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A simple approach to compute integrals in collocation and Galerkin BEMs for 2D problems (**)

1 - Introduction

The Boundary Element Methods (BEM) have become an important technique for solving linear elliptic partial differential equations appearing in many relevant engineering applications (e.g. acustics, elastostatic, elastodynamics, etc.) (see [**18**], [**32**], [**33**]). By means of the foundamental solution of the considered differential equation a large class of both exterior and interior elliptic boundary value problems can be formulated as a linear integral equation on the boundary of the given domain.

The numerical analysis of this method for two dimensional problems is now fairly well studied; see [**5**], [**13**], [**27**], [**31**] for collocation methods and [**15**], [**28**], [**16**] for Galerkin methods.

More recently, in [**30**] for collocation methods and in [**29**] for Galerkin BEM, the analysis has been extended to a nonlinear boundary value problem, where the partial differential equation itself is linear but the boundary conditions are nonlinear. In various applications, the problems involve nonlinearities in the boundary conditions. Among these is the steady-state heat transfer [**6**] where the boundary has a variable thermal heat conductivity or the body obeys the Newton low of cooling. Further applications arise in some electromagnetic problems that contain nonlinearities in the boundary conditions; for instance problems where the electrical conductivity of the boundary is variable.

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The BEM can offer substantial computational advantages over other numerical techniques. However, in order to achieve an efficient numerical implementation of general validity, a number of issues have to be dealt with special attention. One of the most significant and important issue of the practical application of the BEM analysis is the evaluation of weakly singular, Cauchy singular and even hypersingular integrals over boundary elements. The integral with a hypersingular kernel is not defined in the usual sense, neither as a Cauchy principal value, but as the finite-part. Boundary integral equations with hypersingular kernels arise whenever the gradient (or, e.g. the normal derivative) of a classical boundary integral equation is taken.

Recently in ([**2**], [**3**]) we have considered problems on 2D domains. In particular, we have examined all «difficult» integrals whose evaluation is required when the integral equations are solved by a Galerkin BEM based on piecewise polynomial approximants of arbitrary local degree. By «difficult» integrals we mean the singular and the nearly singular ones. To compute these integral we have proposed new numerical integration schemes, which require the user only to define an arbitrary mesh on the boundary and specify the local degrees of the approximant. These formulas enable the construction of adaptive routines for the implementation of the $h-p$ version of the BEM. Of course these schemes can be used also to compute the integrals generated by collocation methods. They are formulas of product-interpolatory type, which integrate exactly the nasty component of the integrand.

In this paper we consider all difficult integrals required by collocation and Galerkin BEMs, when these are applied to 1D weakly singular and singular integral equations. In particular we show that they can be evaluated very efficiently by means of very simple quadrature formulas, thus avoiding the use of special rules, such as those of product type. Numerical evidences of the effectiveness of our approach, as well as some applications, are presented.

2 - Basic quadrature formulas

One of the most frequently used quadrature rule is certainly the well-known Gauss-Legendre formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{n} \lambda_i f(\xi_i) + R_n(f), \tag{2.1}$$

which has the maximum degree of exactness, i.e., $R_n(f) = 0$ whenever $f(x)$ is a polynomial of degree at most $2n-1$.

While it is very accurate when $f(x)$ is *smooth*, its performance becomes poor when $f(x)$ has *irregularities*. Indeed, given any $f \in C^m[-1, +1]$ we have, for example, $R_n(f) = o(1)$ when $m = 0$ and $R_n(f) = O(n^{-m})$ otherwise. We will however show that,using very elemetary tools, with this rule we can also easily handle several other cases where the function $f(x)$ is not smooth at all. For instance, one case of interest in Sect. **3** will be the one where $f(x)$ is very smooth, usually analytic, except at the endpoints ± 1 where it has mild singularities of the type $\log(1 \pm x)$ or $(1 \pm x)\log(1 \pm x)$. Also in such cases we can recover the high accuracy of (2.1) for smooth functions, by introducing in (2.1) a simple change of variable (see [**24**]).

By choosing a new function $\varphi(t)$, with $\varphi'(t) \geqslant 0$, mapping $(-1, 1)$ onto ifself, we obtain

$$\int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} f(\varphi(t))\, \varphi'(t)\, dt \equiv \int_{-1}^{1} F(t)\, dt . \tag{2.2}$$

If furthermore

$$\varphi^{(i)}(-1) = 0 , \qquad \varphi^{(j)}(1) = 0 , \qquad i = 1, \ldots, p_1 - 1 ; \qquad j = 1, \ldots, p_2 - 1 ,$$

then we can make $F(t)$ as smooth as we like simply by taking integers p_1, p_2 sufficiently large.

In [**25**] the following polynomial transformation

$$\varphi(t) = \frac{(p_1 + p_2 - 1)!}{(p_1 - 1)!(p_2 - 1)!} \int_0^t u^{p_1 - 1}(1 - u)^{p_2 - 1}\, du , \qquad p_1, p_2 \geqslant 1 , \tag{2.3}$$

has been proposed. The integral in (2. 3) can be evaluated exactly (up to machine accuracy) by means of (2.1) with $n = \left\lfloor \dfrac{p_1 + p_2}{2} \right\rfloor$. Finally, if we apply the n-point Gauss-Legendre rule (2.1) to the final form of (2.2) we obtain

$$\int_{-1}^{1} f(x)\, dx = \sum_{i=1}^{n} \lambda_i f(\varphi(\xi_i))\, \varphi'(\xi_i) + R_n(F) . \tag{2.4}$$

For this formula, with $p_1 = 1$, we have the following convergence result (see [**22**]).

Theorem 1. *If* $f(x) = (1-x)^m \log(1-x)$, *with* m *being a non negative integer, then we have*

$$R_n(F) = O(n^{-2p_2(m+1)} \log n) . \tag{2.5}$$

The above smoothing procedure has been actually generalized to integrals of type (2.1) with $f(x)$ possibly having also (or only) a fixed number of internal weak singularities (see [**24**]).

Remark 1. If the function $f(x)$ in (2.4) is of the form like that of Theorem 1, but with an internal singularity, for example $f(x) = x^m \log x$, then we only have

$$R_n(F) = O(n^{-p_2(m+1)+1}) .$$

Therefore, when the singularity is at the endpoints of the interval of integration, the rate of convergence is more than twice that one has for corresponding interior singularities.

