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An evolution model for the Ginzburg-Landau equations (**)

1 - Introduction

For a steady problem the Ginzburg-Landau equations (see [**3**], [**4**], [**7**], [**8**], [**9**]) are minimizers of a Gibbs free energy represented as a function of the vector potential $\boldsymbol{A}$, connected with magnetic field by $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and of a complex parameter $\psi = |\psi| e^{i\theta} = fe^{i\theta}$, where $f = |\psi|$ denotes the concentration of superconducting electrons.

A system of equations equivalent to the Ginzburg-Landau one is obtained by means of a free energy represented as a functional of the observable variables $\boldsymbol{J}_s$ and f [**15**]. The aim of this work is to extend this point of view to a time-dependent model.

The dynamic Ginzburg-Landau equations was developed by Gor'kov and Èliashberg [**6**]. In this model the term $\dfrac{\partial \boldsymbol{E}}{\partial t}$ of the first Maxwell equation is assumed negligible. Therefore, this new system represents an approximated problem, which we shall call quasi dynamic model, whose we prove the compatibility with the thermodynamic laws. It is important to observe that the hypotheses considered in [**6**] are not in agreement with the thermodynamics, when the dynamical (not-approximated) case is considered. For this reason, in this work following the point of view considered in [**15**], we introduce a new constitutive hypothesis, which

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is able to provide a model in full agreement with the thermodynamic laws, either for the quasi-dynamic problem or for the general case.

2 - Superconductivity and Ginzburg-Landau model

The most outstanding property of a superconductor is the complete disappearance of the electrical resistivity at some low *critical temperature* T_c, which is characteristic of the material. However, there exists a second effect which is equally meaningful. This phenomenon, called Meissner effect, is the perfect diamagnetism. In other words, the magnetic field is expelled from the superconductor, independently of whether the field is applied in the superconductive state (zero-field-cooled) or already in the normal state (field-cooled).

The London theory [**1**], [**2**] assumes that in a superconductor the current $\boldsymbol{J}$, is the sum of a supercurrent $\boldsymbol{J}_s$ and of a normal current $\boldsymbol{J}_n$. Moreover, it is assumed that $\boldsymbol{J}_n$ obeys Ohm's law

$$\boldsymbol{J}_n = \sigma \boldsymbol{E} \tag{1}$$

where $\boldsymbol{E}$ is the *electric field* and σ is the *electric contuctivity*.

The peculiar equation in the London theory, as proved in [**14**], is that relating $\boldsymbol{J}_s$ with the *magnetic field* $\boldsymbol{H}$

$$\nabla \times \Lambda \boldsymbol{J}_s = -\mu \boldsymbol{H} \tag{2}$$

where $\Lambda(x)$ is a scalar coefficient characteristic of the material, and μ is the *magnetic permeability*. The constitutive equations (1), (2) must be related to Maxwell's equations

$$\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J}_s - \boldsymbol{J}_n, \qquad \nabla \cdot \boldsymbol{E} = \boldsymbol{0} \tag{3}$$

$$\mu \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{H} = 0 \tag{4}$$

where ε is the dielectric constant. For sake of simplicity, we shall suppose ε, μ, σ scalar and constant coefficients.

Equation (2), together with (3) (4) is able to explain the lack of the electrical resistivity (see the proof in [**14**]) and the Meissner effect [**2**]. In the London theory, $\Lambda(x)$ is a constitutive coefficient related to the local density of superconducting electrons n_s by

$$\Lambda(x) = \frac{m_*}{e_*^2 n_s} \tag{5}$$

where m_* and e_* are respectively the mass and the charge of a super-electron.

An important step in the phenomenological description of superconductivity was the introduction of an order parameter distinguishing the superconducting phase from the normal phase by Ginzburg-Landau [3] in 1950. This order parameter was assumed to be a complex field $\psi = fe^{i\theta}$, representing a kind of macroscopic wave function, such that $f = n_s$ and θ is a suitable phase (1).

Ginzburg and Landau [3] proposed a phenomenological extension of the London equations, in order to take into account the parameter ψ as a new unknown.

