Riv. Mat. Univ. Parma (6) **2*** (1999), 215-233

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On maximal ideals of tame near-rings (**)

1 - Notation and results

For the basic notions of near-ring theory, such as *near-ring, zero-symmetric, N-group, ideal*, we refer to [**20**].

Let N be a zero-symmetric near-ring with identity. Then the unital N-group Γ is called k-tame iff for all $\gamma, x_1, x_2, \ldots, x_k \in \Gamma$ and $n \in N$ there is an element $m \in N$ such that

$$n * (\gamma + x_i) - n * \gamma = m * x_i \qquad \text{for all } i \in \{1, 2, \ldots, k\}.$$

As an example, the near-ring of zero-preserving polynomial functions on an Ω-group is k-tame for every natural number k. A 1-tame N-group is simply called *tame* [**23**]. For a tame N-group Γ, every N-subgroup I is an ideal. The zero-symmetric near-ring N with identity is called *k-tame* iff it has a faithful, k-tame N-group. Our first result gives information on unique maximal ideals of a 2-tame near-ring N.

Theorem 1.1. *Let N be a zero-symmetric near-ring with identity. We assume that N is 2-tame and has the DCC on left ideals. If N has precisely one maximal ideal, say I, and if the quotient N/I is not a ring, then $I = 0$.*

Thus a 2-tame near-ring with a unique maximal ideal and without (nontrivial) quotients that are rings has to be simple. The proof is given in section 4.

Our next goal is to relate the maximal ideals of a tame near-ring to certain an-

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(**) Received October 10, 1999. AMS classification 16 Y 30.

nihilators of sections of its tame N-group. We abbreviate the lattice of ideals of Γ by $\mathsf{Id}\,\Gamma$. In the view of lattice theory, $\mathsf{Id}\,\Gamma$ is a complete algebraic modular lattice (cf. [**16**]). We call an ideal M of Γ *strictly meet irreducible* iff for every subset $\mathcal{X}$ of $\mathsf{Id}\,\Gamma$ with $M = \bigwedge_{X \in \mathcal{X}} X$ we have $M \in \mathcal{X}$. Then the ideal M^+ defined by

$$M^+ := \bigwedge_{X \in \mathsf{Id}\,\Gamma,\, X > M} X$$

satisfies $M^+ > M$, and there are no ideals I with $M < I < M^+$. For $I \in \mathsf{Id}\,\Gamma$, the quotient Γ/I is subdirectly irreducible iff I is strictly meet irreducible. We will use the the subdirect representation theorem in the version that every ideal I of Γ is the meet of a set of strictly meet irreducible ideals ([**16**], Theorem 2.19). For two ideals $A, B \in \mathsf{Id}\,\Gamma$ with $A \leqslant B$, we denote the interval $\{I \in \mathsf{Id}\,\Gamma | A \leqslant I \leqslant B\}$ by $I[A, B]$. We write $A \prec B$ iff $A < B$ and there is no $I \in \mathsf{Id}\,\Gamma$ with $A < I < B$. For two ideals A, B of Γ with $A \leqslant B$, we define the set $B/A := \{b + A \,|\, b \in B\}$. On this set we define addition by $(b_1 + A) + (b_2 + A) := (b_1 + b_2) + A$, and the operation of N by $n * (b + A) := (n * b) + A$. If Γ is a tame N-group and A, B are ideals of Γ with $A \leqslant B$, then the N-group B/A is tame, too. If furthermore $A \prec B$ in $\mathsf{Id}\,\Gamma$, then the N-group B/A has no ideals except $0 = A/A$ and B/A. We write $Ann\,(B/A)$ for the annihilator of B/A, which means

$$Ann\,(B/A) = \{n \in N \,|\, n * B \subseteq A\}.$$

The annihilator $Ann\,(B/A)$ is an ideal of the near-ring $(N, +, \circ)$. Suppose that the tame near-ring N has the DCC on left ideals (called DCCL from now on). Tameness yields that for $A, B \in \mathsf{Id}\,\Gamma$ with $A \prec B$ the near-ring $N/Ann(B/A)$ is 2-primitive. Since 2-primitive near-rings with identity and DCCL are simple, the ideal $Ann(B/A)$ is maximal. It is not so clear whether these annihilators of some section in the ideal lattice of Γ account for all maximal ideals; G. A. Cannon's and L. Kabza's solution [**5**] of [**24**], Problem 5 supports this hope. At least for finite and tame N and Γ it can be proved using [**19**], Lemma 1.4 and [**17**], Theorem 6.27 that for every maximal ideal I there are N-subgroups A, B of Γ such that I is the annihilator of B/A. Actually, from [**19**] and [**4**] one obtains that often the maximal ideals of N are in bijective correspondence to the N-isomorphism classes of minimal sections of the ideal lattice of Γ (cf. Corollary 2.3). We will now give a generalization to near-rings without any finiteness or chain conditions. Let Γ be a faithful N-group. We say that a subnear-ring S is *dense* in N iff for every $n \in N$ and for every finite number $\gamma_1, \gamma_2, \ldots, \gamma_k$ of elements of Γ there is an $s \in S$ with $s * \gamma_i = n * \gamma_i$ for $i = 1, 2, \ldots, k$.

Theorem 1.2. *Let N be a zero-symmetric near-ring with identity, let Γ be a faithful tame N-group, and let I be a maximal ideal of N. Then either I is dense in N or there is a strictly meet irreducible ideal E of Γ such that $I = Ann(E^+/E)$.*

In the case that N has the DCCL, the only subnear-ring of N that is dense in N is N itself. So the alternative that a maximal ideal is dense in N cannot occur in this case. We prove this result in section **3**.

We investigate what these results mean for the near-ring $P_0(V)$ of zero-preserving polynomial functions on an algebra V with group reduct. For the notion *Ω-group*, we refer to [**10**]; polynomial functions are defined, e.g., in [**11**] or [**16**], Definition 4.4. The near-ring operations of $P_0(V)$ are pointwise addition of functions, and functional composition.

