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GIUSEPPE TOMASSINI (*)

Levi type extremal operators (**)

1 - Introduction

Let $z = x + iy$, $w = u + iv$ be complex coordinates in $\mathbb{C}^2$. Given a smooth hypersurface M we denote by $\mathcal{L}_M$ the distribution of the complex tangent lines to M. $\mathcal{L}_M$ is called the *Levi distribution* on M. It is well known that $\mathcal{L}_M$ is completely integrable if and only if

$$k_L = -\det\begin{pmatrix} 0 & F_z & F_u + i \\ F_{\bar{z}} & F_{\bar{z}z} & F_{\bar{z}u} \\ F_u - i & F_{zu} & F_{uu} \end{pmatrix} \tag{1.1}$$

vanishes on M, $\varrho = 0$ being a local equation for M.

In this situation, by virtue of the Frobenius Theorem, M is foliated by holomorphic curves and is said to be *Levi flat.*

If M is a graph $v = F(x, y, u)$ we have

$$k_L = \frac{1}{4}\{(1 + F_u^2)(F_{xx} + F_{yy}) + (F_x^2 + F_y^2)\,F_{uu}$$

$$+ 2(F_y - F_x F_u)\,F_{xu} - 2(F_x + F_y F_u)\,F_{yu}\}.$$

(*) Scuola Normale Superiore, P.zza dei Cavalieri 7, 56100 Pisa, Italy.
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Thus Levi flat graphs are solutions of the *Levi equation*

$$L(F) = \frac{1}{4}\{(1+F_u^2)(F_{xx}+F_{yy}) + (F_x^2+F_y^2)\,F_{uu} + 2(F_y - F_x F_u)\,F_{xu} - 2(F_x + F_y F_u)\,F_{yu}\} = 0\,. \tag{1.2}$$

$L(F)$ is a degenerate elliptic operator. It has been introduced in Complex Analysis in the context of the Dirichlet problem for Levi flat graphs ([**BG**], [**DG**], [**T**], [**ST 1**]).

In the last years sophisticated techniques of PDE theory have been employed to prove a series of interesting results on the regularity of solutions of the Levi equation ([**C**], [**CLM**], [**CM 1**], [**CM 2**]).

Let $DF = (F_z, F_u)$, $|DF| = (1+|F_z|^2+F_u^2)^{1/2}$ and $\underline{L}(F) = (1+|DF|^2)^{-1} \cdot L(F)$, be the normalized of $L(F)$. Due to the particular form of $\underline{L}(F)$ it is easy to show (Sec. 2) that there are two fully nonlinear degenerate elliptic operators Λ_1, Λ_2 such that

$$\Lambda_1(F) \leqslant \underline{L}(F) \leqslant \Lambda_2(F) \tag{1.3}$$

for every smooth function F.

Λ_1, Λ_2 are «extremal operators», in the sense of Pucci ([**P**]), for a family of degenerate elliptic operators. Precisely consider a domain Ω in $\mathbb{C}\times\mathbb{R}$ and operators $L_{\alpha,\beta}(F)$ of the form

$$L_{\alpha,\beta}(F) = -\det\begin{pmatrix} 0 & \alpha & \beta \\ \overline{\alpha} & F_{\overline{z}z} & F_{\overline{z}u} \\ \overline{\beta} & F_{zu} & F_{uu}\end{pmatrix} \tag{1.4}$$

where α, β are measurable functions of (z,u), F, DF, D^2F with the properties:

1) for fixed (z,u), α, β are differentiable a.e. with respect to F, DF, D^2F;

2) $|\alpha|^2 + |p|^2 = 1$;

3) the hermitian form

$$\frac{\partial L_{\alpha,\beta}}{\partial F_{z\overline{z}}}\xi\overline{\xi} + \frac{\partial L_{\alpha,\beta}}{\partial F_{zu}}\xi\eta + \frac{\partial L_{\alpha,\beta}}{\partial F_{\overline{z}u}}\overline{\xi}\eta + \frac{\partial L_{\alpha,\beta}}{\partial F_{uu}}\eta^2$$

is positive definite ($\geqslant 0$).

Let $\mathcal{F}_\Omega$ be the family generated by the $L_{\alpha,\beta}(F)'\,s$: $\mathcal{F}_\Omega = \left\{\sum_{1\leqslant j\leqslant m}\lambda_j L_{\alpha_j,\beta_j}\right\}$

where λ_j is measurable and nonnegative, $\sum_{1 \leq j \leq m} \lambda_j = 1$ and $m \in \mathbb{N}$. Then Λ_1, Λ_2 are the extremal operators for $\mathcal{F}_\Omega$.

We call them *Levi type extremal operators.*

Extremal operators were introduced to study singularities for solutions of elliptic equations, in particular to produce critical examples. For all this matter as well as for other applications in PDE we refer to [**P**] and [**CC**].

