^{\pm}$ as the zero set of a section of the anticanonical bundle $K_{M^{\pm}}^{-1}$; consider $M_0 = M^{+} \cup_{\Sigma} M^{-}$; then we can find (mod obstructions) a deformation $\{M_t\}$ of M_0 with M_t smooth for $t \neq 0$; more precisely, locally around Σ, we can perform the following construction (and then extend): let $\nu_{\pm}$ be the normal bundle of $\Sigma \subset M^{\pm}$; consider $p = \nu_{+} \oplus \nu_{-} \to \Sigma$; then $p^{*}(\nu_{\pm})$ has a tautological section $\sigma_{\pm}$; for any holomorphic section ε of $\nu_{+} \oplus \nu_{-}$, the equation $\sigma_{+}\sigma_{-} = \varepsilon$ cuts out a 3-dimensional subvariety V_{ε} of $\nu_{+} \oplus \nu_{-}$ (and, of course $M_0 \approx V_0$); if ε has transverse zeroes giving a smooth curve Z in Σ, then V_ε is smooth: for such an ε we can set $M_t := V_{t\varepsilon}$.

Now, a holomorphic bundle over M_0 is given by a pair of bundles $E^\pm$ over $M^\pm$ which are isomorphic over Σ; let $\theta_\Sigma \in \wedge^{2,0}(\Sigma)$ be a never vanishing holomorphic $(2, 0)$-form; let $E \to \Sigma$ be a complex vector bundle and extend it to $M^\pm$; $\mathcal{W}_\Sigma(E)$ is a complex symplectic manifold:

$$q[\![\omega]\!](\alpha, \beta) := \int_\Sigma \alpha \wedge \beta \wedge \theta_\Sigma .$$

Set:

$$L^\pm(E) := \{[\omega] \in \mathcal{W}_\Sigma(E) \,|\, [\omega] \in \mathcal{W}_{M^\pm}(E)\}$$

then:

$$\mathcal{W}_{M_0}(E) = \{([\omega_+], [\omega_-]) \in \mathcal{W}_{M^+}(E) \times \mathcal{W}_{M^-}(E) \,|\, [\omega_+] = [\omega_-] \;\; on \;\; \Sigma\}$$

therefore (cf. (1) and (2)):

$$\mathcal{W}_{M_0}(E) \approx L^+(E) \cap L^-(E) .$$

We have the following ansatz

$$L^+(E) \cap L^-(E) = \lim_{t \to 0} \mathcal{W}_{M_t}(E)$$

i.e. the intersections points $L^+(E) \cap L^-(E)$ appears as the limit of the $\mathcal{W}(E)$ on the Calabi-Yau manifold M_t as the complex structure degenerates.

b) We want to show that, in some sense, the previous version of Casson Invariant Theory cannot be further extended; in fact, assume M is a real n-dimensional manifold and let $\eta \in \wedge^{n-3}(M)$, $d\eta = 0$ and let $(E, h) \to M$ be a *rank* r, Hermitian bundle; consider:

$$GCS([\omega]) := \int_M tr\left(\Omega_\omega \wedge \omega - \frac{1}{6}\, \omega \wedge [\omega, \omega]\right) \wedge \eta$$

defined on a suitable covering of $\widetilde{\mathfrak{M}}(C(E))$; then:

$$GCS'[\![\omega]\!](\alpha) = \int_M tr(\Omega_\omega \wedge \alpha) \wedge \eta$$

the induced linearized complex is essentially

$$\wedge^0 \xrightarrow{d} \wedge^1(M) \xrightarrow{\eta \wedge d} \wedge^{n-1}(M) \xrightarrow{d} \wedge^n(M) \tag{7}$$

equivalently, considering a curve $\omega : S^1 \to \widetilde{\mathfrak{M}}(C(E))$, in the same notations as before, we have:

$$\Omega_{\widetilde{\omega}} = -\,\widetilde{*}(\Omega_{\widetilde{*}} \wedge \eta)$$

$$\Updownarrow$$

$$\begin{cases} \dfrac{\mathrm{d}\omega}{\mathrm{d}t} \wedge \eta = -\,*\,\Omega_\omega \\ \dfrac{\mathrm{d}\omega}{\mathrm{d}t} = -\,*\,(\Omega_\omega \wedge \eta) \end{cases} \tag{8}$$

the system (8) reduces to a single equation if and only if the map $\alpha \mapsto *(\alpha \wedge \eta)$ is invertible (and this is also equivalent to the ellipticity of (7)); it is well known that this corresponds to the fact that $*\,\eta$ induces a non degenerate cross-product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and this is possible if and only if $n = 3, 7$.
$n = 3$, $\eta = 1$ is the standard case, $n = 7$ corresponds to $Spin(7)$-manifolds, with G_2-manifolds as a special case.