Theorem 1.3. *Let V be an Ω-group whose ideal lattice satisfies both the DCC and the ACC. We assume that $P_0(V)$ has the DCCL. If $P_0(V)$ has precisely one maximal ideal, then V is either simple and not abelian, or V is nilpotent.*

Here, being *abelian* and *nilpotent* is defined via the commutator operation used in universal algebra ([**16**], Definition 4.150). In Ω-groups, we work with ideals rather than with congruences. For two ideals A, B of V, the *commutator* $[A, B]$ is the ideal generated by all elements $p(a, b)$, where $a \in A$, $b \in B$ and p is a binary polynomial function on V with $p(v, 0) = p(0, v) = 0$ for all $v \in V$. This ideal product has been defined in [**25**]; it is equal to the commutator studied in universal algebra (cf. [**7**], Theorem 2.6; [**2**]). An Ω-group V is *nilpotent* if the series $D_0 = V$, $D_m = [D_{m-1}, V]$ is 0 from some $k \in \mathbb{N}$ onward. It is *abelian* if $[V, V] = 0$. As an example for our techniques, we study when the inner automorphism near-ring of a group or the near-ring of polynomial functions on a ring with unit have precisely one maximal ideal. This is done in section 5. In section 6, we shall investigate endomorphism near-rings with precisely one maximal ideal; this also provides a description of finite groups G for which $I(G)$ ($A(G)$, $E(G)$) is simple.

2 - Preliminaries

Let N be a near-ring which is tame on Γ. In the sequel, we state two conditions on ideals A, B, C, D with $A \leqslant B$ and $C \leqslant D$ that imply that the N-groups B/A and D/C are N-isomorphic. We say that two N-groups G and H are *N-isomorphic* iff there is a group isomorphism $\varphi\colon G \to H$ with $n * \varphi(g) = \varphi(n * g)$ for all $n \in N$, $g \in G$. For the first condition, we need some concepts of lattice theory. For $A, B, C, D \in \mathsf{Id}\,\Gamma$ with $A \leqslant B$, $C \leqslant D$, we say that the interval $I[A, B]$ *projects up* to $I[C, D]$ iff $A = B \wedge C$ and $D = B \vee C$, and we write $I[A, B] \nearrow I[C, D]$ or $I[C, D] \searrow I[A, B]$. The smallest equivalence relation that contains $\nearrow$ will be abbreviated by $\leftrightsquigarrow$. If $I[A, B] \leftrightsquigarrow I[C, D]$, we say that the two intervals are *projective*.

Proposition 2.1. *Let N be a zero-symmetric near-ring, and let Γ be a tame N-group. Let A, B, C, D be ideals of Γ with $A \leqslant B$, $C \leqslant D$ such that the intervals $I[A, B]$ and $I[C, D]$ are projective. Then the N-groups B/A and D/C are N-isomorphic.*

Proof. We assume $I[A, B] \nearrow I[C, D]$. Then every element in $d \in D$ can be written as $d = b + c$ with $b \in B$, $c \in C$. The mapping $\varphi \colon D/C \to B/A$ with $\varphi((b + c) + C) = b + A$ is an isomorphism. ■

The result is actually well-known as the homomorphism theorem $(B + C)/B \cong_N B/(C \cap B)$ ([**17**], Theorem 2.28).
The next method to find isomorphic N-groups is a generalization of the known fact that for a finite simple ring with unit R, all faithful simple unital R-modules are isomorphic (cf. [**22**], Proposition 2.1.15, p. 154; [**4**], Theorem 4.3). We need the following version:

Proposition 2.2. *Let N be a zero-symmetric near-ring, let I be an ideal of N, and let Γ be an N-group that satisfies $Ann(\Gamma) = I$ and $N * \gamma = \Gamma$ for all $\gamma \in \Gamma$ with $\gamma \neq 0$.*

We assume that we have a left ideal L of N such that $L > I$ and there is no left ideal L' of N with $L > L' > I$.

Then the N-group Γ is N-isomorphic to the N-group L/I.

For $I = 0$, this is [**20**], Theorem 4.56(a).

Proof. Since $L \nleqslant Ann(\Gamma)$, we have elements $l_0 \in L$, $\gamma_0 \in \Gamma$ with $l_0 * \gamma_0 \neq 0$. We define a mapping φ by

$$\varphi \colon L \to \Gamma, \qquad l \mapsto l * \gamma_0.$$

It is easy to see that φ is an N-homomorphism from the N-group L to the N-group Γ. Since $l_0 * \gamma_0 \neq 0$, the assumptions on Γ yield $N * l_0 * \gamma_0 = \Gamma$. Since $N * l_0 \subseteq L$, we get $L * \gamma_0 = \Gamma$, and hence φ is surjective. We take L' to be the kernel of φ, i.e.

$$L' = \{ l \in L \mid l * \gamma_0 = 0 \}.$$

We check that L' is a left ideal of N. Furthermore, every element of $I = Ann(\Gamma)$ lies in L'. So we have

$$I \leqslant L' \leqslant L .$$

Since by the assumptions L covers I, L' has to be either L or I. The element l_0 shows $L' < L$, and so $L' = I$. The homomorphism theorem yields that the N-group $L/\ker \varphi = L/I$ is N-isomorphic to Γ. ■

If two N-groups are N-isomorphic, they have the same annihilators. Sometimes, the converse is true:

Corollary 2.3. *Let N be a zero-symmetric near-ring with identity, and let Γ be a tame N-group. Let A, B, C, D be ideals of Γ with $A \prec B$, $C \prec D$, and $Ann(B/A) = Ann(D/C)$.*

If the near-ring $N/Ann(B/A)$ has a minimal left ideal, the N-groups B/A and D/C are N-isomorphic.