In absence of an applied magnetic field, there is no latent heat associated with the transition, which will be called second-order phase change. If the transition occurs in presence of a magnetic field at $T < T_c$, there exists a latent heat corresponding to an absorbing of heat when the sample goes normal. This transition is a first-order phase change. In this case theoretical arguments lead one to expect breakup into a domain structure, with alternating normal and superconducting regions. Landau suggests to represent this «intermediate state» through the introduction of a positive surface energy associated with the creation of an interface between a normal and a superconducting region.

3 - Steady state

In [3], Ginzburg and Landau consider only the steady case with the electric field $\boldsymbol{E} = 0$. Therefore, from the Maxwell equation (3) we have

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_s \tag{6}$$

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(1) Various authors use a standard normalization, for which

$$\frac{n_s}{n_0} = f^2$$

where n_0 is the value of n_s at $T = 0° K$, so that $f^2 = 1$ at $T = 0° K$ and $f^2 = 0$ at $T = T_c$.

from which $\nabla \cdot \boldsymbol{J}_s = 0$. Since $\boldsymbol{B} = \mu \boldsymbol{H}$ is solenoidal, it is possible to introduce a vector potential $\boldsymbol{A}$ satisfying

$$\nabla \times \boldsymbol{A} = \boldsymbol{B}\,, \qquad \nabla \cdot \boldsymbol{A} = 0\,. \tag{7}$$

Comparing (7) with (2) we got

$$\Lambda \boldsymbol{J}_s = -\boldsymbol{A} + \nabla \varphi \tag{8}$$

where φ is a smooth scalar function. Clearly (2) is equivalent to (8).

Following a general theory for phase transitions of the second kind, Ginzburg-Landau [**3**] assume that near the critical temperature T_c the Gibbs free energy of a superconducting material, occupying the domain $\Omega \subset \mathbb{R}$, is given by

$$\begin{aligned}\int_\Omega e(\psi, \boldsymbol{A})\,\mathrm{d}x = \int_\Omega \Big(-\alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2\mu} |\nabla \times \boldsymbol{A}|^2 \\ + \frac{1}{2m_*} |(-i\hbar\nabla - e_*\boldsymbol{A})\,\psi|^2 \Big) dx - \int_{\partial\Omega} \boldsymbol{A} \times \boldsymbol{H}_{ex} \cdot \boldsymbol{n}\,\mathrm{d}\sigma\end{aligned} \tag{9}$$

where α and β are coefficients depending on the temperature T, such that $\alpha(T) > 0$ (< 0) for $T < T_c$ $(T > T_c)$ and $\beta(T) > 0$, and $\hbar$ is the Planck's constant. The vector $\boldsymbol{H}_{ex}$ represents the *magnetic field on the boundary.*

Let us consider the new functional

$$\begin{aligned}\int_\Omega \varepsilon(\psi, \boldsymbol{A})\,\mathrm{d}x = \int_\Omega \Big(e(\psi, \boldsymbol{A}) + \frac{\alpha^2}{2\beta} \Big)\,\mathrm{d}x = \int_\Omega \Big(\frac{1}{2} \Big(\sqrt{\beta}|\psi|^2 - \frac{\alpha}{\sqrt{\beta}}\Big)^2 \\ + \frac{1}{2\mu} |\nabla \times \boldsymbol{A}|^2 + \frac{1}{2m_*} |(-i\hbar\nabla - e_*\boldsymbol{A})\,\psi|^2 \Big)\,\mathrm{d}x - \int_{\partial\Omega} \boldsymbol{A} \times \boldsymbol{H}_{ex} \cdot \boldsymbol{n}\,\mathrm{d}\sigma\end{aligned} \tag{10}$$

where $\boldsymbol{n}$ is the local outer unit normal to the boundary $\partial\Omega$.

The functionals (9) and (10) are equivalent in the sense that they present the same critical points, but the functional (10) is non-negative.