In this paper we state some simple property for weak (viscosity) solutions of Λ_1, Λ_2. In particular in Section **3** we prove a special form of «maximum principle» (Prop. 3.5) and as a consequence we obtain that weak solutions of Λ_1, Λ_2 and L satisfy a «weak Hartogs property» (Cor. 3.6).

Finally in Section **4** we deal with the Dirichlet problem for Λ_1, Λ_2. After shown that solutions of that provide barriers for the Levi operator we prove that this problem translates into a Dirichlet problem for a Bellman equation (Prop. 4.2)

2 - Extremal operators

1 - Consider the Levi operator $L(F)$ and let $\underline{L}(F) = |DF|^{-2} L(F)$ be the normalized of $L(F)$, F smooth.

Let $\mathcal{H} = \mathcal{H}(F)$ denote the hermitian matrix

$$\begin{pmatrix} F_{\bar{z}z} & F_{\bar{z}u} \\ F_{zu} & F_{uu} \end{pmatrix}. \tag{2.1}$$

Given $(z^0, u^0) \in \mathbb{C} \times \mathbb{R}$ there exists a unitary matrix $A = (a_{\alpha\bar{\beta}})$ such that $\overline{A}^t \mathcal{H} A = \operatorname{diag}(\Lambda_1(F), \Lambda_2(F))$ at (z^0, u^0). Therefore the matrix

$$B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \tag{2.2}$$

diagonalizes $L(F)$ i.e.

$$\overline{B}^t \begin{pmatrix} 0 & F_z & F_u + i \\ F_{\bar{z}} & F_{\bar{z}z} & F_{\bar{z}u} \\ F_u - i & F_{zu} & F_{uu} \end{pmatrix} B = \begin{pmatrix} 0 & l_1'(F) & l_2'(F) \\ \bar{l}_1'(F) & \Lambda_1(F) & 0 \\ \bar{l}_2'(F) & 0 & \Lambda_2(F) \end{pmatrix} \tag{2.3}$$

where $\Lambda_1(F) \leq \Lambda_2(F)$ and

$$\begin{pmatrix} l_1'(F) \\ l_2'(F) \end{pmatrix} = A^t \begin{pmatrix} F_z \\ F_u + i \end{pmatrix}. \tag{2.4}$$

It follows that

$$\underline{L}(F) = |l_2(F)|^2 \Lambda_1(F) + |l_1(F)|^2 \Lambda_2(F)$$

with $l_j(F) = (1 + |DF|^2)^{-1/2} l_j'(F)$, $j = 1, 2$.

Since $|l_1(F)|^2 + |l_2(F)|^2 = 1$ and A is unitary one has

$$\Lambda_1(F) \leqslant \underline{L}(F) \leqslant \Lambda_2(F) \tag{2.5}$$

for every $F \in C^{1,1}$.

We define $\Lambda_1(F)$, $\Lambda_2(F)$ *Levy type extremal operators.*

They have the following explicit forms:

$$\Lambda_1(F) = \frac{1}{2}\{F_{z\bar{z}} + F_{uu} - [(F_{z\bar{z}} + F_{uu})^2 - 4(F_{z\bar{z}} F_{uu} - |F_{zu}|^2)]^{1/2}\}$$

$$\Lambda_2(F) = \frac{1}{2}\{F_{z\bar{z}} + F_{uu} + [(F_{z\bar{z}} + F_{uu})^2 - 4(F_{z\bar{z}} F_{uu} - |F_{zu}|^2)]^{1/2}\}.$$

The linearized operators are rspectively

$$\Lambda_{1,F}(G) = aG_{z\bar{z}} + 2\, Re\, bG_{zu} + cG_{uu} \tag{2.6}$$

$$\Lambda_{2,F}(G) = cG_{z\bar{z}} - 2\, Re\, bG_{zu} + aG_{uu} \tag{2.7}$$

where

$$a = (\delta(F)^{-1/2}/2)(\delta(F)^{1/2} - F_{z\bar{z}} + F_{uu}), \tag{2.8}$$

$$b = -\delta(F)^{-1/2} F_{\bar{z}u}, \tag{2.9}$$

$$c = (\delta(F)^{-1/2}/2)(\delta(F)^{1/2} + F_{z\bar{z}} - F_{uu}), \tag{2.10}$$

$$\delta(F) = (F_{z\bar{z}} - F_{uu})^2 + 4|F_{zu}|^2 \tag{2.11}$$

and $a \geqslant 0$, $c \geqslant 0$, $ac = |b|^2$.

In particular, $\Lambda_{j,F}(F) = \Lambda_j(F)$, $j = 1, 2$.