Proof. Since $A \prec B$, and since Γ is a tame N-group, B/A has no N-subgroups. For every $\beta \in B/A$, the set $N * \beta$ forms an N-subgroup of B/A, and since for the identity of id of N we have $id * \beta = \beta$, we see $N * \beta = B/A$ for every non-zero $\beta \in B/A$.

The fact that $N/Ann(B/A)$ has a minimal left ideal gives us the left ideal L required in the assumptions of Proposition 2.2. Now the result follows from this Proposition. ■

We need the following elementary fact about density and DCCL:

Lemma 2.4. *Let G be a near-ring that is faithful on the G-group Γ, and let F be a subnear-ring of G which is dense in G. If one of the two near-rings F and G has the DCCL, then the near-rings are equal.*

Proof. For simplicity, we assume that both near-rings are subnear-rings of the near-ring $M(\Gamma)$ of all mappings on Γ. For every subnear-ring R of $M(\Gamma)$, and for every natural number $n \in \mathbb{N}$, we introduce the near-ring $\mathsf{L}_n R := \{ l \in M(\Gamma) \mid \forall S \subseteq G \colon |S| \leqslant n \Rightarrow \exists r \in R : r|_S = l|_S \}$ and let $\mathsf{L}R := \bigcap_{n \in \mathbb{N}} \mathsf{L}_n R$. The fact that F is dense in G can now be written as

$$F \subseteq G \subseteq \mathsf{L}F\,. \tag{2.1}$$

We say that $D \subseteq \Gamma$ is a base of equality for R iff every function in R that is zero at all elements of D is zero everywhere on Γ. For every subset C of Γ the set $\{r \in R \mid r(c) = 0 \text{ for all } c \in C\}$ is a left ideal of R. If no finite base of equality for R exists, then we can construct an infinite descending chain of such left ideals. So every subnear-ring of $M(\Gamma)$ with DCCL has a finite base of equality.

If we assume that F has the DCCL, then it has a finite base of equality. By [**1**], Proposition 2, it follows that $\mathsf{L}F = F$, and therefore the inclusions of (2.1) yield $F = G$.

If G has the DCCL, then we show $G \subseteq F$ as follows: Take $g \in G$, and let D be the finite base of equality for G. By density, we have $f \in F$ such that $f|_D = g|_D$. Both functions lie in G and agree on D, and so $f = g$. This shows $g \in F$. ■

3 - Maximal ideals are dense or annihilators

Proof of Theorem 1.2. We assume that I is not dense in N. Then there is a a finite subset D of Γ and there is an element $n \in N$ such that there is no $i \in I$ that satisfies $\forall d \in D : i * d = n * d$.

We take a set D of minimal cardinality with this property, and note that D cannot be the empty set. We choose $d \in D$. We define two sets S and B by

$$S := \{ i * d \mid i \in I,\ i * \delta = 0 \ \text{ for all } \ \delta \in D - \{d\} \}$$

$$B := \{ n * d \mid n \in N,\ n * \delta = 0 \ \text{ for all } \ \delta \in D - \{d\} \}.$$

Let $n_0 \in N$ be such that there is no $i \in I$ with $n_0 * \delta = i * \delta$ for all $\delta \in D$. Since D is minimal, there is an element $i_0 \in I$ such that $n_0 * \delta = i_0 * \delta$ for all $\delta \in D - \{d\}$. The element $(n_0 - i_0) * d$ obviously lies in B. It does not lie in S: suppose it did. Then we have an element $i_1 \in I$ such that $(n_0 - i_0) * d = i_1 * d$ and $i_1 * \delta = 0$ for all $\delta \in D - \{d\}$. Thus we have $n_0 * \delta = (i_1 + i_0) * \delta$ for all $\delta \in D$, a contradiction to the choice of n_0. Therefore S is a subset of B, but not equal to B.

Since I is an ideal of N, both sets S and B are N-subgroups of Γ, and hence, by tameness, ideals. We will now study the N-group B/S. We know that $i \circ n \in I$ for all $i \in I$, $n \in N$. So $i * b \in S$ for all $i \in I$, $b \in B$. Therefore the ideal I satisfies

$$I \leq Ann\,(B/S)\,.$$

Let $\mathcal{X}$ be the set of all strictly meet irreducible ideals E of Γ with the property

$E \geqslant S$. By [**16**], Theorem 2.19, the intersection $\bigwedge_{X \in \mathcal{X}} X$ is equal to S. Since $B \nleqslant S$, there is at least one element M in $\mathcal{X}$ with $M \ngeqslant B$. So we have found a meet irreducible ideal M of Γ that satisfies $M \geqslant S$ and $M \ngeqslant B$. So we have

$$S \leqslant M \wedge B \leqslant M^+ \wedge B \leqslant B .$$

Now we show that the interval $I[M \wedge B, M^+ \wedge B]$ projects up to $I[M, M^+]$ in the lattice $\mathsf{Id}\,\Gamma$. To this end, we have to prove

$$(M^+ \wedge B) \wedge M = M \wedge B \tag{3.1}$$

and

$$(M^+ \wedge B) \vee M = M^+ . \tag{3.2}$$

Property (3.1) is true since $M^+ \wedge M = M$. For (3.2), we observe that $(M^+ \wedge B) \vee M \leqslant M^+$ and $(M^+ \wedge B) \vee M \geqslant M$. Since $M < M^+$, (3.2) can only fail if $(M^+ \wedge B) \vee M = M$. But we show

$$(M^+ \wedge B) \vee M > M . \tag{3.3}$$

We obviously have $\geqslant$, so suppose that we have equality in (3.3). Since the lattice $\mathsf{Id}\,\Gamma$ is modular, $(M^+ \wedge B) \vee M = M^+ \wedge (B \vee M)$, and so we obtain

$$M^+ \wedge (B \vee M) = M .$$

Since M is meet irreducible, this implies $B \vee M = M$, and thus $B \leqslant M$ which contradicts the fact that M was chosen such that $B \nleqslant M$. This proves (3.3). The property (3.3) also implies $M \wedge B < M^+ \wedge B$.