The search for a minimum of (9) or (10) by means of the variations with respect to ψ^* and $\boldsymbol{A}$, leads to the so called Ginzburg-Landau equations

$$\frac{1}{2m_*} (i\hbar\nabla + e_*\boldsymbol{A})^2 \psi - \alpha\psi + \beta|\psi|^2\psi = 0 \tag{11}$$

$$\boldsymbol{J}_s = \mu^{-1} \nabla \times \nabla \times \boldsymbol{A} = -\frac{i\hbar e_*}{2m_*} (\psi^* \nabla\psi - \psi\nabla\psi^*) - \frac{{e_*}^2}{m_*} |\psi|^2 \boldsymbol{A} \tag{12}$$

while the boundary conditions appropriated at an insulating surface are

(13) $$(i\hbar\nabla + e_*\boldsymbol{A})\,\psi\cdot\boldsymbol{n}\,|_{\partial\Omega} = 0\,, \qquad (\nabla\times\boldsymbol{A})\times n\,|_{\partial\Omega} = \mu\boldsymbol{H}_{ex}\times\boldsymbol{n}\,.$$

Equations (11)-(13), taken along with (6), form the basis of the Ginzburg-Landau theory. These equations lead to the usual London theory, when the density of superconducting electrons f^2 is a given function.

Equation (12) can be written as

(14) $$\Lambda(f)\boldsymbol{J}_s = -\left(\frac{\hbar}{e_*}\nabla\theta + \boldsymbol{A}\right).$$

Therefore (14) becomes identical to (8), if we set $\varphi = -\dfrac{\hbar}{e_*}\theta$.

Moreover, the free energy (9) and the Ginzburg-Landau equations (11), (12) are invariant under gauge transformations of the form

(15) $$(\psi, \boldsymbol{A}) \leftrightarrow (\psi e^{i\kappa\chi}, \boldsymbol{A} + \nabla\chi)$$

where $\kappa = \dfrac{e_*}{\hbar}$.

It is natural to ask whether such functional may be made independent of $\boldsymbol{A}$ and θ, but dependent on observable variables as $\boldsymbol{J}_s$ and f. In order to have this kind of representation we show (see [8]) that

(16) $$\begin{aligned}|i\hbar\nabla\psi + e_*\boldsymbol{A}\psi|^2 &= \hbar^2(\nabla f)^2 + f^2(\hbar\nabla\theta + e_*\boldsymbol{A})^2\\ &= \hbar^2(\nabla f)^2 + \Lambda\boldsymbol{J}_s^2 = \hbar^2(\nabla f)^2 + \Lambda^{-1}\boldsymbol{p}_s^2\end{aligned}$$

where if $\boldsymbol{v}_s$ is the *supercurrent velocity* and $f>0$, $\mathbf{p}_s = \dfrac{m_*}{e_*}\boldsymbol{v}_s = \Lambda\boldsymbol{J}_s$ is the *momentum field*, which satisfies the equation

(17) $$\nabla\times\boldsymbol{p}_s = -\mu\boldsymbol{H}\,.$$

When $f=0$, the meaning of $\boldsymbol{p}_s$ is given by the equation (17). In other words

$$\boldsymbol{p}_s = -\left(\frac{\hbar}{e_*}\nabla\theta + \boldsymbol{A}\right).$$

In any case, we can consider on the boundary

(18) $$\boldsymbol{p}_s\cdot\boldsymbol{n}\,|_{\partial\Omega} = 0\,.$$

Hence, we consider the free energy (9) written in the form

(19)
$$\int_{\Omega} e(f, \boldsymbol{H})\, dx = \int_{\Omega} \left(-\alpha f^2 + \frac{\beta}{2} f^4 + \frac{1}{2\mu} |\nabla \times \boldsymbol{p}_s|^2 + \frac{\hbar^2}{2m_*} (\nabla f)^2 + \frac{1}{2} \Lambda^{-1}(f)\, \boldsymbol{p}_s^2 \right) dx - \int_{\partial\Omega} \boldsymbol{p}_s \times \boldsymbol{H}_{ex} \cdot \boldsymbol{n}\, d\sigma .$$

The theory is now essentially completed by minimizing the total free energy for variations in f and $\boldsymbol{p}_s$. This finally leads to the system

(20)
$$\frac{\hbar^2}{2m_*} \nabla^2 f - \frac{e_*}{2m_*} f \boldsymbol{p}_s^2 + \alpha f - \beta f^3 = 0$$