The effectiveness of this approach is shown by the following two examples, which can be taken as test cases for the applications we will consider in the next section.

$$I_1 = \int_0^1 x \log x \, dx \qquad I_2 = \int_0^1 [2 \log x + \log(1-x)] \, dx .$$

Some numerical results are reported in Tables I, II. Here and in the following examples, all computation has been performed on a PC using double precision (16 digits) arithmetic. The sign «–» means that full accuracy has been achieved.

In spite of the error estimate we have recalled at the beginning of the section for the Gauss-Legendre rule, in practice also the requirement that $f(x)$ is analityc in $[-1, 1]$ is not sufficient to guarantee a high accuracy to (2.1). The presence of (real or complex conjugate) poles very close to the interval of integration usually gives rise to a poor performance of (2.1), at least when the number of nodes is not

TABLE I. – *Relative errors given by rule* (2.4) *applied to* I_1.

p_1	p_2	$n=2$	$n=4$	$n=8$	$n=16$	$n=32$	$n=64$
2	1	$1.2E-2$	$3.2E-5$	$1.5E-7$	$6.7E-10$	$2.9E-12$	$1.2E-14$
3	1	$7.9E-2$	$1.4E-5$	$2.5E-9$	$6.9E-13$	—	—
4	1	$1.1E-1$	$9.6E-5$	$1.7E-10$	—	—	—
5	1	$1.0E-1$	$2.9E-3$	$4.0E-11$	—	—	—

TABLE II. – *Relative errors given by rule* (2.4) *applied to* I_2.

p_1	p_2	$n=2$	$n=4$	$n=8$	$n=16$	$n=32$	$n=64$	$n=128$
2	2	$4.3E-1$	$2.5E-2$	$1.8E-3$	$1.2E-4$	$8.1E-6$	$5.2E-7$	$3.3E-8$
3	3	$4.7E-1$	$1.8E-2$	$2.7E-4$	$4.6E-6$	$7.7E-8$	$1.2E-9$	2.0E – 11
4	4	$1.6E-1$	$5.5E-2$	$7.7E-5$	$3.0E-7$	$1.2E-9$	$5.1E-12$	2.8E – 14
5	5	$2.8E-1$	$1.7E-1$	$4.5E-5$	$3.0E-8$	$3.1E-11$	$3.6E-14$	—

very high. Indeed, although the estimate $|R_n(f)| \leqslant Cn^{-k}$, with k arbitrarily large, holds for analytic functions, the constant C can be quite large when f has poles very close to $[-1, 1]$. This behaviour is clearly shown by the following simple example (see Table III):

$$I_3 = \int_{-1}^{1} \frac{e^x}{x^2+\varepsilon^2}\, \mathrm{d}x \,.$$

For such type of integrals a smoothing procedure can be, however, easily suggested. To illustrate it we consider the following integral

$$I_4 = \int_0^1 \frac{f(x)}{[(x-a\varepsilon)^2 + b^2\varepsilon^2]^{1/2}}\, \mathrm{d}x \,, \tag{2.6}$$

where a, b are two real constants, while ε is a positive parameter which may assume very small values. This is the form of integrals that we shall encounter in our BEM applications. Recalling some elementary complex variable calculus, it is quite natural to set

$$x = a\varepsilon + t^q$$

TABLE III. – *Relative errors given by the Gauss-Legendre rule applied to* I_3.

n	$\varepsilon = 1$	$\varepsilon = 0.1$	$\varepsilon = 0.01$
4	$6.5E-4$	0.59	0.96
8	$5.8E-7$	0.32	0.92
16	$4.1E-13$	$7.4E-2$	0.84
32	—	$3.1E-3$	0.69
64	—	$5.3E-6$	0.43

with q positive integer, odd when $a\varepsilon > 0$. Indeed, in the new form

$$
\begin{aligned}
I_4 &= q \int_{-(a\varepsilon)^{1/q}}^{(1-a\varepsilon)^{1/q}} \frac{f(a\varepsilon + t^q)\, t^{q-1}}{(t^{2q} + b^2 \varepsilon^2)^{1/2}} \, dt\,, \qquad q \text{ odd, when } a\varepsilon > 0\,, \\
&= q \int_{(-a\varepsilon)^{1/q}}^{(1-a\varepsilon)^{1/q}} \frac{f(a\varepsilon + t^q)\, t^{q-1}}{(t^{2q} + b^2 \varepsilon^2)^{1/2}} \, dt\,, \qquad \text{when } a\varepsilon < 0\,,
\end{aligned}
\tag{2.7}
$$

the presence of $(2q)$ poles in the integrand function is less adverse as q increases, up to a certain value after which there is no improvement. The case $a\varepsilon < 0$ is more favorable and this shows up also in the example we consider next. In this case, as $q \to \infty$ the length of the interval of integration tends to zero, since here we are considering $|a\varepsilon|$ very small.

To show the effectiveness of this procedure, we take in (2.6) first $a = b = 1$ and then $a = -1,\ b = 1$, with $f(x) = e^x$, rewrite the corresponding integrals in the form (2.7) and then apply to them the Gauss-Legendre rule. In Tables IV-VII we report the results we have obtained.

As remarked earlier, the case $\varepsilon < 0$ is more favorable. Moreover, values of q larger than those reported do not seem to produce any further improvement. This phenomenon is a straightforward consequence of the behaviour of the poles in the new form (2.7) (as well as that of its interval of integration).

In some applications of interest, however, the integral is of the form

$$
I_4 = \int_0^1 K(x, \varepsilon)\, f(x)\, dx
\tag{2.8}
$$

where $K(x, \varepsilon)$ has the behavior $[(x - a\varepsilon)^2 + b^2 \varepsilon^2]^{-1/2}$, but we may not want to de-

TABLE IV. – *Relative errors given by the Gauss-Legendre rule applied to* (2.7).