Since $S \leqslant M \wedge B \leqslant M^+ \wedge B \leqslant B$, we know $Ann\,((M^+ \wedge B)/(M \wedge B)) \geqslant Ann\,(B/S)$. But already $Ann\,(B/S)$ contains the maximal ideal I. Since

$$Ann\,((M^+ \wedge B)/(M \wedge B))$$

does not contain the identity function, it cannot be equal to N, so we get

$$Ann\,((M^+ \wedge B)/(M \wedge B)) = I .$$

By Proposition 2.1, the N-groups $(M^+ \wedge B)/(M \wedge B)$ and M^+/M are N-isomorphic. Therefore they have the same annihilator, which gives $Ann\,(M^+/M) = I$. Thus M is the meet irreduible ideal E that we are looking for. ■

4 - 2-tame near-rings with unique maximal ideals

The key to proving Theorem 1.1 is the observation that for two different strictly meet irreducible ideals A, B of the 2-tame module Γ the N-modules A^+/A and B^+/B can only be isomorphic if $N/Ann(A^+/A)$ is a ring. This observation is motivated by [8].

Proposition 4.1. *Let N be a zero-symmetric near-ring with identity, let Γ be a 2-tame N-group, and let A and B be two meet irreducible elements of $\mathrm{Id}\,\Gamma$ such that $A \neq B$. We assume that the N-groups A^+/A and B^+/B are N-isomorphic. Then $N/Ann(A^+/A)$ is a ring.*

Proof. Since $A \neq B$, we either have $A \nleq B$ or $A \ngeq B$. Without loss of generality we assume $A \ngeq B$. Then we have $A \vee B > A$, and so $A \vee B \geq A^+$. We fix $a_1^+, a_2^+ \in A^+$ and $n \in N$. We compute

$$n * (a_1^+ + a_2^+) - n * a_1^+ - n * a_2^+ . \tag{4.1}$$

Since $A^+ \leq A \vee B$, there are $a_1 \in A$, $b_1 \in B$ such that $a_1^+ = a_1 + b_1$. Furthermore, the N-groups A^+/A and B^+/B are N-isomorphic. Let φ be the isomorphism, and let $b_2^+ \in B^+$ be such that $b_2^+ + B = \varphi(a_2^+ + A)$. We will now prove

$$\text{For all } r \in N: (r * b_2^+ \in B \Rightarrow r * a_2^+ \in A). \tag{4.2}$$

For proving (4.2), we assume $r * b_2^+ \in B$. Then in the N-group B^+/B, we have $r * (b_2^+ + B) = 0 + B$. Thus we know $0 + B = r * (b_2^+ + B) = r * (\varphi(a_2^+ + A)) = \varphi(r * (a_2^+ + A))$. Since φ is an isomorphism, $\varphi(x)$ can only be $0 + B$ if the argument x is equal to $0 + A$. So we have

$$r * (a_2^+ + A) = 0 + A .$$

This implies $r * a_2^+ \in A$, and thus finishes the proof of (4.2).

Using the equality $a_1^+ = a_1 + b_1$, the expression (4.1) becomes

$$n * (a_1 + b_1 + a_2^+) - n * (a_1 + b_1) - n * a_2^+ .$$

Modulo A, this is expression is congruent to

$$n * (b_1 + a_2^+) - n * b_1 - n * a_2^+ .$$

Since N is 2-tame on Γ, there is an element $m \in N$ such that

$$m * x = n * (b_1 + x) - n * b_1 \qquad \text{for } x \in \{a_2^+, b_2^+\} . \tag{4.3}$$

Thus we have

$$n * (b_1 + a_2^+) - n * b_1 - n * a_2^+ = m * a_2^+ - n * a_2^+ = (m-n) * a_2^+ .$$

We show

$$(m-n) * a_2^+ \in A . \tag{4.4}$$

By (4.2), it is sufficient to show $(m-n) * b_2^+ \in B$. By (4.3), we have $(m-n) * b_2^+ = m * b_2^+ - n * b_2^+ = n * (b_1 + b_2^+) - n * b_1 - n * b_2^+$. This is congruent to $n * b_2^+ - n * b_2^+ = 0$ modulo B, thus $(m-n) * b_2^+$ lies in B; this proves (4.4). Altogether, we have shown that $n * (a_1^+ + a_2^+) - n * a_1^+ - n * a_2^+$ lies in A.

Taking n to be the identity, we obtain that the group A^+/A has abelian addition; all $n \in N$, $a_1^+, a_2^+ \in A^+$ satisfy

$$n * ((a_1^+ + A) + (a_2^+ + A)) = n * (a_1^+ + A) + n * (a_2^+ + A) .$$

Thus $N/Ann(A^+/A)$ is a ring. ∎

Proposition 4.2. *Let N be a zero-symmetric near-ring with identity and DCCL. We assume that N has a unique maximal ideal, say I. Let Γ be a tame N-group, and let A, B, C, D be ideals of Γ with $A \prec B$, $C \prec D$. Then the N-groups B/A and D/C are N-isomorphic.*

Proof. The annihilator $Ann(B/A)$ is contained in a maximal ideal, and since the only maximal ideal is I, we have

$$Ann(B/A) \leqslant I . \tag{4.5}$$

To prove that equality holds in (4.5), we show that $Ann(B/A)$ is a maximal ideal of I. We know that the near-ring $N/Ann(B/A)$ is 2-primitive on B/A. By [**4**] (or [**18**], Corollary 3), every 2-primitive near-ring with DCCL is simple. The quotient $N/Ann(B/A)$ being simple, $Ann(B/A)$ is a maximal ideal of N and hence equal to I. In the same way, we obtain $Ann(D/C) = I$, and so by Corollary 2.3, the N-groups B/A and D/C are N-isomorphic. ∎

We can now describe 2-tame near-rings with DCCL, a unique maximal ideal and no ring quotients.