Moreover if F and G are smooth functions $\Lambda_j(F) - \Lambda_j(G)$, $j = 1, 2$ we have the formulas

$$\Lambda_1(F) - \Lambda_1(G) = CH_{z\bar{z}} - 2\, Re\, BH_{zu} + AH_{uu} \tag{2.12}$$

$$\Lambda_2(F) - \Lambda_2(G) = AH_{z\bar{z}} + 2\, Re\, BH_{zu} + CH_{uu} \tag{2.13}$$

where

$$A = 1/2\left(1 + \frac{F_{z\bar{z}} - G_{z\bar{z}} + F_{uu} + G_{uu}}{\delta(F)^{1/2} + \delta(G)^{1/2}}\right), \tag{2.14}$$

$$B = \frac{F_{\bar{z}u} + G_{\bar{z}u}}{\delta(F)^{1/2} + \delta(G)^{1/2}}, \tag{2.15}$$

$$C = 1/2\left(1 - \frac{F_{z\bar{z}} - G_{z\bar{z}} + F_{uu} + G_{uu}}{\delta(F)^{1/2} + \delta(G)^{1/2}}\right). \tag{2.16}$$

Since $A \geqslant 0$, $C \geqslant 0$ and $AC = |B|^2$ the hermitian form

$$A\xi\bar{\xi} + 2\,Re\,B\xi\bar{\eta} + C\eta\bar{\eta}$$

is positive definite (with at least one positive eigenvalue).

In particular the extremal operators are of the form

$$\det\begin{pmatrix} 0 & \alpha & \beta \\ \bar{\alpha} & G_{\bar{z}z} & G_{\bar{z}u} \\ \bar{\beta} & G_{zu} & G_{uu} \end{pmatrix} \tag{2.17}$$

where α, β are functions of $D^2 F$.

They are degenerate elliptic in the sense of viscosity ([**J**], [**L**]).

2 - Let $\Omega \subseteq \mathbb{C} \times \mathbb{R}$ be a domain and $\mathcal{F}_\Omega$ the family of the *Levi type* operators as defined in Introduction.

Let Λ_1, Λ_2 be the extremal operators. It is immediate to check that the following properties hold true:

(i) for every $F \in C^{1,1}(\Omega)$ and $L \in \mathcal{F}_\Omega$ there is a linear operator $l \in \mathcal{F}_\Omega$ such that $l(F) = L(F)$ a.e.;

(ii) Λ_1, Λ_2 belong to $\mathcal{F}_\Omega$;

(iii) for every $F \in C^{1,1}(\Omega)$

$$\Lambda_1(F) = \min_{L \in \mathcal{F}_\Omega} L(F), \qquad \Lambda_2(F) = \max_{L \in \mathcal{F}_\Omega} L(F).$$

3 - Maximum principle for weak solutions

1 - Let us recall the definition of weak solution (in the sense of viscosity). Denote $\Lambda(F)$ one of the operators $\Lambda_j(F)$, $j = 1, 2$. Let $\Omega \subset \mathbb{C} \times \mathbb{R}$ be a domain and $F: \Omega \to \mathbb{R}$ a continuous function. F is said to be a *weak subsolution* (respectively a *weak supersolution*) of $\Lambda(F) = 0$ if, for every $p \in \Omega$ and ϕ smooth near p such that $F - \phi$ has a local maximum (respectively a local minimum) at p one has $\Lambda(\phi)(p) \geq 0$ (respectively $\Lambda(\phi)(p) \leq 0$).

A *weak solution* is a continuous function which is both a weak subsolution and a weak supersolution.

Observe that in view of the definition of Λ_1 and Λ_2, weak subsolutions (supersolutions) of Λ_1 (Λ_2) are subsolutions (supersolutions) of all operators $L \in \mathcal{F}_\Omega$.

Proposition 3.1. *Let F be a continuous function in $\Omega \subset \mathbb{C} \times \mathbb{R}$. If F is a weak subsolution (weak supersolution) of $\Lambda_1(F) = 0$ ($\Lambda_2(F) = 0$) then F has no local maximum point (local minimum point) in Ω.*

Proof. We may assume that Ω is bounded with regular boundary and that $F \in C^0(\overline{\Omega})$.

Let F be a weak subsolution $\Lambda_1(F) = 0$. It is immediate to check that F is a weak subsolution of $F_{z\bar{z}} + F_{uu} = 0$ so that (passing to new variables $z' = z$, $u' = u/2$) we may assume that F is a weak subsolution of $\Delta F = 0$. F is subharmonic. To prove this let U be the harmonic function which coincides with F on $b\Omega$ and assume that $F - U$ has a positive maximum a at a point $p = (z^0, u^0) \in \Omega$. Let ε be sufficiently small to have $\varepsilon(|z - z^0|^2 + |u - u^0|^2) < a$ on $b\Omega$. It follows that $F - U + \varepsilon(|z - z^0|^2 + |u - u^0|^2)$ has some maximum point in Ω, say q. Since F is a weak subsolution of $\Delta F = 0$ and $V = U - \varepsilon(|z - z^0|^2 + |u - u^0|^2)$ is smooth in Ω, by definition of weak subsolution we must have $\Delta(U - V) \geq 0$ at q. On the other hand $\Delta(V) = -6\varepsilon$, and this gives a contradiction. Therefore $F \leq U$ on whole Ω. Thus F is subaharmonic and hence has no local maximum point in Ω.