(21)
$$\mu^{-1} \nabla \times \nabla \times \boldsymbol{p}_s + \Lambda^{-1}(f)\, \boldsymbol{p}_s = 0$$

on Ω, and to the boundary conditions

(22)
$$\nabla f \cdot \boldsymbol{n} = 0\,, \qquad (\boldsymbol{\nabla} \times \boldsymbol{p}_s) \times \boldsymbol{n} = -\boldsymbol{H}_{ex} \times \boldsymbol{n}$$

on $\partial\Omega$. It is easy to observe that equation (20) is equal to the real part of the Ginzburg-Landau equation (11), while equation (21) corresponds to the Maxwell equation

(23)
$$\nabla \times \boldsymbol{H} = \Lambda^{-1} \boldsymbol{p}_s$$

which is equivalent to the restriction

(24)
$$\nabla \cdot \boldsymbol{J}_s = 0$$

because we have from (21)

$$0 = \nabla \cdot \boldsymbol{J}_s = \frac{{e_*}^2}{m_*} f \nabla f \cdot \boldsymbol{p}_s + \Lambda^{-1} \nabla \cdot \boldsymbol{p}_s .$$

Besides, as we can see, the system of four scalar equations (20)-(21), written in terms of real variables, is absolutely equivalent to the system (11), (12) {see [**8**] eq. (23), (24), and [**12**] eq. (6)-(8)}. Namely, the case for which the gauge is fixed by the problem

(25)
$$\nabla \cdot \boldsymbol{A} = 0\,; \qquad \boldsymbol{A} \cdot \boldsymbol{n} = \boldsymbol{0}\,, \ \text{on } \partial\Omega\,.$$

It is possible to obtain an analogous result for the system (20), (21), starting from the new form of free energy

$$\begin{aligned}
&\int_{\Omega} \tilde{e}(f, \boldsymbol{H})\,\mathrm{d}x \\
(26)\qquad &= \int_{\Omega} \left\{ -\alpha f^2 + \frac{\beta}{2} f^4 - \frac{1}{2}\mu \boldsymbol{H}^2 + \frac{\hbar^2}{2m_*}(\nabla f)^2 - \frac{1}{2}\Lambda(f)(\nabla\times\boldsymbol{H})^2 \right\} \mathrm{d}x \\
&\quad - \int_{\partial\Omega} \boldsymbol{p}_s^{\sigma} \times \boldsymbol{H}\cdot n\,\mathrm{d}\sigma
\end{aligned}$$

where $\boldsymbol{p}_s^{\sigma}$ is the momentum field on the boundary $\partial\Omega$.

Minimizing the total free energy for variations in f and $\boldsymbol{H}$, we obtain the system

$$\frac{\hbar^2}{2m_*}\nabla^2 f - \frac{m_*}{2e_*^{\,2}} f^{-3}(\nabla\times\boldsymbol{H})^2 + \alpha f - \beta f^3 = 0 \qquad (27)$$

$$\nabla\times(\Lambda(f)\,\nabla\times\boldsymbol{H}) = -\,\mu\boldsymbol{H} \qquad (28)$$

on Ω, and the boundary conditions

$$\nabla f\cdot\boldsymbol{n} = 0\,, \qquad (\Lambda(f)\,\nabla\times\boldsymbol{H})\times\boldsymbol{n} = \boldsymbol{p}_s^{\sigma}\times\boldsymbol{n} \qquad (29)$$

on $\partial\Omega$. It is possible to prove that the system (27), (28) is equivalent to (20), (21).

Let briefly discuss the boundary conditions (29). First, $(29)_1$ is equivalent to $(13)_1$, when the boundary condition $\boldsymbol{A}\cdot\boldsymbol{n} = \boldsymbol{0}$ holds. Next, $(29)_2$ is different from $(13)_2$, but its physical meaning is clear and related with $(13)_2$.

