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Up and down along rays (**)

1 - Introduction and definitions

Among the interesting problems arising in Banach spaces, the following two attracted much interest during the last two-three decades; the first one is related to extension of maps in some optimal way, while the second one is important in approximation theory.

Problem A). Suppose a contractive map, or a lipschitz map, is defined in the unit ball (or on a convex body); is it possible to extend it to the whole space preserving contractiveness, or the lipschitz constant? Otherwise, what it the smallest lipschitz constant such an extension can have?

Problem B). Is a Chebyshev set in a Hilbert space necessarily convex? Recall that a Chebyshev set is a set admitting a single best approximation from any point in the space.

Both Problem A and Problem B are related to properties of «retractions» of the space onto the unit ball, or onto the unit sphere. In particular, concerning Problem A, satisfactory results have been given, mainly by considering retractions onto the unit ball along rays (from outside); Problem B remains unsolved, but «inversion» with respect to the unit sphere gives some insight and allows different fomulations of the problem.

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In this paper we want to study the kind of retractions used in the above context: we study in details their properties, we consider the connections between the radial projection and the angular distance and we obtain some new characterizations of inner product spaces. In fact, unfortunately, «best» properties for the radial projection, seem more or less characterize inner product spaces (see also [**A**]). In any case, we shall also try to indicate the best inequalities which can be given in general.

Here we list the notations and the basic facts that shall be used in the paper.

Let $(X, \|.\|)$ be a normed space over the real field R; by S (or, when necessary, by S_X) we shall denote its unit sphere.

We shall consider different ways of moving points along rays from the origin; i.e., transformations T such that for $x \neq \theta$, Tx belongs to the «ray»:

$$R(x) = \{\lambda x;\, \lambda > 0\}.$$

We recall some definitions.

We say that X is an *inner product space*, i.p.s. for short, if the norm is generated by an inner product, denoted by $(.,.)$.

Given x, y in X, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in R$ we shall write

$$x \perp y \qquad (x \text{ is } \textit{orthogonal} \text{ to } y \text{ according to Birkhoff and James}).$$

We say that orthogonality is *symmetric* if $x \perp y$ implies $y \perp x$. Recall the following fact (see e.g. [**A**], § 4):

Proposition 1. *Let X be a normed space with dimension at least* 3; *then orthogonality is symmetric if and only if X is an inner product space.*

The *radial projection* RP is defined in the following way:

$$\mathrm{RP}(x) = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \dfrac{x}{\|x\|} & \text{if } \|x\| > 1\,. \end{cases}$$

Also, set (see e.g. [**T**]):

$$|\mathrm{RP}| = \sup\left\{\frac{\|\mathrm{RP}(x) - \mathrm{RP}(y)\|}{\|x - y\|};\, x, y \in X;\, x \neq y\right\}.$$

Sometimes, to be more precise, we shall also write RP_X, and $|\mathrm{RP}_X|$, to denote the radial projection for a given space X, and its «lipschitz norm».

We recall another definition (see e.g. the end of § 3 in [**A**]); let, for x, $y \neq \theta$:

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

This number is called the *angular distance*, or also the *Clarkson angle* between x and y.

A space is said to be *uniformly non square* if there exists a constant $\varepsilon > 0$ such that for $\|x\| = \|y\| = 1$ we have $\min\{\|x-y\|, \|x+y\|\} < 2 - \varepsilon$.

In this paper we consider radial motions of points in real normed spaces. We study properties of X which depend on the values of $|\mathrm{RP}|$ (for example: X is an i.p.s.); we indicate the connections with the angular distance and we relate several existing results, trying to avoid the use of duality mappings, projection constants, and —as far as possible— of orthogonality.

The angular distance is related to the radial projections onto the unit sphere, both from inside and from outside. We consider other displacements along radial directions, like those sending pairs of points to the same distance from the origin. Also, we try to indicate the best constants which control these radial movements.

All examples are collected in the last but one Section 5.

Finally, in the last Section 6, we indicate some facts concerning «sunny» retractions onto convex bodies and inversions.

2 - The radial projection

It is not difficult to see that in any space we have:

(1) $$1 \leqslant |\mathrm{RP}| \leqslant 2\,.$$

The following facts are well known (see [**T**]); note that they have been rediscovered from time to time, also recently: see e.g. [**H**] and [**H-p**].

Proposition 2. *For a space X the following properties are equivalent*:

(a) $|\mathrm{RP}| = 1$ (*the radial projection is nonexpansive*);
(b) *orthogonality is symmetric.*

Also, $|\mathrm{RP}| < 2$ *if and only if the space is uniformly non square.*
Moreover, we have:

(c) $$|\mathrm{RP}| = \sup\left\{\frac{\left\| x - \dfrac{y}{\|y\|}\right\|}{\|x-y\|}\,;\, x,\, y \in X;\, \|x\| \leqslant 1 < \|y\|\right\};$$

(c′) $$|\mathrm{RP}| = \sup\left\{\frac{\|x-y\|}{\|x-\lambda y\|}\,;\, x,\, y\in X;\, \|x\|\leqslant 1 = \|y\|,\, \lambda > 1\right\}.$$

The equivalence between (c) and (c′) is clear. More precisely: the proof of (c) given in [**T**] also shows that

(c″) $$\sup\left\{\frac{\|\mathrm{RP}(x) - \mathrm{RP}(y)\|}{\|x-y\|}\,;\, x,\, y\in X;\, 1\leqslant \|x\| < \|y\|\right\} = \sup\left\{\frac{\|x - y/\|y\|\|}{\|x-y\|}\,;\, x,\, y\in X;\, \|x\| = 1 < \|y\|\right\}.$$

A slightly stronger statement is the following, that we shall prove directly.