$\varepsilon = 1.E-2$ $I_4 = 7.531164310779$ a = b = 1			
n	$q = 1$	$q = 3$	$q = 5$
4	$2.1E-1$	$1.3E-1$	$3.7E-2$
8	$6.6E-2$	$8.8E-3$	$2.1E-2$
16	$1.7E-2$	$3.1E-4$	$3.1E-4$
32	$6.9E-4$	$9.8E-7$	$6.7E-8$
64	$4.4E-7$	$2.7E-12$	1.7E−11
128	$1.5E-12$	—	—

TABLE V. – *Relative errors given by the Gauss-Legendre rule applied to* (2.7).

$\varepsilon = 1.E-4$ $I_4 = 12.103560321358$ $a = b = 1$				
n	$q=1$	$q=3$	$q=5$	$q=7$
4	$5.5E-1$	$1.2E-1$	$1.7E-1$	$2.7E-1$
8	$4.4E-1$	$1.4E-1$	$5.2E-2$	$3.1E-2$
16	$3.3E-1$	$2.2E-2$	$1.2E-2$	$4.8E-4$
32	$2.0E-1$	$1.2E-3$	$3.2E-4$	$3.7E-4$
64	$3.5E-2$	$4.8E-6$	$2.1E-7$	$6.4E-8$
128	$1.1E-2$	$1.2E-10$	$1.7E-12$	1.8E−12
256	$1.6E-3$	—	—	—

termine the value of a. In this situation one can then set $x = t^q$ in (2.8), obtaining the new form

$$I_4 = q\int_0^1 K(t^q, \varepsilon)\, f(t^q)\, t^{q-1}\,\mathrm{d}t\,. \tag{2.9}$$

This transformation leaves unchanged the interval of integration and it is a bit less effective than the previous one. This is clearly shown by the following Tables VIII-X, which refer to the cases considered in Tables V-VII, respectively. Therefore the explicit evaluation of a turns out to be worthwhile.

A final variant of (2.1) is the corresponding «Gaussian» formula for Hadamard finite part integrals, of the form

$$\ne\!\!\!\!\int_{-1}^{1} \frac{f(x)}{1-x}\,\mathrm{d}x = w_0 f(1) + \sum_{i=1}^{n} \frac{\lambda_i}{1-\xi_i} f(\xi_i) + R_n^{FP}(f)\,, \tag{2.10}$$

where $R_n^{FP}(f) = 0$ whenever $f(x)$ is a polynomial of degree $2n$. Notice that (2.9)

TABLE VI. – *Relative errors given by the Gauss-Legendre rule applied to* (2.7).

$\varepsilon = -1.E-2$ $I_4 = 5.6906262374672$ $a = -1$, $b = 1$						
n	$q=1$	$q=2$	$q=3$	$q=4$	$q=8$	$q=12$
4	$1.1E-1$	$5.1E-3$	$2.5E-3$	$7.0E-4$	$4.3E-4$	$5.7E-4$
8	$1.4E-2$	$4.5E-6$	$1.2E-5$	$1.8E-6$	$5.0E-7$	$6.9E-8$
16	$1.4E-4$	$3.8E-8$	$1.1E-10$	$6.2E-11$	$3.7E-13$	1.0E−12
32	$4.0E-8$	—	—	—	—	—

TABLE VII. – *Relative errors given by the Gauss-Legendre rule applied to* (2.7).

$\varepsilon = -1.E-4$ $I_4 = 10.339112542551$ a = −1, b = 1							
n	$q=1$	$q=2$	$q=3$	$q=4$	$q=6$	$q=8$	$q=12$
4	$4.7E-1$	$1.4E-1$	$1.3E-2$	$8.3E-3$	$8.2E-3$	$5.0E-3$	$1.7E-3$
8	$3.5E-1$	$1.8E-2$	$4.5E-3$	$8.5E-4$	$1.8E-4$	$1.2E-4$	$3.0E-5$
16	$2.2E-1$	$2.4E-3$	$5.5E-5$	$1.3E-6$	$1.2E-9$	$6.4E-8$	$3.3E-9$
32	$1.1E-1$	$1.5E-5$	$1.1E-8$	$8.9E-11$	$5.0E-13$	—	—
64	$2.4E-2$	$8.9E-10$	—	—	—	—	—
128	$3.1E-4$	—	—	—	—	—	—
256	$2.5E-6$	—	—	—	—	—	—

can be viewed as a Gauss-Radau rule associated with the weight function $\dfrac{1}{1-x}$. The coefficient w_0 is given by the expression

$$w_0 = \log 2 - \sum_{i=1}^{n} \frac{\lambda_i}{1-\xi_i}\,.$$

For this rule we have the following result (see [**23**]).

Theorem 2. *Let* $f \in C^m([-1,1])$, $m \geq 1$, *with* $f^{(m)} \in H_\mu[-1,1]$ *for some* $0 < \mu \leq 1$. *Then*

$$R_n^{FP}(f) = O(n^{-m-\mu})\,. \tag{2.11}$$

Our last basic quadrature rule refers to the evaluation of double integrals of

TABLE VIII. – *Relative errors given by the Gauss-Legendre rule applied to* (2.9).

$\varepsilon = 1.E-4$ a = b = 1					
n	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
4	$5.5E-1$	$1.9E-1$	$3.4E-1$	$1.9E-1$	$1.8E-1$
8	$4.4E-1$	$1.2E-1$	$1.1E-1$	$9.5E-2$	$7.4E-2$
16	$3.3E-1$	$7.2E-2$	$1.3E-2$	$1.2E-2$	$2.1E-4$
32	$2.0E-1$	$1.5E-2$	$4.0E-3$	$2.5E-4$	$1.8E-3$
64	$3.5E-2$	$2.4E-4$	$2.7E-5$	$2.0E-5$	$6.3E-6$
128	$1.1E-2$	$2.0E-6$	$1.6E-8$	$9.7E-10$	$1.1E-9$
256	$1.6E-3$	$7.0E-12$	—	—	—

TABLE IX. – *Relative errors given by the Gauss-Legendre rule applied to* (2.9).