The proof for supersolution is similar. ■

In particular under the conditions of the above proposition we have

$$\max_{\overline{\Omega}} F = \max_{b\Omega} F \left\{ \min_{\overline{\Omega}} F = \min_{b\Omega} F \right\}.$$

In order to prove the comparison principle for the extremal operators we recall briefly the main properties of the regularization by «sup and inf» convolution ([**ES**], [**J**], [**Sl**]).

Let $F\colon \mathbb{C}\times\mathbb{R}\to\mathbb{R}$ be continuous and bounded. Set, for $\varepsilon>0$, $z\in\mathbb{C}$ and $u\in\mathbb{R}$

$$F^{\varepsilon}(z, u) = \max\{F(\zeta, v) - \varepsilon^{-1}(|z-\zeta|^2 + (u-v)^2), (\zeta, v)\in\mathbb{C}^2\times\mathbb{R}\}$$

and

$$F_{\varepsilon}(z, u) = \min\{u(\zeta, v) + \varepsilon^{-1}(|z-\zeta|^2 + (u-v)^2), (\zeta, v)\in\mathbb{C}\times\mathbb{R}\}.$$

The above definitions immediately imply that $F_{\varepsilon}\leqslant F\leqslant F^{\varepsilon}$ and $|F_{\varepsilon}|$, $|F^{\varepsilon}|$ are bounded by sup $|F|$; F_{ε}, F^{ε} are Lipschitz and $F^{\varepsilon}\searrow F$, $F_{\varepsilon}\nearrow F$ uniformly on compact subsets as $\varepsilon\to 0$. Moreover, the functions

$$F^{\varepsilon}(z, u) + \varepsilon^{-1}(|z|^2 + u^2)$$

and

$$F_{\varepsilon}(z, u) - \varepsilon^{-1}(|z|^2 + u^2)$$

are respectively convex and concave. In particular they are twice differentiable a.e. ([**K**]).

Let F be a continuous, bounded function in $\mathbb{C}\times\mathbb{R}$ and suppose that F is a weak subsolution of $\Lambda(F)=0$ (respectively a weak supersolution) in a bounded domain Ω of $\mathbb{C}\times\mathbb{R}$. Let

$$\Omega_{\varepsilon} = \{(z, u)\in\Omega : \operatorname{dist}((z, u), bD) > 2\ \sup|F|\varepsilon^{1/2}\}.$$

Then F^{ε} (respectively F_{ε}) is a weak subsolution (respectively a weak supersolution) in Ω_{ε}. Moreover

$$\Lambda(F^{\varepsilon})\geqslant 0 \qquad (\Lambda(F^{\varepsilon})\leqslant 0)$$

at each point of twice differentiability of F^{ε} (F_{ε}).

This can be seen as follows. Let $F^{\varepsilon}-\phi$ have a local maximum at $(z^0, u^0)\in\Omega_{\varepsilon}$, ϕ smooth, and $(\zeta^0, u^0)\in\mathbb{C}\times\mathbb{R}$ be such that

$$F^{\varepsilon}(z^0, u^0) = F(\zeta^0, v^0) - \varepsilon^{-1}(|z^0-\zeta^0|^2 + (u-v)^2).$$

Set

$$\psi(z, u) = \phi(z + z^0 - \zeta^0, u + u^0 - v^0).$$

Then $(\zeta^0, v^0) \in \Omega$ and for all (z, u) near (z^0, u^0) and all $(\zeta, v) \in \mathbb{C} \times \mathbb{R}$ we have

$$F(\zeta, v) - \varepsilon^{-1}(|z-\zeta|^2 + (u-v)^2) - \phi(z, u) \leqslant F^\varepsilon(z, u) - \phi(z, u) \leqslant u(\zeta^0, v^0)$$
$$- \varepsilon^{-1}(|z^0 - \zeta|^2 + (u-v)^2)$$
$$- \phi(z^0, u^0);$$

in particular, for $z = \zeta + z^0 - \zeta^0$, $u = v + u^0 - v^0$ and (ζ, v) near (ζ^0, v^0), we have

$$F(\zeta, u) - \psi(\zeta, v) \leqslant F(\zeta^0, v^0) - \psi(\zeta^0, v^0),$$

i.e. $F - \psi$ has a local maximum at (ζ^0, v^0). Since

$$\psi_\alpha(\zeta^0, v^0) = \phi_\alpha(z^0, u^0), \qquad \psi_{\alpha\bar\beta}(z^0, u^0) = \phi_{\alpha\bar\beta}(\zeta^0, v^0)$$

we obtain $\Lambda(\phi) \geqslant 0$.