4 - Evolution model. Quasi-dynamic case

The generalization of the Ginzburg-Landau theory to the evolution problem was analyzed by Schmid [**5**], Gor'kov and Eliashberg [**6**] in the context of the BCS theory of superconductivity. In addiction to the variables ψ, $\boldsymbol{A}$ a third variable, the *electric potential* ϕ, is necessary to describe the physical state of the evolution system. The potentials $\boldsymbol{A}$ and ϕ are such that

$$\mathbf{E} = -\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi\,, \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}\,.$$

The evolution model of superconductivity is governed by the differential system [**5**], [**6**]

$$\gamma\left(\frac{\partial\psi}{\partial t}+i\kappa\phi\psi\right)=-\frac{1}{2m_*}(i\hbar\nabla+e_*\boldsymbol{A})^2\psi+\alpha\psi-\beta|\psi|^2\psi \tag{30}$$

$$\sigma\left(\frac{\partial\boldsymbol{A}}{\partial t}+\nabla\phi\right)=-\nabla\times\nabla\times\boldsymbol{A}+\boldsymbol{J}_s \tag{31}$$

with

$$\boldsymbol{J}_s=-\frac{i\hbar e_*}{2m_*}(\psi^*\nabla\psi-\psi\nabla\psi^*)-\frac{{e_*}^2}{m_*}|\psi|^2\boldsymbol{A} \tag{32}$$

and γ a suitable coefficient representing a relaxation time. The associated boundary conditions are still given by (13).

Equation (31) is essentially Ampere's law

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_s+\boldsymbol{J}_n+\varepsilon\frac{\partial\boldsymbol{E}}{\partial t}$$

when $\dfrac{\partial\boldsymbol{E}}{\partial t}$ is supposed negligible. For this reason the system (30)-(32) will be called the quasi-steady problem.

Moreover, the system (30)-(32) must be invariant under a gauge trasformation

$$(\psi,\boldsymbol{A},\phi)\leftrightarrow(\psi e^{i\kappa\chi},\boldsymbol{A}+\nabla\chi,\phi-\dot{\chi})$$

where the gauge χ can be any smooth scalar function of (x,t). Various gauges have been considered [**10**], [**11**], [**13**]. In the London gauge, χ is chosen so that $\nabla\cdot\boldsymbol{A}=0$, $\boldsymbol{A}\cdot n|_{\partial\Omega}=\boldsymbol{0}$. In the Lorentz gauge we have

$$\phi=-\nabla\cdot\boldsymbol{A}$$

and in the zero electrical potential gauge $\phi=0$. It is not possible to have both $\phi=0$, and the London gauge simultaneously in $\Omega\times(0,T)$.

We now proceed by relating the phenomenological boundary problem (20),

(21) and (22) to the non stationary model (30)-(32). It is natural to work with the system

$$\gamma \frac{\partial f}{\partial t} = \frac{\hbar^2}{2m^*}\nabla^2 f - \frac{e_*^2}{2m_*}\boldsymbol{p}_s^2 f + \alpha f - \beta f^3 \tag{33}$$

$$\nabla \times \boldsymbol{p}_s = -\mu \boldsymbol{H} \tag{34}$$

$$\nabla \times \boldsymbol{H} = \Lambda^{-1}\boldsymbol{p}_s + \sigma \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{35}$$

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \tag{36}$$

on Ω, and with the boundary conditions

$$\nabla f \cdot \boldsymbol{n}|_{\partial\Omega} = 0\,, \qquad \boldsymbol{H} \times n|_{\partial\Omega} = \boldsymbol{g}\,, \qquad \boldsymbol{p}_s \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \tag{37}$$

and the initial conditions

$$f(x, 0) = f_0(x)\,, \qquad \boldsymbol{p}_s(x, 0) = \boldsymbol{p}_{s0}(x)\,.$$

Actually, equation (33) is the real part of (30), while (34) corresponds to (32). Finally, (35) and (36) are the Maxwell equations.