Proposition 2′. *In any space X, we have*:

(d) $$|\mathrm{RP}| = \sup\left\{\frac{\left\| x - \dfrac{y}{\|y\|}\right\|}{\|x-y\|}\,;\, \|x\| = 1 \leqslant \|y\|\right\} = \sup\left\{\frac{\alpha[x,y]}{\|x-y\|}\,;\, \|x\|\geqslant 1;\, \|y\|\geqslant 1\right\}.$$

Proof. We prove the first equality (which clearly implies the second one). It is clear by the definition that $|\mathrm{RP}|$ is not smaller than the second member of (d); conversely, we have to prove that given $\delta > 0$, we can find a pair x, y with $\|x\| = 1 \leqslant \|y\|$ such that $\dfrac{\|x - y/\|y\|\|}{\|x-y\|} > |\mathrm{RP}| - \delta$. Also, we can assume $|\mathrm{RP}| > 1$ (otherwise the result is trivial); given $\delta \in (0, |\mathrm{RP}| - 1)$, according to (c) we can find x, $y \in X$; $\|x\| \leqslant 1 < \|y\|$, such that

$$\frac{\|x - y/\|y\|\|}{\|x-y\|} > |\mathrm{RP}| - \delta\,.$$

Let $x' = x/\|x\|$; $y' = y/\|x\|\cdot\|y\|$, so that $\|x'\| = 1$; $\|y'\| = 1/\|x\|$; $\mathrm{RP}(y') = y/\|y\|$:

we have

$$(*) \qquad \frac{\|x' - y'\|}{\|x' - y/\|x\|\|} > |\mathrm{RP}| - \delta > 1 .$$

Now consider the convex function of $t : f(t) = \|x' - ty\|$; we have (by $(*)$) $f(1/\|x\| \cdot \|y\|) = \|x' - y'\| > \|x' - y/\|x\|\| = f(1/\|x\|)$; since $\frac{1}{\|y\|} \leqslant \frac{1}{\|x\| \cdot \|y\|} < \frac{1}{\|x\|}$, this implies $f\left(\frac{1}{\|y\|}\right) \geqslant f\left(\frac{1}{\|x\| \cdot \|y\|}\right)$, thus $\left\| x' - \frac{y}{\|y\|} \right\| \geqslant \|x' - y'\|$, so

$$\frac{\left\|\mathrm{RP}(x') - \mathrm{RP}\left(\frac{y}{\|x\|}\right)\right\|}{\left\|x' - \frac{y}{\|x\|}\right\|} = \frac{\left\|x' - \frac{y}{\|y\|}\right\|}{\left\|x' - \frac{y}{\|x\|}\right\|} \geqslant \frac{\|x' - y'\|}{\left\|x' - \frac{y}{\|x\|}\right\|} > |\mathrm{RP}| - \delta .$$

Since $\left\| \frac{y}{\|x\|} \right\| > 1 = \|x'\|$, this proves the proposition. ■

Clearly (d) is equivalent to

$$(\mathrm{d}') \qquad |\mathrm{RP}| = \sup\left\{ \frac{\|x - y\|}{\|x - \lambda y\|} ; \|x\| = \|y\| = 1 ; \lambda \geqslant 1 \right\} .$$

We recall the following result.

Proposition 3 (see [**KX**], Proposition 1, or [$\mathbf{D}_1$], Proposition 8 for X strictly convex). *For a normed space X, the condition* $|\mathrm{RP}| = 1$ *is equivalent to*:

(e) *if* $\|x\| \leqslant \|w\|$, *then* $\|x - w\| \leqslant \|x - \lambda w\|$ *for all* $\lambda > 1$.

Now, it is clear that (e) is equivalent to:

(e′) there exists $r > 0$, such that if $\|x\| = r$ and $\|w\| > r$, then $\|x - w\| \leqslant \|x - \lambda w\|$ for all $\lambda > 1$.

Also (take a sequence $w_n = y(1 + 1/n)$), it implies, so it is equivalent to:

(e″) if $\|x\| = r$ and $\|y\| \geqslant r$, then $\|x - y\| \leqslant \|x - \lambda y\|$ for all $\lambda > 1$ (for some given r, or equivalently, for any $r > 0$).

But the convexity in t of the function:

$$\varphi(t) = \|x - ty\| ,$$

for x, y given, also implies the equivalence of the previous conditions to

(e‴) if $\|x\| = \|y\| = r$, then $\|x - y\| \leqslant \|x - \lambda y\|$ for all $\lambda > 1$.