Proof of Theorem 1.1. Let Γ be a faithful 2-tame N-module, and let E be a strictly meet irreducible ideal of Γ.

Now let A be any other strictly meet-irreducible ideal of Γ. By Proposition 4.2, the N-groups A^+/A and E^+/E are N-isomorphic. We know that $Ann(E^+/E)$ is a maximal ideal of N, and hence equal to I. Since, by assumption, $N/I = N/Ann(E^+/E)$ is not a ring, Proposition 4.1 yields $A = E$.

So E is the only strictly meet-irreducible ideal in $\mathrm{Id}\,\Gamma$. Every ideal of Γ is the intersection of a set of strictly meet-irreducible ideals. But since there is only E to form such an intersection, we get Γ and 0 as the only ideals of Γ. Therefore $E = 0$ and $E^+ = \Gamma$, and thus $I = Ann(E^+/E)$ is equal to $Ann(\Gamma)$. But since Γ is faithful, we obtain $Ann(\Gamma) = 0$, thus $I = 0$, which we had to prove. ■

In particular, we obtain that N is primitive on its N-group Γ, and thus by [**23**], Theorem 8.4, N is dense in $M_0(\Gamma)$. Since N satisfies the DCCL, Lemma 2.4 tells that it is equal to $M_0(\Gamma)$, and again by DCCL, Γ, and thus N, are finite. The finiteness of a near-ring satisfying the assumptions of Theorem 1.1 also follows from [**23**], Theorem 8.5.

5 - Near-rings of polynomials with a unique maximal ideal

We will now see what Theorem 1.1 yields for the near-rings of zero-symmetric polynomial functions on expansions of groups. It is interesting to observe that near-rings such as $E(G)$ can be viewed as near-rings of zero-symmetric polynomial functions. For obtaining $E(G)$, we take V to be the Ω-group $V := (G, +, \{e: G \to G \mid e \text{ is an endomorphism of } G\})$. Then $P_0(V) = E(G)$. Similarly, all *compatible* (cf. [**23**], p. 283) near-rings can be obtained.

Let A, B, C, D be ideals of the Ω-group V with $A \leqslant B$, $C \leqslant D$. The interval $I[A, B]$ is called *abelian* iff $[B, B] \leqslant A$. The *centralizer* of B modulo A, abbreviated by $(A : B)$, is the largest ideal C of V such that $[C, B] \leqslant A$. We will need the following properties of the commutator operation:

Proposition 5.1. *Let A, B, C be ideals of the Ω-group V. Then we have*

1. $[A \vee B, C] = [A, C] \vee [B, C]$
2. $[A, B] = [B, A]$
3. $[A, B] \leqslant A \wedge B$
4. *Let $A \leqslant B$. Then an element $z \in V$ lies in $(A : B)$ iff $s(z, b) \in A$ for all $b \in B$ and for all binary polynomial functions s on V that satisfy $\forall v \in V : s(v, 0) = s(0, v) = 0$.*

The first three properties are well known in commutator theory [**6**]; number (4) follows from [**25**], Proposition 9.5.

Proof. We call a binary polynomial function s a *commutator polynomial* iff $s(v, 0) = s(0, v) = 0$ for all $v \in V$. For (1), we only show $\leqslant$. For $a \in A$, $b \in B$, $c \in C$ and a commutator polynomial s, we have $s(a+b, c) = s(a+b, c) - s(b, c) + s(b, c)$. Considering $s_1(x, y) := s(x+b, y) - s(b, y)$, we see $s(a+b, c) - s(b, c) = s_1(a, c) \in [A, C]$. The second term $s(b, c)$ obviously lies in $[B, C]$.

For (4), we are done if we show that the set

$$Z := \{z \in V \mid s(z, b) \in A \text{ for all } b \in B \text{ and all commutator polynomials } s\}$$

is an ideal of V. To this end, let z be in Z, and let $p \in P_0(V)$. We want to show that $p(z)$ is in Z. We fix $b \in B$ and a commutator polynomial s, and compute $s(p(z), b)$. Since $z \in Z$, we know that $t(z, b)$ lies in A, where $t(x, y) = s(p(x), y)$. Thus $s(p(z), b) \in A$. For showing that Z is closed under addition, let $z_1, z_2 \in Z$. We write $s(z_1 + z_2, b)$ as $s(z_1 + z_2, b) - s(z_2, b) + s(z_2, b)$. Defining $t(x, y) := t(x + z_2, y) - t(z_2, y)$, we see that $s(z_1 + z_2, b) - s(z_2, b)$ lies in A; since $s(z_2, b)$ also lies in A, we get $s(z_1 + z_2, b) \in A$. Hence Z is also closed under addition, and therefore an ideal. ■

Some of the properties given in [**6**] that hold if $I[A, B]$ is projective to $I[C, D]$ still hold if we assume the weaker fact that the $P_0(V)$-groups B/A and D/C are isomorphic.

Proposition 5.2. *Let V be an Ω-group, and let A, B, C, D be ideals of V with $A \leqslant B$, $C \leqslant D$ such that the $P_0(V)$-groups B/A and D/C are $P_0(V)$-isomorphic. Then $(A : B) = (C : D)$.*

We remark that this has been proved in [**25**], Theorem 12.1.