$\varepsilon = -1.E-2$ a $= -1$, b $= 1$			
n	$q = 1$	$q = 2$	$q = 3$
4	$1.1E-1$	$8.4E-3$	$3.4E-3$
8	$1.4E-2$	$6.5E-4$	$4.1E-4$
16	$1.4E-4$	$6.6E-7$	$2.9E-8$
32	$4.0E-8$	$4.2E-13$	$4.7E-13$

the form

$$\int_0^1 \int_0^y K(x, y)\, f(x, y)\, dx\, dy\,, \tag{2.12}$$

where

$$K(uy, y) = y^\alpha K(u, 1)\,, \qquad \alpha \geq -1\,, \tag{2.13}$$

or

$$K(uy, y) = \log y + K(u, 1)\,, \tag{2.14}$$

and both $K(u, 1)$ and $f(x, y)$ are smooth functions, while $K(x, y)$ is singular at $x = y = 0$. Consider for example

$$K(x, y) = (x^2 + y^2)^{-1/2},$$

TABLE X. – *Relative errors given by the Gauss-Legendre rule applied to* (2.9).

$\varepsilon = -1.E-4$ a $= -1$, b $= 1$					
n	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 6$
4	$4.7E-1$	$7.7E-2$	$1.5E-1$	$9.1E-2$	$4.9E-2$
8	$3.5E-1$	$6.8E-2$	$2.6E-2$	$1.0E-2$	$1.4E-2$
16	$2.2E-1$	$4.9E-3$	$7.6E-4$	$7.6E-5$	$3.5E-4$
32	$1.1E-1$	$6.3E-6$	$5.2E-7$	$5.0E-7$	$2.7E-7$
64	$2.4E-2$	$2.6E-8$	$1.5E-11$	$8.4E-13$	$1.2E-13$
128	$3.1E-4$	—	—	—	—
256	$2.5E-6$	—	—	—	—

and

$$K(x, y) = \log|x + y| \ .$$

By introducing the very simple change of variable $x = uy$, usually called «Duffy transformation» (see [**12**]), we obtain

$$\int_0^1 y^{\alpha+1} \int_0^1 K(u, 1)\, f(uy, y)\, \mathrm{d}u\, \mathrm{d}y \tag{2.15}$$

or

$$\int_0^1 y \log y \int_0^1 f(uy, y)\, \mathrm{d}u\, \mathrm{d}y + \int_0^1 y \int_0^1 K(u, 1)\, f(uy, y)\, \mathrm{d}u\, \mathrm{d}y =: I_1 + I_2 . \tag{2.16}$$

To compute (2.15) we use a Gauss-Legendre rule for the inner integral, and a Gauss-Jacobi rule with weight $y^{\alpha+1}$ for the outer. For the evaluation of I_1 we use a Gauss-Legendre rule for the internal integral and (2.4) for the outer. Integral I_2 can be computed by means of the product of two Gauss-Legendre rules or by «optimal» cubatures for the unit square, such as those reparted in [**9**].

3 - Evaluation of the elements of collocation and Galerkin matrices

In this section we consider those integrals which are required by collocation and Galerkin methods, and whose evaluation cannot be performed efficiently by means of Gaussian rules. In the collocation case, these are integrals which refer to collocation points that are inside, or outside but close to, an element of integration. In the Galerkin case, they are double integrals refering either to the same boundary element or to two consecutive ones (see however [**11**]).

Thus, first we examine the case of collocation integrals, assuming that the collocation point belongs to the element of integration. Using the parametric representation of the element, the integrals of interest to us can be written (see [**3**]) in the form

$$\int_0^1 K(u, s)\, f(u)\, \mathrm{d}u\,, \qquad s \in [0, 1] \tag{3.1}$$

with

$$K(u, s) = \log|u - s| K_0(u, s) \tag{3.2}$$

or

$$K(u, s) = \frac{K_0(u, s)}{u - s}, \tag{3.3}$$

where $K_0(u, s)$ is smooth and s is a parameter representing the collocation point.

In the case of kernel (3.2) we can make use of relation (2.2), taking a function $\varphi(t)$ which smooth the log singularity. For example, setting

$$u = \varphi(t) = s + t^q,$$

q being an odd positive integer, we obtain

$$q \int_{-s^{1/q}}^{(1-s)^{1/q}} K(s + t^q, s)\, f(s + t^q)\, t^{q-1}\, \mathrm{d}t\,. \tag{3.4}$$

This integral is then evaluated using the Gauss-Legendre rule (2.1). This approach appears to be cheaper than the classical product rule for the $\log|u - s|$ kernel (see [**20**]).

Remark 2. In the case of an integral equation defined on a curve given by a parametric representation $\Gamma(u) = (\gamma_1(u), \gamma_2(u))$, and with a log-kernel, the form of this latter becomes

$$k(u, s) \log\{[\gamma_1(u) - \gamma_1(s)]^2 + [\gamma_2(u) - \gamma_2(s)]^2\}, \tag{3.5}$$

where $k(u, s)$ is smooth. In this situation there is no need to reduce (3.5) to the form (3.2), by adding and subtracting $\log|u - s|$ to the above log factor, as done in [**3**]. Indeed, one can introduce the proposed change of variable directly into the original expression (3.5).

Kernel (3.3) gives rise to a Cauchy principal value integral (3.1) if $s \in (0, 1)$, and to a finite part integral when $s = 0$ or $s = 1$. In former case we rewrite (3.1) as follows

$$\left(\, ⨎_0^s + ⨎_s^1 \,\right) \frac{K_0(u, s)}{u - s} f(u)\, \mathrm{d}u \tag{3.6}$$

and apply rule (2.9) to each of the two (sub-)integrals. The composite formula is a $(2n + 1)$-point rule which has degree of exactness $2n$; thus it is of interpolatory type and represents, at least in this case, a more convenient alternative to the classical product rule (see [**21**]). See also the formula proposed in Sect. **5** of that paper. When $s=0$ or $s=1$ we apply (2.9) directly to the whole interval.

Next we consider the case of a collocation point that belongs to an element which is consecutive to the element of integration and has the same parametric representation functions. In this circumstance the kernel of integral (3.1) assumes one of the following two forms

$$K(u,s)=\log|u+as|K_0(u,s), \tag{3.7}$$

$$K(u,s)=\frac{K_0(u,s)}{u+as}, \tag{3.8}$$

where a denotes a positive constant, and as is supposed small; let us say $as<0.5$ (1).