Already in [**DK**], Proposition 1, the equivalence between $|\,\mathrm{RP}\,| = 1$ and (e‴) had been indicated, in the following form: if $\|x\| = \|y\|$, then $\min\limits_t \varphi(t)$ is attained for $|t| \leqslant 1$.

Note that, according to (d′), $|\,\mathrm{RP}\,| = 1$ implies (e‴) (with $r = 1$), so Proposition 2′ implies Proposition 3.

Also, the condition $|\,\mathrm{RP}\,| = 1$ is clearly equivalent to the following fact:

(f) given x, y with $\|x\| = \|y\|$ (equivalently: for x, y in S), we have $\|\lambda x - \mu y\| \geqslant \|x - y\|$ for $\lambda, \mu \geqslant 1$.

In fact, (f) clearly implies (e″); conversely, if $\|x\| = \|y\| = 1$ and for example $\lambda \geqslant \mu \geqslant 1$, we obtain from (e″):

$$\left\| \frac{\lambda}{\mu} x - y \right\| \geqslant \|x - y\| \geqslant \left\| \frac{x - y}{\mu} \right\|, \qquad \text{which implies (f)}.$$

3 - Moving up and down; the angular distance

In this section we indicate what happens if we move (radially) points to the unit sphere, not only from outside, but also from inside.

It is simple to see that in general we have (see [**KS**]):

$$\alpha[x, y] \leqslant \frac{4\|x - y\|}{\|x\| + \|y\|} \quad \text{for every } x, y \neq \theta\,; \tag{2}$$

also, we have strict inequality unless $x = y$, and the constant 4 cannot be improved (see Example 1 in Section 5).

Moreover, if X is an i.p.s., then

$$\alpha[x, y] \leqslant \frac{2\|x - y\|}{\|x\| + \|y\|} \quad \text{for every } x, y \neq \theta\,. \tag{2′}$$

The right hand side of (2) (or (2′)) is homogeneous in x, y, thus it is enough to state those inequalities e.g. for pairs x, y with norm $\geqslant 1$.

Note (see [**KS**]) that equality in (2′) is equivalent to: either $\|x\| = \|y\|$, or $y\|x\| + x\|y\| = 0$.

We also have:

Proposition 4 (see [**A**], p. 83). *A space X is an i.p.s. if (and only if)*:

$$(2'') \qquad \alpha[x, y] \leqslant \frac{2\|x-y\|}{\|x\|+\|y\|} \quad \textit{for every } x, y \textit{ in } X \textit{ with } x \perp y .$$

Other estimates concerning $\alpha[x, y]$ are known (see e.g. [**KS**]); in particular, the following inequality, stronger than (2), is always true (see [**A**], p. 32):

$$(3) \qquad \alpha[x, y] \leqslant \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}} .$$

Note that the above inequality is «sharp», in any space (see Example 2 in Section 5); in other terms we always have:

$$(3') \qquad \begin{aligned} s_+(X) &= \sup\left\{\frac{\alpha[x, y]\cdot\max\{\|x\|, \|y\|\}}{\|x-y\|}\,;\, x, y \in S - \{\theta\};\, x \neq y\right\} \\ &= \sup\left\{\frac{\alpha[x, y]\cdot\max\{\|x\|, \|y\|\}}{\|x-y\|}\,;\, 1 = \|x\| \leqslant \|y\|\right\} = 2 . \end{aligned}$$

Note that this constant was considered in [**A-R**] (where it was denoted by $\Psi_\infty(X)$); it was not realized there that it cannot be smaller than 2: for this reason, some results in [**A-R**] are meaningless.

Now take in S a pair $x \perp y$, then consider x/n, y/n; it is clear from this that we always have:

$$\sup\left\{\frac{\alpha[x,y]}{\|x-y\|}\,;\, x,y\in X-\{\theta\};\, x\neq y\right\} = \sup\left\{\frac{\alpha[x,y]}{\|x-y\|}\,;\, x,y\in X-\{\theta\};\, x\perp y\right\} = +\infty .$$

Now set:

$$(4) \qquad \begin{aligned} \sigma_+(X) &= \sup\left\{\frac{\alpha[x, y]\cdot(\|x\|+\|y\|)}{2\|x-y\|}\,;\, x, y \neq \theta;\, x \neq y\right\} \\ &= \sup\left\{\frac{\alpha[x, y]\cdot(\|x\|+\|y\|)}{2\|x-y\|}\,;\, 1 = \|x\| \leqslant \|y\|\right\} \\ &= \sup\left\{\frac{\alpha[x, y]\cdot(\|x\|+\|y\|)}{2\|x-y\|}\,;\, 0 < \|x\| \leqslant \|y\| = 1\right\}; \end{aligned}$$

the last equalities easily follow from homogeneity.