5 - η-antiselfduality

Let $\pi : (E, h) \to (M, g)$ be a unitary bundle of complex *rank* r over a compact Riemannian manifold of dimension n and let $\eta \in \wedge^{n-4}(M)$ with $\mathrm{d}\eta = 0$. We have first the following

Lemma 5.1 ([**11**]). *Let $\omega \in \mathcal{C}(U_h(E))$ such that:*

$$tr\,\Omega_\omega \text{ is a harmonic } 2\text{-form} \tag{9}$$

$$\eta \wedge \left(\Omega_\omega - \frac{1}{r}(tr\,\Omega_\omega)\,I\right) = -\,*\left(\Omega_\omega - \frac{1}{r}(tr\,\Omega_\omega)\,I\right) \tag{10}$$

then ω is a YM-connection and

$$\frac{1}{4\pi^2}\left(\int_M |\Omega_\omega|^2\,\mathrm{d}\mu(g) - \frac{1}{r}\int_M |tr\,\Omega_\omega|^2\,\mathrm{d}\mu(g)\right) = \left(-Ch_2(E) + \frac{1}{r}(c_1(E))^2\right)\cdot[\eta] \tag{11}$$

Proof. From

$$\Omega_\omega = \frac{1}{r}(tr\,\Omega_\omega)\,I + *\left[\eta \wedge \left(\Omega_\omega - \frac{1}{r}(tr\,\Omega_\omega)\,I\right)\right]$$

it follows:

$$\text{(12)}\qquad \begin{aligned} &D_\omega^*\,\Omega_\omega = \frac{1}{r}D_\omega^*(tr\,\Omega_\omega)\,I + *\,D_\omega\left[\eta \wedge \left(\Omega_\omega - \frac{1}{r}(tr\,\Omega_\omega)\,I\right)\right] = \\ &\frac{1}{r}\,\mathrm{d}^*(tr\,\Omega_\omega)\,I + *\left[(-1)^{n-4}\eta \wedge \left(D_\omega\,\Omega_\omega - \frac{1}{r}\,\mathrm{d}(tr\,\Omega_\omega)\,I\right)\right] = 0 \end{aligned}$$

to get (11) simply multiply (10) by $\Omega_\omega - \frac{1}{r}(tr\,\Omega_\omega)\,I$ and integrate. ■

Definition 5.2. *A solution of* (9) *and* (10) *is called an* η*-asd connection* (*or* η*-asd istanton*).

Fundamental Examples (see [**11**]).

a) Let (M, J_M, g) be a compact Kähler manifold of complex dimension m with Kähler form κ or, more in general, let (M, κ) be a $2m$-dimensional compact symplectic manifold equipped with a κ-calibrated complex structure J_M:
set

$$\eta := \frac{\kappa^{m-2}}{(m-2)!}$$

we have the following algebraic facts:

- $\alpha \in \wedge^{2,0}(M) \oplus \wedge^{0,2}(M) \Rightarrow \eta \wedge \alpha = *\,\alpha$
- $\alpha \in \wedge^{1,1}(M),\ \eta \wedge \alpha = -*\,\alpha \Rightarrow \Lambda_\kappa \alpha = 0$ therefore

$$\eta \wedge \left(\Omega_\omega - \frac{1}{r}(tr\,\Omega_\omega)\,I\right) = -*\left(\Omega_\omega - \frac{1}{r}(tr\,\Omega_\omega)\,I\right)$$

$$\Updownarrow$$

$$\text{(13)}\qquad \begin{cases} \Omega_\omega^{0,2} = \dfrac{1}{r}(tr\,\Omega_\omega^{0,2})\,I \\ H_\omega = 0 \end{cases}$$