In the case of (3.7) we proceed as for (3.2); in particular we introduce the change of variable $u=\varphi(t)$ with

$$\varphi(t)=-as+t^q,$$

and use the Gauss-Legendre rule. Notice that also this situation can be treated as described in Remark 2.

The case of (3.8) is even simpler. Since $K_0(u,s)$ and $f(u)$ are both defined also in the element where the collocation point is lying, we rewrite the integral as follows

$$\int_0^1 \frac{K_0(u,s)\,f(u)-K_0(-as,s)\,f(-as)}{u+as}\,\mathrm{d}u+K_0(-as,s)\,f(-as)\log\frac{1+as}{as}$$

and then apply (2.1) to the integral above. Since the pole $-as$ is outside the interval of integration, no significant numerical cancellation is generated by the above divided difference when it is evaluated at the Legendre points.

In the final case the collocation point is assumed lying on an element consecutive to that of integration, but having a different parametrization. In this situation the behaviour of the kernel around the corner is either of the type

$$\log[(u-as)^2+b^2s^2], \tag{3.9}$$

(1) Otherwise the integral can be evaluated efficiently by means of (2.1).

or

$$[(u-as)^2+b^2s^2]^{-1/2}, \tag{3.10}$$

with a and b that can be easily determined (see [3]). Here too we are considering the critical case, i.e., $|b|s$ very small, otherwise also these integrals can be efficiently evaluated using (2.1) directly.

To compute these integrals we suggest to use the procedure we have proposed in the previous section to evaluate I_4. In particular we introduce into the expression of $K(u, s)$ the change of variable we have proposed for the evaluation of I_4. Unless $|b|$, hence $|b|s$, is excessively small, in which case the use of product formulas as those described in [3] appears to be more convenient. By excessively small we mean $|b|s \approx 10^{-6}$.

In the case of integrals arising from the application of Galerkin boundary element methods we basically follow the approach proposed at the end of the previous section to compute integrals over a reference triangle (2.12).

In particular, in the case of kernels (3.2), (3.7) and (3.9), that we denote in the form $k(t, s)\, K_0(t, s)$, we write

$$\int_0^1 \phi_j(s) \int_0^1 k(t, s)\, K_0(t, s)\, \phi_i(t)\, \mathrm{d}t\, \mathrm{d}s = \int_0^1\int_0^s + \int_0^1\int_s^1$$

$$= \int_0^1 \phi_j(s) \int_0^1 k(u, 1)\, K_0(us, s)\, \phi_i(us)\, \mathrm{d}u\, \mathrm{d}s + \int_0^1 \phi_i(t) \int_0^1 k(1, u)\, K_0(t, ut)\, \phi_j(ut)\, \mathrm{d}u\, \mathrm{d}t$$

$$= \int_0^1\int_0^1 F(u, s)\, \mathrm{d}u\, \mathrm{d}s\,,$$

where we have set

$$F(u, s) = \phi_j(s)\, k(u, 1)\, K_0(us, s)\, \phi_i(us) + \phi_i(s)\, k(1, u)\, K_0(s, us)\, \phi_j(us)\,. \tag{3.11}$$

Hence we apply a product of two Gauss-Legendre rules or a (global) rule for the unit square.

In the case of (3.10), $k(u, 1)$ and $k(1, u)$ have poles which are very close to the interval of integration when the angle θ between the left and the right tangents at the corner (see [3]) is very close to 0 or π. In such a circumstance the approach based on product rules, described in [3], can be more efficient.

Notice also that in practice there is no need to rewrite the kernel of the integral, that in the following we shall denote by $K(t, s)$, in the form $k(t, s)\, K_0(t, s)$.

Indeed, in (3.11) $k(u, 1)\, K_0(us, s)$ and $k(1, u)\, K_0(s, us)$ can be replaced by $K(us, s)/s$ and $K(s, us)/s$, respectively.

The treatment of the cases (3.2), (3.7) and (3.9) is very similar and follows from relationship (2.14).

Also in the case of kernel (3.3) we essentially apply the previous procedure, but with some modifications. First we write

$$I = \int_0^1 \phi_j(s) \int_0^1 \frac{K_0(t, s)}{t - s}\, \phi_i(t)\, \mathrm{d}t\, \mathrm{d}s = \int_0^1\int_0^s + \int_0^1\int_s^1 =: I_1 + I_2;$$

then we proceed as follows:

$$I_1 = \int_0^1 \phi_j(s) \int_0^s \frac{K_0(t, s)\, \phi_i(t) - K_0(s, s)\, \phi_i(s)}{t - s}\, \mathrm{d}t\, \mathrm{d}s + \int_0^1 \phi_j(s)\, K_0(s, s)\, \phi_i(s) \int_0^s \frac{\mathrm{d}t}{t - s}\, \mathrm{d}s$$

$$= \int_0^1 \phi_j(s) \int_0^1 \frac{K_0(us, s)\, \phi_i(us) - K_0(s, s)\, \phi_i(s)}{u - 1}\, \mathrm{d}u\, \mathrm{d}s - \int_0^1 \log s \phi_j(s)\, K_0(s, s)\, \phi_i(s)\, \mathrm{d}s\,,$$

$$I_2 = \int_0^1 \phi_j(s) \int_s^1 \frac{K_0(t, s)\, \phi_i(t) - K_0(s, s)\, \phi_i(s)}{t - s}\, \mathrm{d}t\, \mathrm{d}s + \int_0^1 \phi_j(s)\, K_0(s, s)\, \phi_i(s) \int_s^1 \frac{\mathrm{d}t}{t - s}\, \mathrm{d}s$$

$$= \int_0^1\int_0^1 \phi_j(ut)\, \frac{K_0(t, ut)\, \phi_i(t) - K_0(ut, ut)\, \phi_i(ut)}{1 - u}\, \mathrm{d}u\, \mathrm{d}t - \int_0^1 \log(1 - s)\, \phi_j(s)\, K_0(s, s)\, \phi_i(s)\, \mathrm{d}s\,.$$

The expression that finally we propose to evaluate is

$$I = \int_0^1\int_0^1 G(u, s)\, \mathrm{d}u\, \mathrm{d}s + \int_0^1 \log \frac{1 - s}{s}\, \phi_j(s)\, \phi_i(s)\, K_0(s, s)\, \mathrm{d}s\,, \tag{3.12}$$

where we have set

$$G(u, s)$$

$$= \phi_j(s)[K(us, s)\phi(us) - K(s, s)\phi_i(s)]/s + \phi_j(us)[K(s, us)\phi_i(s) - K(us, us)\phi_i(us)]/s\,.$$

Since this function is smooth, we propose to approximate the double integral in (3.12) by the product of two Gauss-Legendre rules, or by an optimal rule for the unit square. The weakly singular integral in (3.12) can be evaluated using (2.4).