In (35), the total current density $\boldsymbol{J}$ is given by $\boldsymbol{J} = \boldsymbol{J}_s + \boldsymbol{J}_n$, where $\boldsymbol{J}_s = \Lambda^{-1}\boldsymbol{p}_s$ and $\boldsymbol{J}_n$ obeys the Ohm's law

$$\boldsymbol{J}_n = \sigma \boldsymbol{E}\,. \tag{38}$$

In the two fluid models, it is assumed that the super and normal current densities $\boldsymbol{J}_s$ and $\boldsymbol{J}_n$ are related to the super and normal charge densities ϱ_s and ϱ_n by means of the conservation law

$$\nabla \cdot \boldsymbol{J}_s + \nabla \cdot \boldsymbol{J}_n = -\frac{\partial \varrho_s}{\partial t} - \frac{\partial \varrho_n}{\partial t}\,.$$

Of course, the total charge density ϱ is such that

$$\varrho = \varrho_s + \varrho_n\,.$$

From the equations (34) and (36), we have

$$\frac{\partial \boldsymbol{p}_s}{\partial t} = \boldsymbol{E} + \nabla \phi_s \tag{39}$$

where $\phi_s(x, t)$ is a smooth scalar function. The equation (39) corresponds to the Euler equation for a non-viscous electronic liquid (see [**2**], pag. 59) «where ϕ_s is the thermodynamic potential per electron; a function, in particular, of the concentrations of the superelectrons and of the normal electrons».

In order to obtain the evolution problem (30)-(32), «the pressure» ϕ_s will be related to the charge density $\varrho = \varrho_s + \varrho_n$ by means of the identity (see [8])

$$\phi_s = \Lambda(f)\, \varrho\,. \tag{40}$$

In the quasi-dynamic approximation, equations (33)-(36) are equivalent to the new system

$$\gamma \frac{\partial f}{\partial t} = \frac{\hbar^2}{2m^*} \nabla^2 f - \frac{{e_*}^2}{2m_*} \boldsymbol{p}_s^2 f + \alpha f - \beta f^3 \tag{41}$$

$$\nabla \times \boldsymbol{p}_s = -\, \mu \boldsymbol{H} \tag{42}$$

$$\nabla \times \boldsymbol{H} = \Lambda^{-1} \boldsymbol{p}_s + \sigma \boldsymbol{E} \tag{43}$$

$$\boldsymbol{E} = \frac{\partial \boldsymbol{p}_s}{\partial t} + \nabla \phi_s. \tag{44}$$

Moreover from (43) and (40), we get

$$\nabla \cdot \boldsymbol{J}_s = \nabla \cdot (\Lambda^{-1}(f)\, \boldsymbol{p}_s) = -\, \sigma \nabla \cdot \boldsymbol{E} = -\, \sigma \varrho = -\, \sigma \Lambda^{-1}(f)\, \phi_s. \tag{45}$$

The system (41)-(44) can be written in the form

$$\gamma \frac{\partial f}{\partial t} = \frac{\hbar^2}{2m^*} \nabla^2 f - \frac{{e_*}^2}{2m_*} \boldsymbol{p}_s^2 f + \alpha f - \beta f^3 \tag{46}$$

$$\nabla \times \nabla \times \boldsymbol{p}_s = -\, \mu \Lambda^{-1}(f)\, \boldsymbol{p}_s - \mu \sigma \boldsymbol{E} \tag{47}$$

$$\boldsymbol{E} = \frac{\partial \boldsymbol{p}_s}{\partial t} + \nabla \phi_s \tag{48}$$

This system and the equation

$$\nabla \cdot (\Lambda^{-1}(f)\, \boldsymbol{p}_s) = \sigma \Lambda^{-1}(f)\, \phi_s \tag{49}$$

provide a system completely equivalent to the equations (30)-(32) (See [8], formulae (26)-(28)). Moreover, as we shall prove in the next section, this model is in agreement with the second law of thermodynamics. On the contrary, the general system (33)-(36) under the condition (40) does not satisfy this law.

5 - Evolution model - General case

For the study of the general evolution model, in a neighborhood of the critical temperature T_c, we consider the equations (33)-(36) or equivalently the system

$$\gamma \frac{\partial f}{\partial t} = \frac{\hbar^2}{2m^*} \nabla^2 f - \frac{e_*^2}{2m_*} \boldsymbol{p}_s^2 f + \alpha f - \beta f^3 \tag{50}$$

$$\nabla \times \boldsymbol{p}_s = -\mu \boldsymbol{H} \tag{51}$$

$$\nabla \times \boldsymbol{H} = \Lambda^{-1} \boldsymbol{p}_s + \sigma \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{52}$$

$$\boldsymbol{E} = \frac{\partial \boldsymbol{p}_s}{\partial t} + \nabla \phi_s . \tag{53}$$