assume now $c_1(E)$ is of type $(1, 1)$; therefore, if ω is η-asd (and so it satisfies (13) and $tr\,\Omega_{\omega}$ is harmonic), we have:

$$tr\,\Omega_{\omega} = \alpha^{1,1} + \mathrm{d}\beta$$

and, by Hodge decomposition, $\mathrm{d}\beta = 0$ and thus $tr\,\Omega_{\omega}^{0,2} = 0$ and η-antiselfduality reduces to

$$\begin{cases} \Omega_{\omega}^{0,2} = 0 \\ H_{\omega} = 0 \end{cases} \tag{14}$$

note that $\omega \in \mathcal{C}(U_h(E))$ satisfiying (14) is η-asd because, by the curvature identity (4), it is an absolute minimum of YM and so, in particular $D_{\omega}^{*}\,\Omega_{\omega} = 0$ and thus $tr\,\Omega_{\omega}$ is harmonic.

b) Let (M, κ, ε) be a CY 4-fold;
set

$$\eta = 4\,\Re e\,\varepsilon + \frac{1}{2}\kappa^2$$

it is easy to check that $\omega \in \mathcal{C}(U_h(E))$ satisfies (10) if and only if

$$\begin{cases} \Omega_{\omega}^{0,2} - \dfrac{1}{r}(tr\,\Omega_{\omega}^{0,2})\,I = -*_{\varepsilon}\left(\Omega_{\omega}^{0,2} - \dfrac{1}{r}(tr\,\Omega_{\omega}^{0,2})\,I\right) \\ \mathrm{d}(tr\,\Omega_{\omega}^{0,2}) = -\,\mathrm{d}(*_{\varepsilon}\,tr\,\Omega_{\omega}^{0,2}) \\ H_{\omega} = 0\,. \end{cases} \tag{15}$$

If $(c_1(E))^{0,2}$ is complex antiselfdual, then $tr\,\Omega_{\omega}^{0,2} = *_{\varepsilon}\,tr\,\Omega_{\omega}^{0,2}$ and so (15) reduces to the istanton equations (6); note that, for any

$$\alpha \in \mathcal{T}^{0,2}(C(E), \mathfrak{gl}(r, \mathbb{C}), ad),$$

from

$$\alpha = -*_{\varepsilon}\alpha + (\alpha + *_{\varepsilon}\alpha) = -\,\overline{*}(\alpha \wedge \varepsilon) + (\alpha + \overline{*}(\alpha \wedge \varepsilon))$$

we obtain:

$$
\begin{aligned}
\|\alpha\|^2 &= \int_M tr(\alpha \wedge \bar{*}\,\alpha) = -\int_M tr(\alpha \wedge \alpha) \wedge \varepsilon + \int_M tr\big(\alpha \wedge \bar{*}\,(\alpha + \bar{*}(\alpha \wedge \varepsilon))\big) \\
&= -\int_M tr(\alpha \wedge \alpha) \wedge \varepsilon + \frac{1}{2}\int_M tr\big((\alpha + \bar{*}(\alpha \wedge \varepsilon)) \wedge \bar{*}\,(\alpha + \bar{*}(\alpha \wedge \varepsilon))\big) \\
&\quad + \frac{1}{2}\int_M tr\big((\alpha - \bar{*}(\alpha \wedge \varepsilon)) \wedge \bar{*}(\alpha + \bar{*}(\alpha \wedge \varepsilon))\big) \\
&= -\int_M tr(\alpha \wedge \alpha) \wedge \varepsilon + \frac{1}{2}\|\alpha + *_\varepsilon \alpha\|^2 + \frac{1}{2}(\alpha, *_\varepsilon \alpha) - \frac{1}{2}(*_\varepsilon \alpha, \alpha)
\end{aligned}
\tag{16}
$$

consequently, if we normalize ε in such a way that

$$\int_M tr(\alpha \wedge \alpha) \wedge \varepsilon \in \mathbb{R}$$

we have $(\alpha, *_\varepsilon \alpha) \in \mathbb{R}$ and so:

$$\|\alpha\|^2 = -\int_M tr(\alpha \wedge \alpha) \wedge \varepsilon + \frac{1}{2}\|\alpha + *_\varepsilon \alpha\|^2$$