4 - Test problems

The first example we consider is the computation of integrals of the form

$$I = \int_{-1}^{1} E_1(|u-s|)\, f(u)\, \mathrm{d}u\,, \qquad s \in [-1, 1]\,, \tag{4.1}$$

where $E_1(t)$ is the exponential integral function (see [**1**]) and $f(u)$ is generally a smooth function, for example a polynomial. Here we will take $f(u) = e^u$. Integrals of this type arise, for instance, in the solution of the linear transport equation in slab geometry (see [**20**]).

An efficient procedure to compute these integrals, suggested in [**20**], is based on the decomposition

$$E_1(t) = k_1(t^2) + tk_2(t) + \log t\,, \tag{4.2}$$

where k_1 and k_2 are entire functions. In this case the quadrature rule used to compute (4.1) requires the (fairly expensive) construction of two product rules based on Gauss-Legendre nodes: one for the kernel $\log|u-s|$ and one for the kernel $|u-s|$.

With our procedure we do not need to use (4.2), neither to construct special product rules. We simply introduce in (4.1) the change of variable $u = s + t^q$, q odd integer, and then approximate the corresponding integral by using our Gauss-Legendre rule (2.1). Routines for the evaluation of $E_1(t)$ are easily available from many standard software libraries. Thus we have

$$I = q \int_{-(1+s)^{1/q}}^{(1-s)^{1/q}} E_1(|t|^q)\, t^{q-1} e^{s+t^q}\, \mathrm{d}t \approx q\frac{\gamma}{2} \sum_{i=1}^{n} \lambda_i E_1(|\eta_i|^q)\, \eta_i^{q-1} e^{s+\eta_i^q}, \tag{4.3}$$

where

$$\gamma = (1-s)^{1/q} + (1+s)^{1/q}$$

and

$$\eta_i = \frac{1}{2}[\gamma\xi_i + (1-s)^{1/q} - (1+s)^{1/q}]\,.$$

In Table XI we have reported some relative errors produced by this rule. The reference value has been computed using (4.3) with $n = 256$ and $q = 9$.

TABLE XI. – *Relative errors produced by rule* (4.3).

$I = 2.45800813061196 \quad s = 0.5$					
n	$q = 1$	$q = 3$	$q = 5$	$q = 7$	$q = 9$
8	$1.5E-1$	$1.6E-2$	$4.2E-4$	$8.9E-4$	$2.7E-3$
16	$3.9E-2$	$1.6E-3$	$5.4E-5$	$2.6E-6$	$2.6E-7$
32	$3.7E-2$	$1.7E-4$	$1.3E-6$	$9.3E-9$	$4.3E-10$
64	$9.9E-3$	$1.1E-6$	$2.2E-9$	$1.7E-10$	—
128	$9.4E-3$	$4.1E-6$	$2.0E-9$	$5.0E-13$	—
256	$2.5E-3$	$3.2E-7$	$5.7E-11$	—	—

The second example refers to a classical weakly singular integral equation of the second kind, that for simplicity we take of the form

$$z(y) + \int_{-1}^{1} \log|y-x|\, z(x)\, dx = 1, \qquad -1 \leqslant y \leqslant 1. \tag{4.4}$$

We consider this equation defined on [-1,1], but our numerical procedure would performe equally well also in the case of a curvilinear domain.

The solution $z(x)$ of this equation is analytic in (-1,1), but not at the endpoints ± 1 where it has a singular behaviour of the form $(1 \pm x)\log(1 \pm x)$. An efficient method to solve (4.3), recently proposed in [**25**], suggests to introduce preliminarly the change of variable

$$y = \phi(s), \qquad x = \phi(u)$$

where ϕ is defined by (2.3) with $p_1 = p_2 = p$. This has the effect of transforming (4.3) into the new equation

$$v(s) + \int_{-1}^{1} \log|\phi(s) - \phi(u)|\, \phi'(s)\, v(u)\, du = \phi'(s), \qquad -1 \leqslant s \leqslant 1,$$

where the new unknown $v(u) = z(\phi(u))\, \phi'(u)$ can be made arbitrarily smooth at ± 1, by taking the integer p sufficiently large. At this point a classical collocation method can be applied, setting

$$v(u) \approx v_N(u) = \sum_{k=0}^{N-1} c_k P_k(u),$$

where $P_k(u)$ denotes the Legendre polynomial of degree k, and taking as collocation points the N zeros of $P_N(s)$.

The entries of the final linear system one has to solve are of the form

$$\phi'(s_i) \int_{-1}^{1} \log |\phi(s_i) - \phi(u)| P_k(u)\, du\,, \qquad k = 0, \ldots, N-1\,. \tag{4.5}$$

To compute these integrals we simply set

$$u = s_i + t^q, \qquad q \text{ odd integer}\,,$$

and apply the n-point Gauss-Legendre rule (2.1) to the resulting form. In our case (4.4), taking as in [**25**] $n = 2N$, and setting, for example, $p = 5$ and $q = 9$, we obtain for v_N the same accuracy that the method proposed in [**25**] gives for the same value of p. But in that method the rule used to compute the quantities (4.4) is more expensive. Incidentally we remark that special care must be taken when $|\phi(s) - \phi(s + t^q)|$ is excessively small; instead of computing the corresponding log term we can neglet it, since it ought to be multiplied by the factor t^{q-1}.