Clearly (take $\|x\| = \|y\|$) $\sigma_+(X) \geqslant 1$ always, so we have (see (3')):

$$(4') \qquad 1 \leqslant \sigma_+(X) \leqslant s_+(X) = 2 .$$

The inequality $|\mathrm{RP}| \leqslant 2$ is usually indicated as a consequence of (2) (see [**T**]). But the implication (2) (or (3)) $\Rightarrow |\mathrm{RP}| \leqslant 2$ is not completely trivial, since also in inner product spaces, for $\|x\| \leqslant 1 \leqslant \|y\|$, we can have $\|x - y/\|y\|\| > \alpha[x, y]$ (see Example 3; see also Remark 3 in Section 4); so the above implication relies on Proposition 2': in fact, according to (d), we obtain by (2):

$$(4'') \qquad |\mathrm{RP}| = \sup\left\{ \frac{\alpha[x, y]}{\|x - y\|} \; ; \; \|x\| = 1 \leqslant \|y\| \right\} \leqslant \sigma_+(X) \leqslant 2 \, .$$

Similarly, the implication $(2') \Rightarrow |\mathrm{RP}| = 1$ is obvious.

The number $\alpha[x, y]$ was considered by J. J. Schäffer, who proved the following (see [**A**], p. 145): in any space, we have (use independent pairs x, y in S):

$$\frac{1}{2} \leqslant \liminf_{\alpha[x, y] \to 0^+} \frac{\|x - y\|}{\alpha[x, y] \cdot \max\{\|x\|, \|y\|\}} \leqslant 1 \; ;$$

moreover, the above limit is equal to 1 if, and when $\dim(X) \geqslant 3$ only if, X is an i.p.s..

We shall also consider the inverse of the above ratio, and set, according to [$\mathbf{D_2}$]:

$$(5) \qquad \begin{aligned} s(X) &= 1 \Big/ \liminf_{\alpha[x, y] \to 0^+} \frac{\|x - y\|}{\alpha[x, y] \cdot \max\{\|x\|, \|y\|\}} \\ &= \limsup_{\alpha[x, y] \to 0^+} \frac{\alpha[x, y] \cdot \max\{\|x\|, \|y\|\}}{\|x - y\|} \in [1, 2] \, . \end{aligned}$$

Remark 1. The analogues of the last two equalities in (4), hold for $s(X)$ too; i.e., for example:

$$(5') \qquad s(X) = \limsup_{\alpha[x, y] \to 0^+} \left\{ \frac{\alpha[x, y] \cdot \|y\|}{\|x - y\|} \; ; \; 1 = \|x\| \leqslant \|y\| \right\} .$$

Also, it is not difficult to see that in the definition of $s(X)$, $\limsup_{\alpha[x, y] \to 0^+}$ can be replaced by $\limsup_{\|x - y\| \to 0}$. Similar remarks apply to $\sigma(X)$ (see (7) below).

Note that $s(X) \leqslant 2$ follows from $s(X) \leqslant s_+(X) = 2$.

In fact, more was proved in [$\mathbf{D_2}$]; i.e.:

Proposition 5 (see [$\mathbf{D_2}$], Lemma 6 + Proposition 4, and Lemma 7). *We al-*

ways have:

(6) $$|RP_X| \leqslant s(X);$$

also, if $\dim(X) < \infty$, *the converse inequality, therefore equality, is true.*

In fact, it easily follows from the above results that

(6′) $$|\mathrm{RP}_X| \leqslant s(X) \quad \text{for any } X,$$

since

$$\begin{aligned} s(X) &= \sup\{s(Y);\ Y \text{ is a two-dimensional subspace of } X\} \\ &= \sup\{|\mathrm{RP}_Y|;\ Y \text{ is a two-dimensional subspace of } X\} = |\mathrm{RP}_X|. \end{aligned}$$

For the sake of completeness, we give a direct, simple proof of the «converse inequality».

Proposition 5′. *We always have*:

(6″) $$s(X) \leqslant |\mathrm{RP}_X|.$$

Proof. Given $\varepsilon \in (0, 1)$, let $\delta = \dfrac{\varepsilon}{3}$ then take $\alpha[x, y] < \delta$.

Given x, y with $1 = \|x\| \leqslant \|y\|$, we obtain:

if $\|x - y\| \leqslant \dfrac{\varepsilon}{2}$, then $\|y\| \leqslant 1 + \dfrac{\varepsilon}{2}$ and so $\dfrac{\alpha[x, y]\cdot\|y\|}{\|x-y\|} \leqslant |\mathrm{RP}|\left(1 + \dfrac{\varepsilon}{2}\right) \leqslant |\mathrm{RP}| + \varepsilon$.

If $\|x - y\| > \dfrac{\varepsilon}{2}$, for $\|y\| \leqslant 3/2$ we have $\dfrac{\alpha[x, y]\cdot\|y\|}{\|x-y\|} \leqslant \dfrac{\delta\cdot(3/2)}{\varepsilon/2} = 1$;

for $\|y\| > 3/2$, we have $\dfrac{\alpha[x, y]\cdot\|y\|}{\|x-y\|} \leqslant \dfrac{\delta\cdot\|y\|}{\|y\| - 1} < \dfrac{\delta\cdot(3/2)}{\frac{3}{2} - 1} = 3\delta < 1$.