Now twice differentiability a.e. of F^ε and F_ε easily implies the statement.

Theorem 3.2. *Let $F \in C^0(\overline{\Omega})$ be a weak subsolution and $G \in C^0(\overline{\Omega})$ a weak supersolution of $\Lambda = 0$ in Ω. Then*

$$\max_{\overline{\Omega}}(F - G) = \max_{b\Omega}(F - G).$$

Proof. We have to prove that if $F \leqslant G$ on $b\Omega$ then $F \leqslant G$.

We may assume that $F < G$ on $b\Omega$. Moreover, since for H smooth and $c \in \mathbb{R}$ positive we have $\Lambda(H \mp c(|z|^2 + u^2/2)) = \Lambda(H) \mp 2c$ we may also assume that F is a weak subsolution of $\Lambda(F) = c$ and G is a weak supersolution of $\Lambda(G) = -c$, c positive.

Let $\max_{\overline{\Omega}}(F - G) = a \geqslant 0$. Extend F, G by continuous functions in such a way to have $F < G$ in $\mathbb{C} \times \mathbb{R} \setminus \overline{\Omega}$ and $F = \text{const}$, $G = \text{const}$ for $|z|^2 + u^2 \gg 0$. Since $\max_{\overline{\Omega}}(F - G) = a > 0$ we have

$$\max_{\mathbb{C} \times \mathbb{R}}(F - G) = a$$

and consequently

$$(\star) \qquad \max_{\mathbb{C} \times \mathbb{R}}(F^\varepsilon - G_\varepsilon) \geqslant a$$

for all $\varepsilon \leqslant \varepsilon_0$.

Now set $\xi = (z, u)$ and define for ξ, $\xi + \eta \in \mathbb{C} \times \mathbb{R}$,

$$\Phi_{\varepsilon,\delta}(\xi, \eta) = F^{\varepsilon}(\xi + \eta) - G_{\varepsilon}(\xi) - \delta^{-1}|\eta|^4 .$$

$\Phi_{\varepsilon,\delta}$ is negative outside of a compact subset K and $\Phi_{\varepsilon,\delta} \geqslant \Phi_{\varepsilon',\delta}$, $\Phi_{\varepsilon,\delta'} \geqslant \Phi_{\varepsilon,\delta}$ provided $\varepsilon' \leqslant \varepsilon$, $\delta' \leqslant \delta$. Moreover, because of ($\star$),

$$\max_K \Phi_{\varepsilon,\delta} \geqslant a .$$

Let $p_{\varepsilon,\delta} = (\xi', \eta')$ be a maximum point for $\Phi_{\varepsilon,\delta}$. By definition $|\eta'| \leqslant C\delta^{1/4}$ (where C depends only on F and G). We claim that for ε, δ near 0, ξ' and $\xi' + \eta'$ belong to Ω.

To prove this we consider a limit point (ξ^0, η^0) of the bounded set $\{p_{\varepsilon,\delta}\}$. Then, since $\Phi_{\varepsilon,\delta}(p_{\varepsilon,\delta}) \geqslant a$, $\delta^{-1}|\eta'|^4$ must be bounded as $\varepsilon, \delta \to 0$ and this forces η^0 to be 0. It follows that

$$F(\xi^0) - G(\xi^0) - B \geqslant a$$

for some positive constant B. Since $F < G$ outside of Ω this proves our claim.

From now on in the proof ε, δ are fixed.

According to convexity and concavity properties of the «sup and inf» convolution the function $\Phi(\xi, \eta) + C(|\xi|^2 + |\eta|^2)$ is convex near (ξ', η') for some sufficiently large constant C. Since Φ takes its maximum at (ξ', η') we can apply the theorem of Jensen [**J**]: there exist sequences $(\xi^k, \eta^k) \to (\xi', \eta')$ and $\varepsilon^k \to 0$ such that $D^2_{\xi,\eta}\Phi(\xi^k, \eta^k) \leqslant \varepsilon^k I$ as $k \to +\infty$.

By definition

$$D^2_{\xi}\Phi(\xi^k, \eta^k) = D^2 F^{\varepsilon}(\xi^k + \eta^k) - D^2 G_{\varepsilon}(\xi^k) = R^k - \overline{R}^k .$$

and $R^k - \overline{R}^k \leqslant \varepsilon^k I$. Moreover, again by virtue of convexity and concavity properties of the «sup and inf» convolution, there exists a constant C such that $R^k \geqslant -CI$ and $\overline{R}^k \leqslant CI$. Consequently, passing to subsequences we obtain $R^k \to R$, $\overline{R}^k \to \overline{R}$ and, since $\Lambda(F^{\varepsilon})(\xi^k + \eta^k) \geqslant c$, $\Lambda(F^{\varepsilon})(\xi^k) \leqslant -c$, $\Lambda(R) \geqslant c$, $\Lambda(\overline{R}) \leqslant -c$ (here we consider Λ as an operator acting on the matrices $M = (M_{ij})$ just replacing second derivatives $\partial_{ij} G$ of a function G by M_{ij}).