The third example we consider the following Dirichlet problem for the Laplace

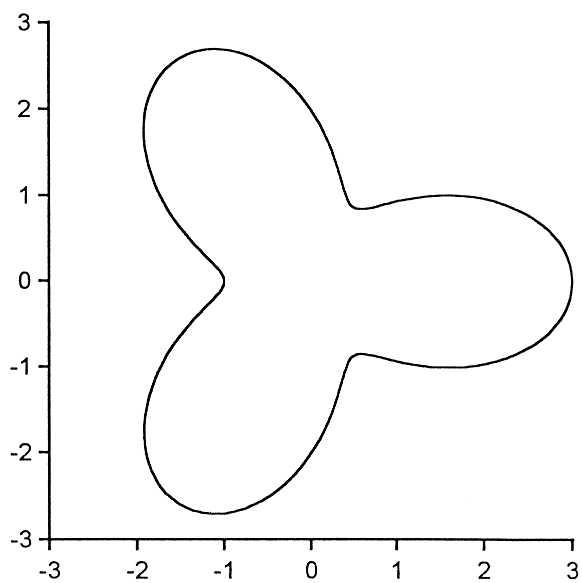

Figure 1 - Domain of Example 3.

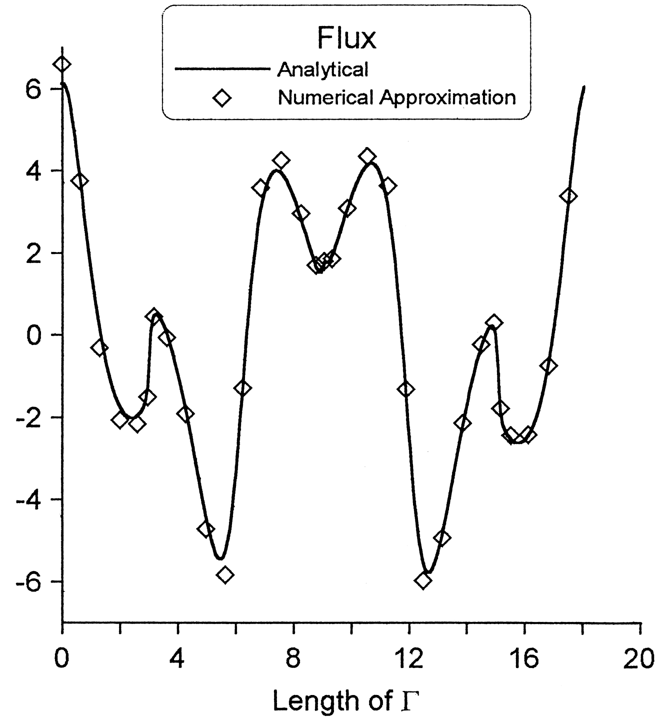

Figure 2 - Analytical gradient and the behaviour of the approximated flux on Γ.

equation

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma \,. \end{cases}$$

This equation stands for many physical problems, such as a steady-state heat conduction, fluid flou in porous media, potential of electrostatic field and more. Figure 1 shows the domain employed, the boundary Γ is parameterised by

$$\gamma(t) = ((2 + \cos 3t)\cos t, (2 + \cos 3t)\,\mathrm{sin} t) \qquad t \in (0,\, 2\pi)\,.$$

The potential distribution over the domain is the harmonic function in $\Re^2$

$$u(x,\, y) = \log\,[(x-4)^2 + (y-4)^2]^{1/2} + x^2 - y^2.$$

The boundary condition provided every where is the potential, with the normal gradients as unknowns. The solution can be obtained applaying the Galerkin

BEM to following integral equation

$$\frac{1}{2}u(x) = \int_{\Gamma} K(x, y)\, q(y)\, \mathrm{d}\Gamma(y) - \fint_{\Gamma} \frac{\partial K(x, y)}{\partial n(y)} u(y)\, \mathrm{d}\Gamma(y) \qquad x \in \Gamma \tag{4.1}$$

where the function $K(x, y) := \frac{1}{2\pi} \log \frac{1}{|x-y|}$ is the fundamental solution of Laplace's equation in a two-dimensional isotropic domain, $q(y) := \frac{\partial u}{\partial n}$ and n is the unit outward normal vector of Γ. Figure 2 shows the analytical gradient and the behaviour of the approximated flux on Γ, obtained with a uniform decomposition of the interval $(0, 2\pi)$ in 32 elemets and using shape functions of degree $p = 1$. Figure 3 shows the behaviour of the relative error obtained with a uniform decomposition of the boundary in 32 elements and using shape functions of degree $p = 1$ and $p = 2$.

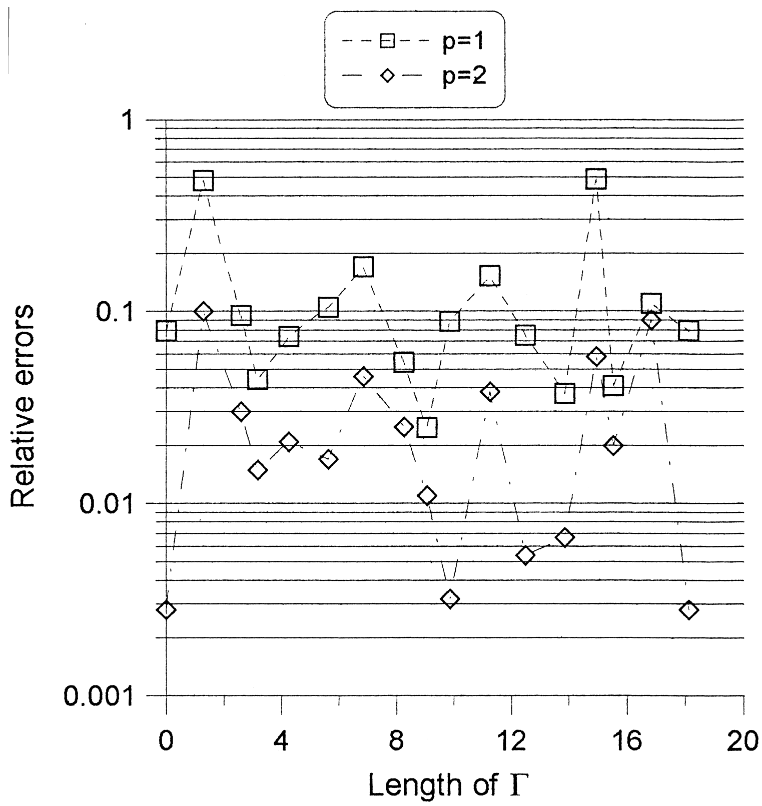

Figure 3 - Relative errors on the boundary of Example 3.