in particular, for $\alpha = \Omega_\omega^{0,2}$, choosing ε in such a way that $-Ch_2(E)\cdot[\varepsilon] \geqslant 0$, we obtain:

$$\|\Omega_\omega^{0,2}\|^2 = -\,Ch_2(E)\cdot[\varepsilon] + \int_M |(1 + *_\varepsilon)\,\Omega_\omega^{0,2}\,|^2 \kappa^n$$

and so

- $Ch_2(E)\cdot[\varepsilon] \neq 0 \Rightarrow \nexists \omega \in \mathcal{C}(U_h(E))$ with $\Omega_\omega^{0,2} = 0$
- $Ch_2(E)\cdot[\varepsilon] = 0 \Rightarrow (1 + *_\varepsilon)\,\Omega_\omega^{0,2} = 0$ if and only if $\Omega_\omega^{0,2} = 0$.

Note also that (4) can be rewritten as:

$$YM(\omega) = -\,Ch_2(E)\cdot([\kappa]^2 + [\varepsilon]) + \frac{r}{2}(c_1(E)\cdot[\kappa^3])^2$$

$$+2\int_M |(1 + *_\varepsilon)\,\Omega_\omega^{0,2}|^2 \kappa^n + \frac{1}{2}\int_M |H_\omega|^2 \kappa^n$$

and so $SU(4)$-istantons are absolute minima for YM and thus they are η-asd.

For example, let (M, κ, ε) be a CY 3-fold and let $(E, h) \to M$ be a Hermitian bundle of *rank* r; consider $\widetilde{M} := M \times \mathbb{T}^2$, $\tilde{\varepsilon} := \varepsilon \wedge \mathrm{d}z$, $\tilde{\kappa} := \kappa + \mathrm{d}z \wedge \mathrm{d}\bar{z}$ ($\mathrm{d}z$, $\mathrm{d}\bar{z}$ being the standard forms on $\mathbb{T}^2$) and let $\widetilde{E} := p^*(E)$ with the induced structure; then:

$$\tilde{\omega} \in \mathcal{C}(U_h(\widetilde{E})) \text{ is } \mathbb{T}^2\text{-invariant}$$

$$\Updownarrow$$

$$\tilde{\omega} = \omega + f\mathrm{d}z - f^{\#}\mathrm{d}\bar{z}$$

for $\omega \in \mathcal{C}(U_h(E))$, $f \in \mathcal{T}^0(C(E), \mathfrak{gl}(r, \mathbb{C}), ad)$ (see Appendix (B)) and so:

$$\Omega_{\tilde{\omega}} = \Omega_{\omega} + D_\omega f \wedge \mathrm{d}z - D_\omega f^{\#} \wedge \mathrm{d}\bar{z} - [f, f^{\#}]\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

now:

$$\tilde{\omega} \ \textit{is a} \ SU(4)\text{-istanton}$$

$$\Updownarrow$$

$$\begin{cases} \Omega_\omega^{0,2} = \bar{\partial}_\omega^* q \\ H_\omega \kappa^3 = [q, q^{\#}] \end{cases} \tag{17}$$

where $q := f^{\#}\bar{\varepsilon}$; but $Ch_2(\widetilde{E}) \cdot [\varepsilon] = 0$ and so $\Omega_\omega^{0,2} = 0$ and $\bar{\partial}_\omega f^{\#} = 0$.

Completely similar results can be obtained for $Spin(7)$- and G_2-manifolds (see again [**11**]).

6 - A more abstract setting

We want to summarise and put everything on a more abstract setting; Let (M, g) be an n-dimensional Riemannian manifold and let (E, h) be a *rank* r Hermitian bundle on it; let L be a section of $Hom_{\mathbb{R}}(\mathfrak{T}^p(U_h(E), \mathfrak{u}(r), ad), \mathfrak{T}^{n-p}(U_h(E), \mathfrak{u}(r), ad))$ such that:

1) for every $\alpha, \beta \in \mathfrak{T}^p(U_h(E), \mathfrak{u}(r), ad)$, $tr(\alpha \wedge L(\beta)) = tr(L(\alpha) \wedge \beta)$
2) $L = L_1 + L_2$ in such a way that:

   (a) for every $\omega \in \mathcal{C}(U_h(E))$, $L_1 \circ D_\omega = D_\omega \circ L_1$

(b) there exists $c \in \mathbb{R}$, $0 < c < 1$, such that, for every $\alpha \in \mathfrak{T}^2(U_h(E), \mathfrak{u}(r), ad)$, $-tr(\alpha \wedge L_2(\alpha)) \leqslant c|\alpha|^2 \mathrm{d}\mu(g)$

(in the previous section we confined ourself to the case $L\colon \alpha \mapsto \alpha \wedge \eta$ with $\eta \in \bigwedge^{n-2p}(M)$, $\mathrm{d}\eta = 0$, $p = 2$);
let ϕ be the section of $End_{\mathbb{R}}(\mathfrak{T}^2(U_h(E), \mathfrak{u}(r), ad))$ defined as $\phi(\alpha) := -*(L(\alpha))$;
then:

$$\langle \alpha, \phi(\beta) \rangle \mathrm{d}\mu(g) = -tr(\alpha \wedge L(\beta)) = -tr(L(\alpha) \wedge \beta) = \langle -L(\alpha), *\beta \rangle = \langle \phi(\alpha), \beta \rangle \mathrm{d}\mu(g)$$

and so ϕ is pointwise symmetric; assume: for every $x \in M$, $1 \geqslant maxsp(\phi[x])$ ($sp(\phi[x])$ being the spectrum of $\phi[x]$); then, writing $\alpha \in \mathcal{T}^2(U_h(E), \mathfrak{u}(r), ad)$ as $\alpha = \phi(\alpha) + (\alpha - \phi(\alpha))$, we obtain:

$$\begin{aligned} \|\alpha\|^2 &= \int_M tr(\alpha \wedge * \alpha) = -\int_M tr(\alpha \wedge L(\alpha)) + \int_M tr(\alpha \wedge *(\alpha - \phi(\alpha))) \\ &= -\int_M tr(\alpha \wedge L(\alpha)) + \frac{1}{2}\int_M tr((\alpha - \phi(\alpha)) \wedge *(\alpha - \phi(\alpha))) \\ &\qquad + \frac{1}{2}\int_M tr((\alpha + \phi(\alpha)) \wedge *(\alpha - \phi(\alpha))) \\ &= -\int_M tr(\alpha \wedge L(\alpha)) + \frac{1}{2}\|\alpha - \phi(\alpha)\|^2 + \frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi(\alpha)\|^2 \end{aligned} \tag{18}$$

by the assumption on the spectrum of ϕ we have:

$$\|\alpha - \phi(\alpha)\|^2 + \|\alpha\|^2 - \|\phi(\alpha)\|^2 \geqslant 0$$

with equality if and only if $\phi(\alpha) = \alpha$;
in the special case $\alpha = \Omega_\omega$ for $\omega \in \mathcal{C}(U_h(E))$, we have:

$$\begin{aligned} -\int_M tr(\Omega_\omega \wedge L(\Omega_\omega)) &= -\int_M tr(\Omega_\omega \wedge L_1(\Omega_\omega)) - \int_M tr(\Omega_\omega \wedge L_2(\Omega_\omega)) \\ &\leqslant -\int_M tr(\Omega_\omega \wedge L_1(\Omega_\omega)) + c\int_M |\Omega_\omega|^2 \mathrm{d}\mu(g) \end{aligned} \tag{19}$$

now $-\int_M tr(\Omega_\omega \wedge L_1(\Omega_\omega))$ is independent of ω : call it $c_{L_1}(E)$; consequently:

- if $\phi(\Omega_\omega) = \Omega_\omega$, then

$$(1 - c)\|\Omega_\omega\|^2 \leqslant c_L(E)$$

- if $L = L_1$ then

$$\|\Omega_\omega\|^2 = c_L(E) + \frac{1}{2}\|\Omega_\omega - \phi(\Omega_\omega)\|^2 + \frac{1}{2}\|\Omega_\omega\|^2 - \frac{1}{2}\|\phi(\Omega_\omega)\|^2 \geqslant c_L(E)$$

with equality if and only if $\phi(\Omega_\omega) = \Omega_\omega$.