Now observe that $\delta(R) + \delta(\overline{R}) > 0$ (otherwise $Tr(R) > Tr(\overline{R})$ which is absurd since $R \leqslant \overline{R}$) and put $S = R - \overline{R}$ to have

$$0 < \Lambda(R) - \Lambda(\overline{R}) = \alpha(S_{11} + S_{22}) + 2\, Re\, \beta(S_{13} + iS_{23}) + \gamma S_{33}$$

where $\alpha \geqslant 0$, $\gamma \geqslant 0$ and $\alpha\gamma = |\beta|^2$.

For $X_1 = X_2 = (2|\beta|)^{-1}\gamma^{1/2}$, $X_3 = (2|\beta|)^{-1}\alpha^{1/2}$ (if $\beta \neq 0$) the above inequality rewrites

$$0 < \Lambda(R) - \Lambda(\overline{R}) = S_{11}X_1^2 + S_{22}X_2^2 + 2S_{13}X_1X_3 + S_{33}X_3^2$$

and this is a contradiction since the matrix S is nonpositive. ■

For smooth solutions strong maximum and minimum principle hold true. Namely

Theorem 3.3. *Let $F \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a nonconstant solution of $\Lambda(F) = 0$. Then F has no interior maximum or minimum.*

In view of what is preceding we only have to show that F has no interior maximum. This is actually a direct consequence of the following Hopf Lemma which can be proved in the same way as for elliptic operators ([**GT**]):

Lemma 3.4. *Let Ω' be a bounded domain with smooth boundary and let $F \in C^1(\overline{\Omega}' \cap C^2(\Omega')$ be a nonconstant solution of $\Lambda(F) = 0$. Assume that $x^0 \in b\Omega'$ is a peak point for F (i.e. $F(x) < F(x^0)$ for $x \neq x^0$). Then the outer normal derivative of F at x^0 is positive.*

2 - The maximum principle for weak solutions of $\Lambda(F) = 0$ as well as of $\underline{L}(F) = 0$ can be refined. Let us discuss it in a simple situation namely when Ω is a bounded convex domain in $\mathbb{C} \times \mathbb{R}$.

Proposition 3.5. *Let $\Omega_c = \Omega \cap \{u < c\}$, $\Sigma_c = b\Omega \cap \{u < c\}$, $c \in \mathbb{R}$, and $F \in C^0(\overline{\Omega})$. Then*

(i) *if F is a weak subsolution of $\Lambda_1(F) = 0$ or $\overline{L}(F) = 0$*

$$\max_{\overline{\Omega}_c} F = \max_{\overline{\Sigma}_c} F\,;$$

(ii) *if F is a weak supersolution of $\Lambda_2(F) = 0$ or $\underline{L}(F) = 0$*

$$\min_{\overline{\Omega}_c} F = \min_{\overline{\Sigma}_c} F\,;$$

(iii) *if F is a weak solution of $\underline{L}(F) = 0$*

$$\min_{\overline{\Sigma}_c} F \leqslant F(z, u) \leqslant \max_{\overline{\Sigma}_c} F$$

for every $(z, u) \in \overline{\Omega}_c$.

Proof. Let $G(z, u) = \max_{\bar{\Sigma}_u} F$. G is constant with respect to z and $F \leqslant G$ on $b\Omega$.

G is a weak supersolution of Λ_1.

To show this let ϕ be smooth and $F - \phi$ have a local minimum at (z^0, u^0), i.e.

$$G(u) - \phi(z, u^0) \geqslant G(u^0) - \phi(z^0, u^0)$$

near (z^0, u^0). Then $\phi(z, u^0) \leqslant \phi(z^0, u^0)$ near (z^0, u^0) so that $\phi_{z\bar{z}}(z^0, u^0) \leqslant 0$. By definition

$$\Lambda_1(\phi)(z^0, u^0) = \phi_{z\bar{z}}(z^0, u^0) + \phi_{uu}(z^0, u^0) - \delta(\phi)(z^0, u^0)$$

where $\delta(\phi) = (\phi_{z\bar{z}} - \phi_{uu})^2 + 4\,|\phi_{zu}|^2$.

If $\phi_{uu}(z^0, u^0) \leqslant 0$ then $\Lambda_1(\phi)(z^0, u^0) \leqslant 0$.