Proof. We show $(C : D) \leqslant (A : B)$. Let z be an element of $(C : D)$. We fix a binary polynomial s of the Ω-group V with $s(v, 0) = s(0, v) = 0$ for all $v \in V$, and we also fix $b \in B$. We compute $s(z, b)$. Since $[(C : D), D] \leqslant C$, the polynomial $p(x) := s(z, x)$ has the property $p(D) \subseteq C$. Since D/C and B/A are $P_0(V)$-isomorphic, we have $p(B) \subseteq A$. This implies $p(b) \in A$, which means $s(z, b) \in A$. So, by Proposition 5.1 (4), the element z lies in the centralizer $(A : B)$, and we have $(C : D) \leqslant (A : B)$. Interchanging the roles of A, B with those of C, D, we obtain the required equality. ■

Proposition 5.3. *Let V be a Ω-group, and let A, B, C, D be ideals of V with $A \prec B$, $C \prec D$ such that the $P_0(V)$-groups B/A and D/C are $P_0(V)$-isomorphic. If $I[A, B]$ is abelian, then $I[C, D]$ is abelian.*

Proof. We assume $[D, D] \nleq C$. Then we have $d_1, d_2 \in D$ and a binary polynomial s of the Ω-group V with $s(v, 0) = s(0, v) = 0$ for all $v \in V$ and $s(d_1, d_2) \notin C$. Since $C \prec D$, the $P_0(V)$-group D/C has only two $P_0(V)$-subgroups, namely $0 = C/C$ and D/C. Therefore, $P_0(V) * (d_1 + C) = D/C$. Hence we have a polynomial function $p \in P_0(V)$ such that $p(d_1) \in d_2 + C$. We consider the polynomial $t(x) := s(x, p(x))$. We know that $t(d_1) = s(d_1, p(d_1))$ is congruent to $s(d_1, d_2)$ modulo C; thus we get $t(d_1) \notin C$. So we have $t(D) \not\subseteq C$. Since the $P_0(V)$-groups D/C and B/A are isomorphic, we have $t(B) \not\subseteq A$. Therefore there is an element $b \in B$ such that $t(b) = s(b, p(b)) \notin A$. But $s'(x, y) := s(x, p(y))$ is 0 whenever one of its arguments is 0; so $t(b)$ lies in $[B, B]$. This shows $[B, B] \nleq A$, and thus $I[A, B]$ is not abelian. ■

If $P_0(V)$ has only one maximal ideal, then all minimal sections in the ideal lattice of V have to be $P_0(V)$-isomorphic. As suggested by [8] and Proposition 4.1, being isomorphic is particularly hard for non-abelian sections.

Proposition 5.4. *Let V be an Ω-group. We assume that $P_0(V)$ has the DCCL and precisely one maximal ideal. If there are ideals C, D of V such that $C \prec D$ and $I[C, D]$ is not abelian, then V is simple.*

Proof. Let A and B be two strictly meet irreducible ideals of V. By Proposition 4.2, A^+/A and B^+/B are $P_0(V)$-isomorphic to D/C. Thus, by Proposition 5.3, the interval $I[A, A^+]$ is not abelian. Hence we have $(A : A^+) \ngeq A^+$. Since A is meet irreducible, each ideal E with $E \geqslant A$ satisfies either $E = A$ or $E \geqslant A^+$. Hence we have $(A : A^+) = A$. In the same way we obtain $B = (B : B^+)$. So Proposition 5.2 yields $A = (A : A^+) = (B : B^+) = B$. But if all strictly meet irreducible ideals of V are equal, then V is simple. ■

This result is particularly suitable for treating polynomial functions on rings with unit: every ring R with unit has a maximal ideal. On a ring $(R, +, \cdot)$ with unit 1, we consider the mapping $s(x, y) := x \cdot y$. Since $s(1, 1) = 1$, we have $[R, R] = R$. So for a maximal ideal C of the ring R, we know that the interval $I[C, R]$ is not abelian.

Corollary 5.5. *Let R be a ring with unit. If $P_0(R)$ has the DCCL and precisely one maximal ideal, then R is simple.*

Proof. Let C be a maximal ideal of R. Since R/C is a ring with unit, it has nonzero multiplication, and thus the interval $I[C, R]$ is not abelian. By Proposition 5.4 we conclude that R is a simple ring. ■

If for a finite Ω-group V the near-ring $P_0(V)$ has only one maximal ideal, then V is simple or nilpotent.

Proposition 5.6. *Let V be an Ω-group whose ideal lattice satisfies both the DCC and the ACC. We assume that $P_0(V)$ has the DCCL. If $P_0(V)$ has precisely one maximal ideal, then V is either simple and not abelian, or V is nilpotent.*

Proof. Let I be the maximal ideal of $P_0(V)$, and let A be a meet irreducible ideal of V. If $I[A, A^+]$ is not abelian, then Proposition 5.4 yields that V is simple. Since then $A = 0$ and $A^+ = V$, it follows that $[V, V] \neq 0$, and thus V is not abelian.

Now we assume that $I[A, A^+]$ is abelian. We show that then the commutator operation satisfies $[V, D] < D$ for every ideal D of V. Therefore we have $[V, V] > [V,[V, V]] > [V,[V,[V, V]]] > \dots$, hence by the DCC on ideals of V, this chain reaches 0, making V nilpotent. Seeking a contradiction, we suppose that we have an ideal D with $[V, D] = D$. Let E be a subcover of D in $\mathsf{Id}\, V$, i.e., an ideal of V with $E \prec D$. Since $[V, D] \nleq E$, we have $(E : D) < V$. Let M be a maximal ideal of V. Then by Proposition 4.2, V/M and D/E are $P_0(V)$-isomorphic. This shows that the centralizer $(M : V)$ is equal to $(E : D)$, and thus we have $(M : V) < V$. So the interval $I[M, V]$ is not abelian. But by Proposition 4.2, the $P_0(V)$-groups V/M and A^+/A are $P_0(V)$-isomorphic, and so by Proposition 5.3, the interval $I[A, A^+]$ is not abelian, a contradiction. ■

An important fact in this proof is that all B/A, where A, B are ideals of Γ with $A \prec B$, are $P_0(V)$-isomorphic. If $I[A, B]$ is abelian, then using a maximal ideal of V, one concludes that for all ideals with $A \prec B$ the centalizer $(A : B)$ is equal to V. If $P_0(V)$ has the DCC on ideals, Theorem 1.3 now also follows from [**25**], Theorem 16.8.

But what if $P_0(V)$ is simple?