Therefore, in any case, $\dfrac{\alpha[x, y]\cdot\|y\|}{\|x-y\|} \leqslant |\mathrm{RP}| + \varepsilon$ for $\alpha[x, y] < \delta$; $1 = \|x\| \leqslant \|y\|$, and this proves (6″). ■

As a consequence of (6′), we have the following result:

Proposition 6 (see [**A**], p. 145). *If* $\dim(X) \geqslant 3$, *then* X *is an i.p.s. if and only if* $s(X) = 1$.

In other terms, inner product spaces of dimension $\geqslant 3$ are characterized by the following property:

given $\delta > 0$, there exists $\varepsilon > 0$ such that $\alpha[x, y] \leqslant (1+\delta)\dfrac{\|x-y\|}{\max\{\|x\|, \|y\|\}}$ for $\alpha[x, y] \leqslant \varepsilon$.

Now set

$$\sigma(X) = \limsup_{\alpha[x, y]\to 0^+} \frac{\alpha[x, y]\cdot(\|x\|+\|y\|)}{2\|x-y\|}. \tag{7}$$

Clearly we have:

$$1 \leqslant \sigma(X) \leqslant s(X). \tag{8}$$

To see that the left inequality is true, it is enough to take $x, y \in S$, $x \neq y$.

But a careful reading of the proof of Lemma 6 in [**D**$_2$] shows that $|\mathrm{RP}_X| \leqslant \sigma(X)$ always, therefore, according to (8) and (6′):

$$|\mathrm{RP}_X| = \sigma(X) = s(X) \quad \text{in any space } X. \tag{8′}$$

We know (see Proposition 4) that $\sigma_+(X) = 1 \Leftrightarrow X$ is an inner product space, so (see (3′)) $s_+(X) = 1 \Leftrightarrow X$ is an inner product space. Therefore, if we take a 2-dimensional space X where orthogonality is symmetric, but which is not an i.p.s., then we have (see (8′)):

$$|\mathrm{RP}_X| = 1 = s(X) = \sigma(X) < \sigma_+(X).$$

We also have:

Proposition 7. *The condition $\sigma_+(X) < 2$ is equivalent to: X is uniformly non square.*

Proof. If X is not uniformly non square, then we have (see Proposition 2 and (4)): $2 = |\mathrm{RP}_X| \leqslant \sigma_+(X) \leqslant 2$, so $\sigma_+(X) = 2$.

Now let $\sigma_+(X) = 2$; given any $\varepsilon > 0$ we can find x, y, say with $1 = \|x\| \leqslant \|y\|$, such that (use also (3)):

$$2 - \frac{\varepsilon}{2} < \frac{\alpha[x, y]\cdot(\|x\|+\|y\|)}{2\|x-y\|} \leqslant \frac{2\|x-y\|}{\max\{\|x\|, \|y\|\}}\cdot\frac{\|x\|+\|y\|}{2\|x-y\|} = 1 + \frac{1}{\|y\|};$$

this implies $\|y\| < \dfrac{2}{2-\varepsilon}$, so we obtain

$$\frac{\mathrm{RP}(x) - \mathrm{RP}(y)}{\|x-y\|} = \frac{\alpha[x,y]}{\|x-y\|} \geq \frac{4-\varepsilon}{1+\|y\|} > \frac{4-\varepsilon}{1+\dfrac{2}{2-\varepsilon}} = 2-\varepsilon\,;$$

since $\varepsilon > 0$ is arbitrary, this shows that $|\mathrm{RP}| = 2$. ∎

We indicate some other estimates.

Proposition 8 (see (3) in [**KS**]). *In any space, we always have*

$$\alpha[x,y] \leq \frac{\|x-y\| + \big|\|x\| - \|y\|\big|}{\max\{\|x\|, \|y\|\}} \qquad \textit{for every } x, y \neq \theta\,. \tag{9}$$

Proof. Let $\max\{\|x\|, \|y\|\} = \|x\|$; we have:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} + \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\|$$

$$\leq \frac{\|x-y\|}{\|x\|} + \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\| = \frac{\|x-y\|}{\|x\|} + \|y\| \cdot \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| = \frac{\|x-y\| + \|x\| - \|y\|}{\|x\|},$$

which implies the thesis. ∎

Remark 2. It easy to see that inequality (9) implies the following one: if $\|x\| \leq 1$, $\|y\| \leq 1$, then $\alpha[x,y] \leq \|x-y\| + 2 - (\|x\| + \|y\|)$.

Also this estimate is —in a sense— sharp: see e.g. Example 1 in Section 5.

Corollary. *In any space, we always have*

$$\alpha[x,y] \leq \frac{4 \cdot \|x-y\|}{\|x\| + \|y\| + \|x-y\|} \qquad \textit{for every } x, y \neq \theta\,. \tag{9'}$$

Proof. It is enough to prove that for every x and $y \neq \theta$ we have:

$$\frac{\|x-y\| + \big|\|x\| - \|y\|\big|}{\max\{\|x\|, \|y\|\}} \leq \frac{4 \cdot \|x-y\|}{\|x\| + \|y\| + \|x-y\|}\,.$$