If $\phi_{uu}(z^0, u^0) \geqslant 0$ we have

$$\delta(\phi)^{1/2} = [(\phi_{z\bar{z}} - \phi_{uu})^2 + 4\,|\phi_{zu}|^2]^{1/2}(z^0, u^0)$$

$$\geqslant (\phi_{uu} - \phi_{z\bar{z}})(z^0, u^0)$$

and consequently

$$\Lambda_1(\phi)(z^0, u^0) \leqslant -\phi_{uu}(z^0, u^0) \leqslant 0\,.$$

The inequality (i) is now a direct consequence of theorem 3.2.

The proofs of (ii) and (iii) are similar. ■

Remark 3.1. The «minimum (maximum) statement» for weak solutions (solutions) of Λ_1 (Λ_2) fails to be true: e.g. let $F = |z|^2 - u$ ($F = |z|^2 - u$) and $\Omega = \{u > |z|^2\}$.

Corollary 3.6. *Let K be a compact convex subset of Ω. Then*

(i) *continuous weak subsolutions of $\Lambda_1(F) = 0$ or $\underline{L}(F) = 0$ in $\Omega \backslash K$ are bounded from above near K;*

(ii) *continuous weak supersolutions of $\Lambda_2(F) = 0$ or $\underline{L}(F) = 0$ in $\Omega \backslash K$ are bounded from below near K;*

(iii) *continuous weak solutions of $\underline{L}(F) = 0$ in $\Omega \backslash K$ are bounded near K;*

Proof. Let $U \subset \Omega$ be a relatively compact convex neighbourhood of K. For each point $p \in bK$ there is a «support hyperplane» T for K ($p \in T$). By a holomorphic change of coordinates we may assume that $T = \{u = c\}$ and $\{u < c\} \cap K = \emptyset$.

Let $\Sigma = bU$ and $\Sigma_c = bU \cap \{u < c\}$. In view of the above proposition we have respectively

$$F(z, u) \leqslant \max_{\overline{\Sigma}_c} F \leqslant \max_{bU} F, \qquad \min_{bU} F \leqslant \min_{\overline{\Sigma}_c} F \leqslant F(z, u)$$

and

$$\min_{bU} F \leqslant \min_{\overline{\Sigma}_c} F \leqslant F(z, u) \leqslant \max_{\overline{\Sigma}_c} F \leqslant \max_{bU} F$$

for $(z, u) \in U \cap \{u \leqslant c\}$.

This proves the corollary. ■

Remark 3.2. The above statement gives rise to the following general question: what closed sets can be singular for weak solutions of $\Lambda(F) = 0$ or $\underline{L}(F) = 0$?

Using the results stated in [**ST 2**] it can be proved, for instance, that every continuous function F which is a weak solution of $\underline{L}(F) = 0$ in $\Omega \setminus K$ is a weak solution in all of Ω.

4 - Dirichlet problem for extremal operators

1 - Solutions of the Dirichlet problem for the Levi type extremal operators provide bounds for solutions of the Dirichlet problem for *Levi type operators* (i.e belonging to $\mathcal{F}_\Omega$); in particular for the Levi operator.

Precisely consider the Dirichlet problem for the complete Levi equation

$$\begin{cases} \underline{L}(F) = k(\cdot, F)(1 + |DF|^2)^{1/2} & \text{in } \Omega \\ F = f & \text{on } b\Omega \end{cases} \tag{4.1}$$

where $k = k(z, u, t)$ is continuous in $\Omega \times \mathbb{R}$ and suppose that $F \in C^0(\overline{\Omega})$ is a weak solution of (4.1) ([**ST 1**]). Let $F_1, F_2 \in C^0(\overline{\Omega})$ be the weak solutions of the corresponding Dirichlet problems for Λ_1 and Λ_2 respectively.

Then F_1 is a weak subsolution of $\Lambda_1 - k(\cdot, F)(1 + |DF|^2)^{1/2}$ and F_2 is a weak supersolution of $\Lambda_2 - k(\cdot, F)(1 + |DF|^2)^{1/2}$. Now we invoke the maximum principle to derive that $F_1 \leqslant F$ in $\overline{\Omega}$, if k is non-decreasing with respect to t, and $F \leqslant F_2$ if k is non-increasing.

In particular, if k does not depend on t we have

$$F_1 \leqslant F \leqslant F_2$$

in Ω and $F_1 = F = F_2$ on $b\Omega$ i.e. F_1, F_2 are barriers for the complete Levi equation.

In what follows we formulate the Dirichlet problem for the extremal operators and we translate it into a Dirichlet problem for a Bellman equation.

For simplicity we restrict ourselves to the homogeneous case.

2 - Let us consider the following operators

$$\tilde{\Delta}(F) = F_{z\bar{z}} + F_{uu},$$

$$MA(F) = F_{z\bar{z}}F_{uu} - |F_{zu}|^2.$$

A continuous function F is said to be a *weak subsolution* of $MA(F) = 0$ if, for every p and ϕ smooth near p such that $F - \phi$ has a local maximum at p, one has $MA(\phi)(p) \geqslant 0$, $\tilde{\Delta}(\phi)(p) \geqslant 0$.