Instead of (40), which leads to a system that is not in agreement with thermodynamics, it seems natural to assume the following constitutive relation:

$$\phi_s = \varrho \, . \tag{54}$$

This condition can be represented, by means of the equation (52), as

$$\nabla \cdot \boldsymbol{J}_s = -\sigma \phi_s - \varepsilon \dot{\phi}_s . \tag{55}$$

Now, we are able to prove that the dynamical model described by the equations (50)-(55) satisfies the Second Law of Thermodynamics. Under the hypothesis of processes near the transition temperature T_c, this law reduces to

Dissipation Principle. If Ω denotes the domain occupied by the superconductor body, for every closed cycle, the inequality

$$\oint_0^{d_p}\int_\Omega (\dot{\boldsymbol{B}}\cdot\boldsymbol{H}+\dot{\boldsymbol{D}}\cdot\boldsymbol{E}+\boldsymbol{J}\cdot\boldsymbol{E})\,\mathrm{d}x\,\mathrm{d}t\geqslant 0 \tag{56}$$

holds, where d_p is the time duration of the cycle.

Theorem 1. *Under the hypothesis* (54), *the inequality* (56) *holds for any cycle only if*

$$\gamma\geqslant 0\,,\qquad \sigma\geqslant 0\,. \tag{57}$$

Proof. Consider the integral

$$\Im(\Omega)=\oint_0^{d_p}\int_\Omega (\dot{\boldsymbol{B}}\cdot\boldsymbol{H}+\dot{\boldsymbol{D}}\cdot\boldsymbol{E}+(\boldsymbol{J}_s+\boldsymbol{J}_n)\cdot\boldsymbol{E})\,\mathrm{d}x\,\mathrm{d}t$$

$$=\oint_0^{d_p}\int_\Omega \left\{\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\boldsymbol{E}^2+\mu\boldsymbol{H}^2)+\boldsymbol{J}_s\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t}\,(\Lambda(f)\,\boldsymbol{J}_s)+\nabla\phi_s\right)+\sigma\boldsymbol{E}^2\right\}\mathrm{d}x\,\mathrm{d}t \tag{58}$$

$$=\oint_0^{d_p}\int_\Omega \left\{\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\varepsilon\boldsymbol{E}^2+\mu\boldsymbol{H}^2+\Lambda(f)\,\boldsymbol{J}_s^2)+\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\Lambda(f))\,\boldsymbol{J}_s^2+\nabla\phi_s\cdot\boldsymbol{J}_s+\sigma\boldsymbol{E}^2\right\}\mathrm{d}x\,\mathrm{d}t\,.$$

It follows from $(37)_3$, (33) and (55) that (58) reduces to

$$\Im(\Omega)=\oint_0^{d_p}\int_\Omega\left(-\frac{e_*^{\,2}}{m_*}\,(\Lambda(f)\,\boldsymbol{J}_s)^2 f\dot{f}-\phi_s\nabla\cdot\boldsymbol{J}_s+\sigma\boldsymbol{E}^2\right)\mathrm{d}x\,\mathrm{d}t$$

$$=\oint_0^{d_p}\int_\Omega(2\gamma\dot{f}^2+\sigma\phi_s^2+\varepsilon\,\dot{\phi}_s\phi_s+\sigma\boldsymbol{E}^2)\,\mathrm{d}x\,\mathrm{d}t \tag{59}$$

$$=\oint_0^{d_p}\int_\Omega(2\gamma\dot{f}^2+\sigma\phi_s^2+\sigma\boldsymbol{E}^2)\,\mathrm{d}x\,\mathrm{d}t\geqslant 0$$

from which we have the conditions (57).

Let now consider the hypothesis (40) and its compatibility with the Dissipation Principle.

Theorem 2. *Under the hypothesis* (40), *the quasi-steady problem satisfies the inequality* (56) *if the conditions* (57) *hold. On the contrary, the evolution problem* (50)-(53) *and* (55) *is not in agreement with* (56).