Note that both members of the previous inequality are homogeneous (in the sense that for any $\lambda > 0$, if they are true for a pair x, y, then they are also true for λx, λy); so it is enough to prove it for $0 < \|y\| \leqslant \|x\| = 1$. With these assumptions, it becomes:

$$\|x - y\| + 1 - \|y\| \leqslant \frac{4 \cdot \|x - y\|}{\|x - y\| + 1 + \|y\|},$$

or

$$(\|x - y\| + 1)^2 - \|y\|^2 \leqslant 4\|x - y\|;$$

by setting $\|x - y\| = a$, this can be written as

$$a^2 - 2a + 1 - \|y\|^2 \leqslant 0.$$

Since we have equality here for $a = 1 \pm \|y\|$, and our assumptions imply $1 - \|y\| \leqslant \|x - y\| = a \leqslant 1 + \|y\|$, the last inequality is true, and this proves the corollary. ■

Among the different estimations we have given for $\alpha[x, y]$, we have the following implications:

(9) ⇒ (9′) ⇒ (3) ⇒ (2).

As we said, in general, the constant 4 in (2) cannot be lowered: this implies that in (3) the constant 2 cannot be lowered in general; more precisely, we observed that in (3), in any space, we cannot change 2 with a smaller constant. But something more than 1 in (3) is enough in inner product spaces, if $\|x - y\|$ is small with respect to $\|x\|$ and $\|y\|$ (Proposition 6).

Note that equality in (3) does not hold if $\|x\| = \|y\|$, $x \neq y$ (see [**G**]: in [**A**], p. 32, there is a «misprint» concerning the last statement, probably due to a wrong translation of [**G**]). More precisely, it was shown in [**G**] that both in (2) and in (3), given any $\varepsilon \neq 0$, there exist pairs x, y such that $\|y\| = (1 + \varepsilon) \cdot \|x\|$, which realize equality in (3) and «almost» equality in (2).

For the previous reason, we cannot hope to decrease the constant 4 in (9′), or to put a constant smaller than 1 in the right hand side of (9), in any space: whichever the shape of the unit sphere in span $\{x, y\}$ is, (9′) becomes an equality if we take $\|x\| = \|y\| = 1$ and $x = -y$. (9) is an equality whenever $\|x\| = \|y\|$ (in these cases we have inequality in (3)).

Equality in (9) for a pair x, y with $0 < \|y\| \leq \|x\| = 1$ implies (see the proof of (9)):

$$(\P) \qquad \left\| x - \frac{y}{\|y\|} \right\| = \|x - y\| + 1 - \|y\| = \|x - y\| + \left\| y - \frac{y}{\|y\|} \right\|;$$

if x and y are independent, this implies that X is not strictly convex.

Now assume that we have equality in (9′) for a pair x, y, $x \neq \pm y$, with $0 < \|y\| \leq \|x\| = 1$; this implies equality in (9), so (¶), and moreover (see the proof that (9) implies (9′)): either $1 - \|y\| = \|x - y\|$ which implies $\left\| x - \frac{y}{\|y\|} \right\| = 2\|x - y\|$, so equality in (3), therefore $\|x\| \neq \|y\| \neq 1$; or $1 + \|y\| = \|x - y\|$, so $\left\| x - \frac{y}{\|y\|} \right\| = 2$ (see also Example 4 in Section 5).

4 - Averaging distances: escaping from the center of the unit ball

We consider now displacements both from inside and from outside the unit sphere. Revisit now inequality (2); since both members are homogeneous in x, y, it is equivalent to:

$$(10) \qquad \alpha[x, y] \leq 2\|x - y\| \quad \text{whenever } \|x\| + \|y\| = 2\,.$$

In fact, switch any pair x, y to $x' = \frac{2x}{\|x\| + \|y\|}$, $y' = \frac{2y}{\|x\| + \|y\|}$ where $\|x'\| + \|y'\| = 2$; we have

$$\frac{\alpha[x', y']}{\|x' - y'\|} = \frac{\alpha[x, y]}{\|x - y\|}\,.$$

Or also, we can consider displacements along rays sending any pair of different points x, y with $\|x\| = d_1$, $\|y\| = d_2$ to points at the same distance from the origin, equal to their «average» distance from it:

$$(\text{A}) \qquad A_{x,y}(x) = \frac{d_1 + d_2}{2d_1}\, x, \qquad A_{x,y}(y) = \frac{d_1 + d_2}{2d_2}\, y \quad \left(\|A_{x,y}(x)\| = \|A_{x,y}(y)\| = \frac{d_1 + d_2}{2} \right).$$

We have (see (2)):

$$(10') \qquad \|A_{x,y}(x) - A_{x,y}(y)\| = \alpha[x, y]\, \frac{d_1 + d_2}{2} \leq 2\|x - y\| \quad \text{for every pair } x, y\,.$$

Again by homogeneity, (10′) is equivalent to (10).

Accordingly, Proposition 4, characterizing inner product spaces (X of arbitrary dimension), can be formulated by using only pairs x, y with $\|x\| + \|y\| = 2$ (pairs x, $y \in S$ are not enough!) and saying that $\alpha[x, y] \leq \|x - y\|$ for them, or that the above transformations $A_{x,y}$ are nonexpansive for every pair x, y (compare with Proposition 2).