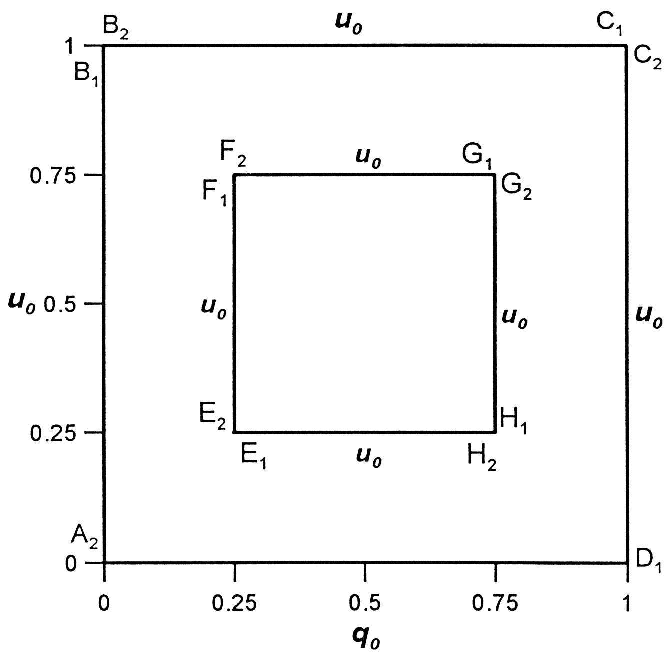

Figure 4 - Domain of Example 4 and boundary conditions.

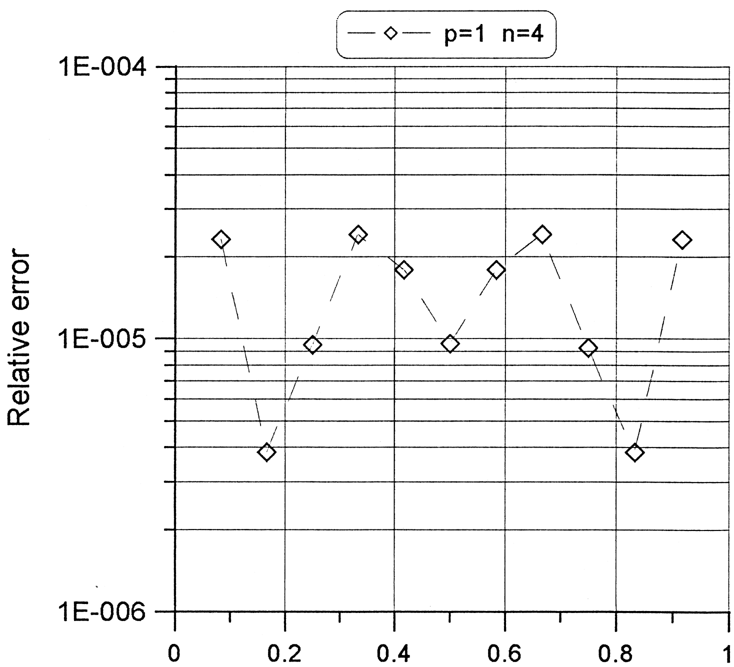

Figure 5 - Relative error on the side AD of Example 4.

TABLE XII. – *Nodal fluxes at the endpoints of each side of the domain of the Example* 4.

Point	Exact	$p = 3$	$p = 4$	$p = 5$
A_2	−3.1416	−3.1341	−3.1431	−3.1415
B_1	−36.4172	−36.2917	−36.4101	−36.4167
B_2	0.0	0.1	0.01	0.0001
C_1	0.0	0.1	0.01	0.0001
C_2	−36.4172	−36.2917	−36.4101	−36.4167
D_1	−3.1416	−3.1341	−3.1431	−3.1415
E_1	1.9297	1.9298	1.9297	1.9297
E_2	2.9425	2.9420	2.9426	2.9425
F_1	11.8242	11.8203	11.8241	11.8242
F_2	−11.6136	−11.6082	−11.6137	−11.6136
G_1	−11.6136	−11.6082	−11.6137	−11.6136
G_2	11.8242	11.8203	11.8241	11.8242
H_1	2.9425	2.9420	2.9426	2.9425
H_2	1.9297	1.9298	1.9297	1.9297

In this final example we consider the Dirichlet-Neumann problem for the Laplace equation

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial n} \equiv q = q_0 & \text{on } \Gamma_2 \end{cases}$$

where Ω is a bounded domain in $\Re^2$ shown in Figure 4 with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and we choose as boundary data

$$\begin{cases} u_0(x,\, y) = \sin \pi x \cosh \pi y & \text{on } \Gamma_1 \\ q_0(x,\, y) = \dfrac{\partial u_0}{\partial n} & \text{on } \Gamma_2 . \end{cases}$$

The function u_0 is harmonic, it also serves as a reference analytical solution. The solution can be obtained applying the Galerkin BEM as in the Example 3. The unknowns of our equation (4.1) are therefore the potential u on the side AD and the flux q on the remaining part of the countour. In a corner point, e.g., B with the potential prescribed over both elements forming the corner, the potential is approximated by continous elements while the flux should have a jump at B.

Hence two unknown fluxes (q_B^1, q_B^2) and one known potential u_B are associated with this node. In Figure 4 the numbers 1 and 2 attached to letter e.g., B_1 and B_2, associated the vertex denoted by the letter B with the left side (B_1) and the right side (B_2), respectively. Table 12 contains analytical results compared with the results produced by $p-$version of the Galerkin BEM, with local polynomial degree $p = 3, 4, 5$, at the endpoints of each side of the domain. Figure 5 shows the behaviour of the relative error on Γ_2 obtained by using local Lagrange basis of degree $p = 3$ on a uniform decomposition ($n = 4$) of the side AD.

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Abstract

In this paper we consider all difficult integrals required by collocation and Galerkin BEMs, when these are applied to weakly singular and singular 1D integral equations. In particular we show that they can be evaluated very efficiently by using very simple quadrature formulas. Several numerical examples are presented.

* * *