Proposition 5.7. *Let V be an Ω-group that has at least one maximal ideal. We assume that $P_0(V)$ has the DCCL. If $P_0(V)$ is simple, then V is either simple or abelian.*

Proof. Let M be a maximal ideal of the Ω-group V. If the interval $I[M, V]$ is not abelian, then Proposition 5.4 yields that V is simple.

So we assume $[V, V] \leqslant M$. Since $P_0(V)$ is simple, its ideal $Ann\,(V/M)$ is either $P_0(V)$ or 0, and since the identity mapping on V does not lie in $Ann\,(V/M)$, we have $Ann\,(V/M) = 0$. Now we show $[V, V] = 0$. To this end, let s be a binary polynomial function on V with $s(v, 0) = s(0, v) = 0$ for all $v \in V$. We fix $w_1, w_2 \in V$. Since $[V, V] \leq M$, we know that the polynomial $t(x) := s(w_1, x)$ has the property $t(V) \subseteq M$, and thus t lies in $Ann(V/M)$. Since $Ann(V/M) = 0$, we get $t = 0$, and thus $s(w_1, w_2) = t(w_2) = 0$, which proves $[V, V] = 0$. ■

Slightly more information can be obtained for finite Ω-groups V:

Proposition 5.8. *Let V be a finite Ω-group. If $P_0(V)$ has precisely one maximal ideal then V is either simple and not abelian, or V is nilpotent and the cardinality of V is a prime power.*

Proof. Let $M_0, M_1, \dots, M_r$ be ideals of V with $V = M_0 > M_1 > \dots > M_r = 0$ such that for all i, M_i is a subcover of M_{i-1} in the ideal lattice of V.

If the interval $I[0, M_{r-1}]$ is not abelian, then Proposition 5.4 yields that V is simple; since then $M_{r-1} = V$, the Ω-group V is non-abelian.

Hence we assume $[M_{r-1}, M_{r-1}] = 0$. Since then V cannot be simple and nonabelian, Proposition 5.6 tells that V is nilpotent. We will now prove that its cardinality is a prime power. Since M_{r-1} has no $P_0(V)$-subgroups, $R := P_0(V)/Ann(M_{r-1})$ is 2-primitive on M_{r-1}. From $[M_{r-1}M_{r-1}] = 0$, we obtain that $a + b = b + a$, and $p(a + b) = p(a) + p(b)$ for all for all $a, b \in M_{r-1}$, $p \in P_0(V)$. (To see this, define $s(x, y) := p(x + y) - p(y) - p(x)$, and observe that s is zero whenever one of its arguments is 0. Thus $s(a, b)$ lies in $[M_{r-1}M_{r-1}] = 0$.) Therefore R is a primitive ring, and by Jacobsons's density theorem ([**9**], p. 28), M_{r-1} can be seen as a vector space over the finite field $D := End_R(M_{r-1})$, which shows that M_{r-1} has p^k elements for some $p, k \in \mathbb{N}$, p prime. By Proposition 4.2, every quotient M_i/M_{i+1} has p^k elements, and thus V has p^{kr} elements. ■

Proposition 5.7 can also be sharpened if V is finite.

Proposition 5.9. *Let V be a finite Ω-group. If $P_0(V)$ is simple, then V is simple and not abelian, or $(V, +)$ is the additive group of a finite vector space.*

Proof. We assume that V is not both nonabelian and simple. We take a minimal ideal M_{r-1} of V. As in the proof of Proposition 5.8, we see that M_{r-1} is the additive group of a vector-space, and hence of exponent p for a prime number p. Therefore the polynomial function $x \mapsto \underbrace{x + x + \dots + x}_{p \text{ times}}$ lies in $Ann(M_{r-1})$. By the simplicity of $P_0(V)$, we have $Ann(M_{r-1}) = 0$, and thus $(V, +)$ is of exponent p. Since, by Proposition 5.7, the Ω-group V is abelian, its addition satisfies $v_1 + v_2$

$= v_2 + v_1$ for all $v_1, v_2 \in V$, and thus $(V, +)$ is abelian. Hence $(V, +)$ is elementary abelian, which whe had to prove. ■

6 - Endomorphism near-rings with unique maximal ideals

We will now study for which finite groups G the near-rings $I(G)$, $A(G)$, and $E(G)$ have precisely one maximal ideal, and when they are simple. In [**13**], it is proved that if $I(G)$ $(A(G), E(G))$ is *local* [**15**], then G is a p-group. It is known that a local near-ring N has precisely one maximal ideal, say I, and the quotient N/I is a near-field. Our contribution is that we do not put any restrictions on the maximal ideal; nevertheless, several steps work as in [**13**]. We can then also tell for which finite groups G the near-ring $I(G)$ $(A(G), E(G))$ is simple.

Although the results of this section follow from the results of section 5, the considerable interest that these near-rings have received justifies to give proofs that use the language of group theory.

Proposition 6.1. *Let G be a group with a finite principal series. We assume that $I(G)$ has the DCCL.*

1) *If $I(G)$ has precisely one maximal ideal, then G is either simple or a p-group.*

2) *If $I(G)$ is simple, then G is simple or an abelian group of prime exponent.*

We remark that S. D. Scott has proved that if $I(G)$ has the DCCL, it is finite ([**23**], Theorem 10.4).

Proof. We choose a maximal normal subgroup M_1 of G. For proving item (1), we assume first that G/M_1 is not abelian. Then by Proposition 5.4 the group G is simple.

If G/M_1 is abelian, then being simple it has p elements for some prime number p. Let $G = M_0 > M_1 > M_2 > M_3 > \ldots > M_r = 0$ be a principal series of G. Since $I(G)$ has only one maximal ideal, Proposition 4.2 gives that all M_i/M_{i+1} have p elements, so G has p^r elements, and is thus a p-group. This finishes the proof of item (1).