Similarly (see Proposition 7), not uniformly non square spaces are characterized by the existence of pairs x, y such that $\|A_{x,y}(x) - A_{x,y}(y)\|$ is approximately equal to $2\|x - y\|$ (see Example 1 in Section **5**).

Remark 3. Let X be an i.p.s.; then the condition $|\,\mathrm{RP}\,| = 1$ says that in this case, if we project radially from outside two points in the unit ball, we act nonexpansively.

This is no more true if we project radially onto the unit sphere two points x, y such that $\min\{\|x\|, \|y\|\} < 1$, or only the point of smaller norm, unless $\|x\| + \|y\| = 2$: (2′) indicates what constant control the distances (see also Example 4). In these cases, we cannot say if we increase or decrease the distance; for example, in many cases $\alpha[x, y]$ is larger than $\|x - y\|$, but it can be smaller (see Example 3 in Section **5**).

Consider points x, y inside the unit ball; in inner product spaces we have:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq \|x - y\| \Leftrightarrow \left(\frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|} \right) \geq (x - y, x - y)$$

$$\Leftrightarrow \frac{2(x, y)}{\|x\| \cdot \|y\|} \leq 2 - \|x - y\|^2 \Leftrightarrow \frac{2(x, y)}{\|x\| \cdot \|y\|} (1 - \|x\| \cdot \|y\|) \leq 2 - \|x\|^2 - \|y\|^2$$

$$\Leftrightarrow \cos(x, y) \leq \frac{2 - \|x\|^2 - \|y\|^2}{2(1 - \|x\| \cdot \|y\|)};$$

in particular, it is easy to see that this is true when $\max\{\|x\|, \|y\|\} \leq 1$ and

$$\cos(x, y) = \frac{(x, y)}{\|x\| \cdot \|y\|} \leq \frac{1}{2},$$

which is also clear from its geometrical meaning.

In general the relations between $\alpha[x, y]$ and $\|x - y\|$ are controlled by the inequalities (2) and (3), and the above remark applies. But something can be said also in general when $\|x - y\|$ is not small with respect to $\|x\|$, $\|y\|$; some simple results of this type appear in the literature: we recall a couple among them.

(i) (see [**YZW**], Lemma 1). Let $x,\ y \in X$; $\lambda,\ \mu \in R$; $\|x-y\| \geqslant \max\{\|x\|,\ \|y\|\}$; then we have

$$\|\lambda x - \mu y\| \geqslant \|x-y\| \cdot \min\{\lambda,\ \mu\}.$$

(ii) (see [**MP**], Lemma 3.1). Let $x \in X$, $\|y\| \leqslant \mu$ $(\mu \neq 0,\ y \neq \theta)$ and $\|x\| \leqslant \|x-y\|$. Then we have

$$\|x-y\| \leqslant \|x - \mu y/\|y\|\|.$$

They easily follow from the convexity of the function

$$\varphi(t) = \|x - ty\|.$$

In fact, $\|x-y\| \geqslant \|x\|$ implies $\|x-ty\| \geqslant \|x-y\|$ for $t \geqslant 1$. More precisely, we have:

Proposition 9. *Let* $0 \leqslant \|x\| \leqslant \|y\|$; *then*

$$\alpha[x,\ y] \geqslant \frac{\|y-x\|}{\|x\|} + 1 - \frac{\|y\|}{\|x\|} \tag{11}$$

Proof. Let $\alpha = \varphi\left(\dfrac{\|y\|}{\|x\|}\right)$; by convexity, we have:

$$\frac{\alpha - \varphi(0)}{\|y\|/\|x\|} \geqslant \varphi(1) - \varphi(0) \Leftrightarrow \alpha - \varphi(0) \geqslant \frac{\|y\|}{\|x\|}(\|x-y\| - \|x\|)$$

$$\Leftrightarrow \alpha[x,\ y] = \frac{\alpha}{\|y\|} \geqslant 1 + \frac{\|y-x\|}{\|x\|} - \frac{\|y\|}{\|x\|},$$

which is (11). ■

Remark 4. If $\|x-y\| \geqslant \|y\| = \max\{\|x\|,\ \|y\|\}(x,\ y \neq \theta)$, then easy computations show that (11) implies (compare with (3)):

$$\alpha[x,\ y] \geqslant \frac{\|x-y\|}{\max\{\|x\|,\ \|y\|\}}. \tag{11$'$}$$

Note that (11′) follows also from (i), by setting $\lambda = \dfrac{1}{\|x\|}$; $\mu = \dfrac{1}{\|y\|}$.

5 - A few simple examples

The first two examples show that inequalities (2) and (3) are «sharp».