F is said to be a *weak supersolution* if for every p and ϕ smooth near p such that $F - \phi$ has a local minimum at p either $MA(\phi)(p) \leqslant 0$ or $MA(\phi)(p) > 0$ and $\tilde{\Delta}(\phi)(p) < 0$.

A *weak solution* is a continuous function which is both a weak subsolution and a weak supersolution.

We have the following

Proposition 4.1. *Let F be continuous in Ω. Then*

(i) *F is a weak solution of $\Lambda_1 = 0$ if and only if is a weak solution of $MA(F) = 0$;*

(ii) *F is a weak solution of $\Lambda_2 = 0$ if and only if $-F$ is a weak solution of $MA(F) = 0$.*

The proof is straightforeward.

Now we observe that if F is regular and the matrix

$$\mathcal{H}(F) = \begin{pmatrix} F_{\bar{z}z} & F_{\bar{z}u} \\ F_{zu} & F_{uu} \end{pmatrix}$$

is positive definite at each point $(z, u) \in \Omega$ then

$$MA(F)(z, u)^{1/2} = (\det \mathcal{H}(F)(z, u))^{1/2} = \inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(F)(z, u)$$

where $\boldsymbol{V}$ is the set of the positive definite hermitian matrices with $\det A = 1/4$.

Furthermore the differential operator $\inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(F)$ is degenerate elliptic.

This reduces the Dirichlet problem for $MA(F) = 0$ to a Dirichlet problem for a Bellman equation. Precisely

Proposition 4.2. *The Dirichlet problem: $MA(F) = 0$ in Ω, $F = f$ on $b\Omega$ and $F \in C^0(\Omega)$ is solvable if and only if is solvable in $C^0(\overline{\Omega})$ the Dirichlet problem*

$$\begin{cases} \inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(F) = 0 & \text{in } \Omega \\ F = f & \text{on } b\Omega\,. \end{cases} \tag{4.2}$$

Proof. Let F be a weak subsolution of $MA(F) = 0$ and let $F - \phi$, ϕ smooth, have a local maximum at $p \in \Omega$. Then $MA(\phi)(p) \geqslant 0$ and $\widetilde{\Delta}(\phi)(p) \geqslant 0$. It follows that $\mathcal{H}(\phi)(p)$ is positive definite and consequently $\inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(F) \geqslant 0$. Let F be a weak supersolution and $F - \phi$ have a local minimum at $p \in \Omega$. If $MA(\phi)(p) = 0$, 0 is an eigenevalue of $\mathcal{H}(\phi)(p)$; therefore $\inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(F) \leqslant 0$. If $MA(\phi)(p) < 0$ one eigenvalue is negative and again $\inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(\phi)(p) \leqslant 0$. Finally assume that $MA(\phi)(p) > 0$ and $\widetilde{\Delta}(\phi)(p) < 0$. Then $\mathcal{H}(\phi)(p)$ is negative definite and $\inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(\phi)(p) = -\infty$.

This proves that F is a weak solution of $\inf_{A \in \boldsymbol{V}} Tr A \mathcal{H}(F) = 0$.

The proof of the converse is similar. ∎

Remark 4.1. For each $A \in \boldsymbol{V}$ the operator

$$\mathcal{B}_A(F) := Tr A \mathcal{H}(F)$$

is elliptic.

In order to solve (4.2) we fix a dense subset $\{A_\nu\}$ of matrices of $\boldsymbol{V}$ and for every $m \in \mathbb{N}$ we solve the Dirichlet problem

$$\begin{cases} \inf\limits_{1 \leqslant \nu \leqslant m} \mathcal{B}_{A_\nu}(F) = 0 & \text{in } \Omega \\ F = f & \text{on } b\Omega\,. \end{cases} \tag{4.3}$$

Next uniform (with respect to m) a priori estimates insure that the sequence $\{F_m\}$ of the corresponding solutions has a subsequence converging in $\mathrm{Lip}(\overline{\Omega})$ to (the unique) weak solution of (4.2).

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Summary

We introduce maximal and minimal operators Λ_1, Λ_2 *(in the sense of Pucci) for the family* $\mathcal{F}_\Omega$ *of the «Levi type operators»* $L_{\alpha,\beta}$ *on a domain* Ω *of* $\mathbb{C}\times\mathbb{R}$. *We state some simple property for (weak, viscosity) solutions of* Λ_1, Λ_2. *In particular we prove a special form of the maximum principle. As a consequence we obtain that solutions of* Λ_1, Λ_2 *satisfy a «weak Hartogs property». We are also dealing with the Dirichlet problem for* Λ_1, Λ_2. *After shown that solutions of that provide barriers for the Levi operator we prove that this problem translates into a Dirichlet problem for a Bellman equation.*

* * *