Finally, let

$$\widetilde{\mathcal{M}}_L(E) := \{\omega \in \widehat{\mathcal{C}}(U_h(E)) \,|\, \phi(\Omega_\omega) = \Omega_\omega\} / \mathcal{G}^*(U_h(E))$$

then set

$$\mathcal{T}^2_+(U_h(E), \mathfrak{u}(r), ad) := (I - \phi)\,\mathcal{T}^2(U_h(E), \mathfrak{u}(r), ad)$$

and let

$$S_\omega \colon \mathcal{T}^1(U_h(E), \mathfrak{u}(r), ad) \to \mathcal{T}^0(U_h(E), \mathfrak{u}(r), ad) \oplus \mathcal{T}^2_+(U_h(E), \mathfrak{u}(r), ad)$$

be defined as

$$S_\omega(\alpha) := (D^*_\omega \alpha, D_\omega \alpha - \phi(D_\omega \alpha))$$

then

$$T_{[\omega]}\widetilde{\mathcal{M}}_L = Ker\, S_\omega$$

and the ellipticity condition (cf. (7)) is given by the following: for every $x \in M$, $\phi[x]$ has 1 as eigenvalue of multiplicity $\dfrac{(n-1)(n-2)}{2}$.

Appendix A. Regular manifolds

Let (M, J_M, g) be a compact Hermitian manifold of complex dimension n with Kähler form κ and let $\pi\colon (E, h) \to (M, g)$ be a Hermitian bundle of $rank\, r$.

In order to guarantee that the map

$$Ch_2 \colon \mathcal{C}(C(E)) \to \mathbb{C}$$

$$\omega \mapsto -\frac{n(n-1)}{12}\int_M tr(\Omega \wedge \Omega) \wedge \kappa^{n-2}$$

is constant, we need $d\kappa^{n-2} = 0$ and this condition sounds somehow too restrictive; we set:

Definition A.1. *An n-dimensional Hermitian manifold (M, J_M, g) with Kähler form κ is said to be regular if*

1) $d\kappa^{n-1} = 0$
2) $d\kappa^{n-2} = (d\kappa^{n-2})^{n,\, n-3} + (d\kappa^{n-2})^{n-3,\, n}$.

Remark A.2. *It is easy to check that*:

a) *if* $n = 3$, *then* (2) $\Rightarrow$ (1)
b) $(S^6, \mathbb{C}ay, std)$ *is regular.*

In general, given $\omega, \tilde{\omega} \in \mathcal{C}(U_h(E))$, $\tilde{\omega} = \omega + \alpha$, we have:

$$tr(\Omega_{\tilde{\omega}} \wedge \Omega_{\tilde{\omega}}) - tr(\Omega_{\omega} \wedge \Omega_{\omega}) = d\left[tr(\alpha \wedge \Omega_{\tilde{\omega}}) + tr(\alpha \wedge \Omega_{\omega}) - \frac{1}{6} tr(\alpha \wedge [\alpha, \alpha])\right]$$

therefore if $\Omega^{0,2}_{\tilde{\omega}} = \Omega^{0,2}_{\omega} = 0$ and (M, J_M, g) is compact and regular, then:

$$Ch_2(\tilde{\omega}) - Ch_2(\omega) = -\frac{1}{6} \int_M tr(\alpha \wedge [\alpha, \alpha]) \wedge d\kappa^{n-2}$$

$$= \frac{1}{6} \int_M [tr(\overline{A}(\alpha^{1,0}) \wedge [\alpha, \alpha]^{2,0}) + tr(A(\alpha^{0,1}) \wedge [\alpha, \alpha]^{0,2})] \wedge \kappa^{n-2}$$

therefore, if $[\omega], [\tilde{\omega}] \in \mathcal{W}(E)$, then we have:

$$YM(\tilde{\omega}) = Ch_2(\tilde{\omega}) + \frac{r}{2} (c_1(E) \cdot [\kappa^{n-1}])^2$$

$$= Ch_2(\omega) + (Ch_2(\tilde{\omega}) - Ch_2(\omega)) + \frac{r}{2} (c_1(E) \cdot [\kappa^{n-1}])^2 = C + \lambda(\alpha)$$

where λ is of order zero; this provides a local L^2-bound for the curvature of elements of $\mathcal{W}(E)$.