Now let $I(G)$ be simple. If G is not simple, then by (1) G is nilpotent, hence the quotient G/M_1 is isomorphic to the cyclic group of order p for some prime p. For proving that G has exponent p, we consider the polynomial function $p \cdot id$ that maps x to $\underbrace{x + x + \ldots + x}_{p \text{ times}}$. This function lies in $Ann\,(G/M_1)$, and since $I(G)$ is simple, we have $Ann(G/M_1) = 0$. So $p \cdot id$ is the zero function, and thus G is of exponent p.

For showing that G is abelian, we show that every element of G lies in the centre. To this end, we fix $g \in G$, and consider the mapping $c(x) := -g - x + g + x$. It lies in $Ann(G/M_1)$, and thus c is the zero function. So g lies in the centre of G. ■

On the other hand, [**13**], Corollary 3.3 tells that for a finite p-group G, the near-ring $I(G)$ is even local; in particular it has a unique maximal ideal. Also if G is finite, simple and not abelian, then $I(G)$ is simple. For sake of completeness, we summarize the argument that for a p-group G the near-ring $I(G)$ has a unique maximal ideal: In G, all sections B/A, where A and B are normal subgroups of G with $A < B$, are $I(G)$-isomorphic; thus they all have the same annihilator $Ann(B/A)$. Since the annihilators account for all maximal ideals, there can be only one maximal ideal.

We will now switch to the near-rings $A(G)$ and $E(G)$. The near-ring $A(G)$ is exactly the near-ring of zero-preserving polynomial functions of the Ω-group $G_A := (G, +, \{a \colon G \to G \mid a \in Aut\, G\})$. The ideals of G_A are precisely the characteristic subgroups of G.

Proposition 6.2. *Let G be a finite group.*

1) *If $A(G)$ has precisely one maximal ideal, then G is either characteristically simple or a p-group.*

2) *If $A(G)$ is simple, then G is characteristically simple.*

Proof. Let $G = C_0 > C_1 > \ldots > C_r = 0$ be a principal characteristic series ([**21**], p. 63) of G. We take $H := C_{r-1}$ to be a minimal characteristic subgroup of G. By [**21**], 1.5.6, H is characteristically simple. If H is not an abelian group, then by Proposition 5.4, we get $G = H$, and thus G is characteristically simple. If H is an abelian group, then by [**21**], 3.3.15 there is a prime p such that H is elementary abelian of exponent p. By Proposition 4.2, all sections C_i/C_{i+1} have the same number of elements, and thus G is a p-group. This proves (1).

For (2), we first assume that H is not abelian. As above, we see that this implies that $G = H$, and G is thus characteristically simple.

If H is abelian, then by Proposition 4.2, the quotient G/C_1 is isomorphic to H, an hence elementary abelian. As in the proof of Proposition 6.1, we see that G is abelian of prime exponent, and therefore characteristically simple. ■

We view the near-ring $E(G)$ as the near-ring of zero-preserving polynomial functions of the Ω-group $G_E := (G, +, \{e: G \to G \mid e \in End\, G\})$. The ideals of G_E are precisely the fully-invariant subgroups of G.

Proposition 6.3. *Let G be a finite group.*

1) *If $E(G)$ has precisely one maximal ideal, then G is either invariantly simple or a p-group.*

2) *If $E(G)$ is simple, then G is invariantly simple.*

Proof. Let $G = I_0 > I_1 > \ldots > I_r = 0$ be a principal fully-invariant series ([**21**], p. 63) of G. We take $H := C_{r-1}$ to be a minimal fully-invariant subgroup of G. By [**21**], 1.5.6, H is invariantly simple. If H is not an abelian group, then by Proposition 5.4, we get $G = H$, and thus G is invariantly simple. If H is an abelian group, then (since for every $n \in \mathbb{N}$ the subgroup $n \cdot H$ is fully invariant), there is a prime p such that H is elementary abelian of exponent p. Now the remainder of the proof is an adaption of the last seven lines of the proof of Proposition 6.2. ■

Let us make some remarks about the reversions of Proposition 6.3 and Proposition 6.2. For every finite characteristically simple group G, the near-ring $A(G)$ is either a primitive ring or a 2-primitive near-ring, and thus simple ([**12**], 3.6). In the same way, $E(G)$ is simple for every finite invariantly simple group G. But there is a p-group G such that $A(G)$ has more than one maximal ideal. As an example, we consider the quaternion group Q_8 with principal characteristic series $Q_8 > \Phi(Q_8) \cong \mathbb{Z}_2 > 0$. Now in $A(Q_8)$, the annihilators $Ann\,(Q_8/\Phi(Q_8))$ and $Ann\,(\mathbb{Z}_2/0)$ account for two different maximal ideals. Since $A(Q_8) = E(Q_8)$ [**14**], this also gives an example of a non-abelian p-group G for which $E(G)$ has more than one maximal ideal.

On the other hand, $A(D_8)$, where D_8 is the dihedral group with eight elements, has precisely one maximal ideal. This follows from the fact that D_8 has the principal characteristic series $D_8 > \mathbb{Z}_4 > \mathbb{Z}_2 > 0$, with all the quotients of size 2, and (hence) $A(D_8)$-isomorphic. In fact, $A(D_8)$ is even local [**13**].

Acknowledgements. The author wishes to thank S.D.Scott for interesting discussions. Examples have been checked using SONATA [**3**].

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Abstract

Let N be a zero-symmetric near-ring with identity, and let Γ be a faithful tame N-group. We prove that every maximal ideal of N is either dense in N or equal to the annihilator of a section in the submodule lattice of Γ. We study the case that there is precisely one maximal ideal: often this maximal ideal has to be 0. As a consequence, we see that if the near-ring of zero-preserving polynomial functions on a finite Ω-group V has precisely one maximal ideal, then V is either simple or nilpotent. Finally, we look at groups G for which the near-rings $I(G)$, $A(G)$, and $E(G)$ have precisely one maximal ideal, or are even simple.

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