Proof. Under the quasi-steady approximation, the integral (59) takes the form

$$\mathfrak{J}(\Omega) = \oint_0^{d_p}\int_\Omega (\dot{\boldsymbol{B}}\cdot\boldsymbol{H} + \dot{\boldsymbol{D}}\cdot\boldsymbol{E} + (\boldsymbol{J}_s + \boldsymbol{J}_n)\cdot\boldsymbol{E})\,\mathrm{d}x\,\mathrm{d}t \tag{60}$$
$$= \oint_0^{d_p}\int_\Omega (2\gamma\dot{f}^2 + \sigma\Lambda^{-1}(f)\,\phi_s^2 + \sigma\boldsymbol{E}^2)\,\mathrm{d}x\,\mathrm{d}t$$

which is compatible with (56), if the constitutive equation (40) holds.

Consider now the evolution problem represented by the system (41)-(45). In such case, from (58) we have

$$\mathfrak{J}(\Omega) = \oint_0^{d_p}\int_\Omega (2\gamma\dot{f}^2 + \sigma\Lambda^{-1}(f)\,\phi_s^2 + \varepsilon\varrho\varphi_s + \sigma\boldsymbol{E}^2)\,\mathrm{d}x\,\mathrm{d}t\,.$$

This integral is not positive, because of the term

$$\oint_0^{d_p}\int_\Omega \varepsilon\dot{\varrho}\varphi_s\,\mathrm{d}x\,\mathrm{d}t = \oint_0^{d_p}\int_\Omega \varepsilon(\Lambda^{-1}(f)\,\varphi_s)^{\cdot}\,\varphi_s\,\mathrm{d}x\,\mathrm{d}t - \oint_0^{d_p}\int_\Omega \varepsilon(\Lambda^{-1}(f)\,\varphi_s)\,\varphi_s\,\mathrm{d}x\,\mathrm{d}t$$

whose sign is related to the chosen process.

Remark 1. *If we compare the two constitutive equations* (40), (54), *we observe that the latter brings to a full thermodynamic compatibility, while the first is in agreement with thermodynamics only for the quasi-dynamic approximation.*

Thanks for the previous remark, we believe that working with the condition (54) is more suitable also in the quasi-dynamic approximation. With such a choice we have

(61) $$\gamma \frac{\partial f}{\partial t} = \frac{\hbar^2}{2m^*} \nabla^2 f - \frac{{e_*}^2}{2m_*} \boldsymbol{p}_s^2 f + \alpha f - \beta f^3$$

(62) $$\nabla \times \boldsymbol{p}_s = -\mu \boldsymbol{H}$$

(63) $$\nabla \times \boldsymbol{H} = \Lambda^{-1} \boldsymbol{p}_s + \sigma \boldsymbol{E}$$

(64) $$\boldsymbol{E} = \frac{\partial \boldsymbol{p}_s}{\partial t} + \nabla \phi_s .$$

Moreover, from (63) we get

$$\nabla \cdot \boldsymbol{J}_s = \sigma \nabla \cdot \boldsymbol{E} = \sigma \varrho \, .$$

Therefore, in the quasi-steady approximation the system (61)-(64) is now implemented with the equation

$$\nabla \cdot (\Lambda(f)\, \boldsymbol{p}_s) = \sigma \phi_s$$

instead of equation (49).

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Abstract

The aim of this paper is addressed to the study of the Ginzburg-Landau theory that yields the behaviour of a superconductor near to the transition phase. We show that the pertinent equations can be derived starting from the representation of the free energy in terms of the magnetic field $\boldsymbol{H}$, *instead of the vector potential* $\boldsymbol{A}$, *such that* $\nabla \times \boldsymbol{A} = \mu \boldsymbol{H}$, *and the modulus* $|\psi|$, *which denotes the concentration of superconducting electrons, instead of the complex parameter* ψ. *Such a representation gives also a conceptual simplification of the model since it makes use of observable quantities. In addition, the free energy in terms of* $\boldsymbol{H}$ *and* $|\psi|$ *makes the theory gauge-invariant in that is free from the vector potential* $\boldsymbol{A}$ *and the phase of* ψ. *The compatibility with thermodynamics is examined and it follows that the generality of the second law is related to the specific approximation of the model.*

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