A similar setting arises if we perform the construction of the example at the end of Section 5, starting from a symplectic 6-fold (M, κ) with a κ-calibrated complex structure J for which there exists $\varepsilon \in \wedge^{3,0}_J(M)$ such that $\varepsilon \wedge \bar{\varepsilon} = \dfrac{\kappa^3}{6!}$ and $d\,\Re e\, \varepsilon = 0$.

Appendix B. Some constructions involving invariant connections

Let $\pi\colon P \to M$ be be a principal G-bundle and let $p\colon \widetilde{M} \to M$ be a principal K-bundle over the differentiable manifold M, where G and K are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ respectively.

Consider:

$$\widetilde{P} := p^*(P) = \{(\tilde{x}, u) \in \widetilde{M} \times P \,|\, p(\tilde{x}) = \pi(u)\}$$

K acts on the right on $\widetilde{P}$ and so $\widetilde{P}$ is

1) a principal G-bundle over $\widetilde{M}$
2) a principal K-bundle over P with bundle projection r;

when considered as in 2), call it $\widehat{P}$; $\widetilde{\omega} \in \mathcal{C}(\widetilde{P})$ is K-invariant if and only if

$$\widetilde{\omega} = r^*(\omega) + (q \circ r)(o)$$

where

- $\omega \in \mathcal{C}(P)$
- $o \in \mathcal{C}(\widehat{P})$
- $q \in \mathcal{T}(P, \mathfrak{g}, ad) \otimes \mathfrak{k}^*$

for a K-invariant $\widetilde{\omega} \in \mathcal{C}(\widetilde{P})$, setting $\widehat{q} := (q \circ r)(o)$, we have:

$$\Omega_{\widetilde{\omega}} = r^*(\Omega_{\omega}) + D_\omega \widehat{q} + \frac{1}{2}[\widehat{q}, \widehat{q}].$$

Consider, as a special case, $\widetilde{M} = M \times K$: let $\{\zeta_1^*, \dots, \zeta_d^*\}$ be a basis of $\mathfrak{k}^*$, and identify every ζ_j^* with the corresponding invariant section of T^*K; $\widetilde{\omega} \in \mathcal{C}(\widetilde{P})$ is K-invariant if and only if

$$\widetilde{\omega} = \omega + \sum_{h=1}^{d} q_h \zeta_h^*$$

with $\omega \in \mathcal{C}(P)$, $q_1, \dots, q_d \in \mathcal{T}^0(P, \mathfrak{g}, ad)$;
for a K-invariant $\widetilde{\omega} \in \mathcal{C}(\widetilde{P})$, we have:

$$\Omega_{\widetilde{\omega}} = \Omega_{\omega} + \sum_{h=1}^{d} D_\omega q_h \wedge \zeta_h^* + \sum_{r<s} [q_r, q_s]\, \zeta_r^* \wedge \zeta_s^*.$$

For example, if $K = \mathbb{T}^2$, $G = U(r)$, then $\widetilde{\omega} \in \mathcal{C}(\widetilde{P})$ is $\mathbb{T}^2$-invariant if and only if

$$\widetilde{\omega} = \omega + a dx + b \mathrm{d} y$$

with $\omega \in \mathcal{C}(P)$ and $a, b \in \mathcal{T}^0(P, \mathfrak{u}(r)\, ad)$; now setting $f := a - ib$ and so $f \in \mathcal{T}^0(P, \mathfrak{gl}(r, \mathbb{C}), ad)$, we obtain:

$$\widetilde{\omega} = \omega + f\,\mathrm{d}z - f^{\#}\,\mathrm{d}\bar{z}$$

and

$$\Omega_{\widetilde{\omega}} = \Omega_{\omega} + D_{\omega} f \wedge \mathrm{d}z - D_{\omega} f^{\#}\,\mathrm{d}\bar{z} - [f, f^{\#}]\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,.$$

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Abstract

The aim of this paper is to echo, discuss and explain some of the attemps to define, by gauge theory, a Casson-like invariant for complex manifolds. One of the basic settings is the description of moduli spaces of bundle complex structures stemming from [**5**].

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