Example 1 (inequality (2) is sharp). Let $X = R^2$ with the sum norm (this space is not uniformly non square, cf. Proposition 7). Let $\varepsilon > 0$, small; $x = (1-\varepsilon, 0)$ $\left(\text{so } \|x\| = 1-\varepsilon;\ \dfrac{x}{\|x\|} = (1,0)\right)$; $y = (1-\varepsilon, \varepsilon)$ (so $\|y\| = 1$). We have $\|x-y\| = \varepsilon$; $\|x\| + \|y\| = 2-\varepsilon$; $\alpha[x, y] = 2\varepsilon = \dfrac{\|x-y\|}{\|x\|+\|y\|} \cdot 2(2+\varepsilon)$. With relation to Remark 2 in Section 3, note that $\|x-y\| + 2 - (\|x\| + \|y\|) = 2\varepsilon = \alpha[x, y]$.
Also:

$$A_{x,y}(x) = \left(\frac{2-\varepsilon}{2}, 0\right); \qquad A_{x,y}(y) = \left(\frac{(2-\varepsilon)(1-\varepsilon)}{2}, \frac{(2-\varepsilon)\,\varepsilon}{2}\right);$$

$$A_{x,y}(x) - A_{x,y}(y) = \left(\frac{\varepsilon(2-\varepsilon)}{2}, \frac{\varepsilon(2-\varepsilon)}{2}\right); \qquad \frac{\|A_{x,y}(x) - A_{x,y}(y)\|}{\|x-y\|} = \frac{\varepsilon(2-\varepsilon)}{\varepsilon}.$$

Example 2 (inequality (3) is sharp). Take, in any space, x such that $\|x\| = 1$ and $y = -nx$. We have:

$$\alpha[x, y] = 2; \qquad \|x-y\| = n+1; \qquad \frac{\alpha[x, y] \cdot \max\{\|x\|, \|y\|\}}{\|x-y\|} = \frac{2n}{n+1}.$$

Example 3. In any space we can take $x \in S$; y near the origin and so that $y/\|y\|$ is near to x: so

$$(\ddagger) \qquad \alpha[x, y] < \|x-y\| \qquad (0 < \|y\| < \|x\| = 1).$$

Of course, the same can happen also for a pair such that $0 < \|y\| < 1 < \|x\|$.

But we can give examples showing that we can have ($\ddagger$) also for points x, y which are «near». For example, take in the plane $x = (1, 0)$; $y = (1-\varepsilon, \varepsilon^2)$. If we endow the plane with the euclidean norm, then by setting $\delta = \sqrt{(1-\varepsilon)^2 + \varepsilon^4} (\cong 1 - \varepsilon)$ we have:

$$\alpha[x, y] = \left\| \left(1 - \frac{1-\varepsilon}{\delta}, \frac{\varepsilon^2}{\delta} \right) \right\| \cong c\varepsilon^{3/2}; \qquad \|x - y\| = \sqrt{\varepsilon^2 + \varepsilon^4} \cong \varepsilon .$$

If we endow instead the plane with the sum norm, then

$$\alpha[x, y] = \left\| (1, 0) - \left(\frac{1-\varepsilon}{1-\varepsilon+\varepsilon^2}, \frac{\varepsilon^2}{1-\varepsilon+\varepsilon^2} \right) \right\| = \frac{2\varepsilon^2}{1-\varepsilon+\varepsilon^2} \cong 2\varepsilon^2; \qquad \|x - y\| \cong \varepsilon .$$

Example 4. Let be $X = R^2$ with the max norm; $x = (1, 1)$; $y = (1/2, -1/2)$; then we have $\|x - y\| = 3/2$; $\dfrac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}} = 2 = \alpha[x, y]$ (so equality in (9′) and in (9)); $\dfrac{2\|x-y\|}{\max\{\|x\|, \|y\|\}} = 3/2$ (so inequality in (3)). Note that $x \perp y$; $y \perp x$; $\|x + y\| = \|x - y\|$.

If we take $u = (1, 2)$, then $\|x - u\| = 1 = \|x\|$; $\|u\| = 2$; $\alpha[x, y] = 1$, so equality in (3).

6 - Other radial displacements

Among the different ways to move points of normed spaces along rays, we also recall the *inversion* I, defined by $I\theta = \theta$ and, for $x \neq \theta$, $I(x) = \dfrac{x}{\|x\|^2}$.

Also this transformation is well behaved only when X is an inner product space; something similar can be said for similar maps, e.g. when exponents different from 2 are considered: see [**A**], § 11 and [**F**].

Given a set C, assume that T from $\mathrm{dom}(T)$ onto C is a *retraction*; i.e., $Tx = x$ for all $x \in C$; we say that T is *sunny* if $x \in \mathrm{dom}(T)$ and $x' = Tx$ implies $T(x' + t(x - x')) = x'$ whenever $t > 0$ and $x' + t(x - x') \in \mathrm{dom}(T)$. Some results concerning sunny retractions are indicated in [**P**], §§ 5 and 7; see also [**KT**], [**TT**] and [**ST**].

We can project radially, instead of over the unit ball of X, onto a starshaped set, with at least one point interior to its kernel: for results concerning these projections see [**V**], and [**SP**], § 2.

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Abstract

We consider radial motions of points in real normed spaces. Radial projections, i.e. retractions of points of the space onto the unit ball, have been studied extensively: in fact, the radial projection constant is, among all parameters, one of the most studied. Recall that radial projections are seldom nonexpansive (outside the case of norms defined by inner products). We consider here also other displacements along radial directions: for example, radial projections onto the unit sphere both from inside and from outside (or sending points at the same distance from the origin). We study the best constants by which these radial movements can be controlled.

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