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Regularity results for some degenerate parabolic equation (**)

1 - Introduction

Let $d \in \mathbb{N}$ and set $H = \mathbb{R}^d$. We denote by $|\cdot|$ the norm, and by $\langle \cdot, \cdot \rangle$ the inner product in H. By $L(H)$ we mean the algebra of all linear operators from H into itself, and by $L_+(H)$ the subset of $L(H)$ of all symmetric nonnegative linear operators.

Moreover for any function $\varphi: H \to \mathbb{R}$, $D\varphi$ is its gradient and D_i, $i = 1, \ldots, d$, is its partial derivative with respect to x_i.

We are concerned with the parabolic equation

$$\begin{cases} D_t u(t, x) = Nu(t, \cdot)(x), \ x \in H, \ t > 0, \\ u(0, x) = \varphi(x), \ x \in H, \end{cases} \tag{1.1}$$

where N is the differential operator

$$N\varphi(x) = \frac{1}{2} \mathrm{Tr}\,[CD^2 \varphi(x)] + \langle Ax + F(x), D\varphi(x) \rangle, \qquad x \in H \, . \tag{1.2}$$

We recall that a *strict* solution of (1.1) is a function $u: [0, +\infty) \times H \to H$, $(t, x) \to u(t, x)$ that is continuously differentiable with respect to t, twice continuously differentiable with respect to x and fulfills (1.1). We shall assume,

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Hypothesis 1.1. (i) $A \in L(H)$, $C \in L_+(H)$.

(ii) $F \in C_b^2(H; H)$ (1).

The following result is well known, see e.g. [**6**], however we shall give a sketch of the proof for the reader's convenience.

Proposition 1.2. *Assume that Hypothesis* 1.1 *holds. Then for all* $\varphi \in C_b^2(H)$, *problem* (1.1) *has a unique strict solution* u. u *is given by the formula*

$$u(t, x) = \mathbb{E}[\varphi(X(t, x))], \qquad t \geq 0, \qquad x \in H, \tag{1.3}$$

where $X(\cdot, x)$ *is the solution of the differential stochastic equation*

$$\begin{cases} dX(t) = (AX(t) + F(X(t))\, dt + C^{1/2}\, dW(t)) \\ X(0) = x, \end{cases} \tag{1.4}$$

W *is a standard Brownian motion in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values on* $\mathbb{R}^d$, *and* $\mathbb{E}$ *means expectation.*

Proof. The differential stochastic equation (1.4) can be solved by a fixed point argument. Moreover, since $F \in C_b^2(H; H)$, $X(t, x)$ is twice differentiable in x, and the partial derivatives:

$$X_x(t, x) \cdot h = \eta^h(t, x), \qquad t \geq 0, \quad x, h \in H,$$

and

$$X_{xx}(t, x)(h, h) = \zeta^h(t, x), \qquad t \geq 0, \quad x, h \in H,$$

(1) If H and K are Hilbert spaces we denote by $C_b(H; K)$ (resp. $B_b(H; K)$) the Banach space of all uniformly continuous (resp. Borel) and bounded mappings from H into K, endowed with the sup norm $\|\cdot\|_0$. Moreover, for any $k \in \mathbb{N}$, $C_b^k(H; K)$ will represent the Banach space of all mappings from H into K, that are uniformly continuous and bounded together with their Fréchet derivatives of order less or equal to k endowed with their natural norm $\|\cdot\|_k$. Finally we set $C_b^\infty(H; K) = \bigcap_{k=1}^{\infty} C_b^k(H; K)$. If $K = \mathbb{R}$ we set $C_b(H; K) = C_b(H)$ (resp. $B_b(H; K) = B_b(H)$) and $C_b^k(H; K) = C_b^k(H)$, $C_b^\infty(H) = C_b^\infty(H; K)$.

are the solutions to the following differential stochastic equations

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}t}\eta^h(t,x) = (A + DF(X(t,x))\,\eta^h(t,x)\\ \eta^h(0,x) = h\,,\end{array}\right. \tag{1.5}$$

and

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}t}\zeta^h(t,x) = (A + DF(X(t,x))\,\zeta^h(t,x)) + D^2F(X(t,x))(\eta^h(t,x), \eta^h(t,x))\\ \zeta^h(0,x) = 0\,,\end{array}\right. \tag{1.6}$$

respectively, see e.g. [**6**]. From (1.5) and the Gronwall lemma, it follows

$$|\eta^h(t,x)| \leqslant e^{(\|A\| + \|F\|_1)t}\,|h|\,, \qquad x, h \in H\,. \tag{1.7}$$

Therefore, from (1.6) we find

$$\begin{aligned}|\zeta^h(t,x)| &\leqslant \int_0^t e^{(t-s)(\|A\| + \|F\|_1)}\,|D^2F(\eta^h(s,x), \eta^h(s,x))|\,\mathrm{d}s\\ &\leqslant \int_0^t e^{(t-s)(\|A\| + \|F\|_1)}\,\|F\|_2\, e^{s(\|A\| + \|F\|_1)}\,\mathrm{d}s\\ &= \|F\|_2 \int_0^t e^{(t+s)(\|A\| + \|F\|_1)}\,\mathrm{d}s\,.\end{aligned} \tag{1.8}$$

It follows that $u(t,\cdot) \in C_b^2(H)$ and

$$\langle Du(t,x), h\rangle = \mathbb{E}[\langle D\varphi(X(t,x)), X_x(t,x)\,h\rangle]\,, \qquad t \geqslant 0,\ x, h \in H\,, \tag{1.9}$$

and

$$\begin{aligned}\langle D^2u(t,x)\,h, h\rangle &= \mathbb{E}[\langle D^2\varphi(X(t,x))\,X_x(t,x)\,h, X_x(t,x)\,h\rangle]\\ &\quad + \mathbb{E}[\langle D\varphi(X(t,x)), X_{xx}(t,x)(h,h)\rangle] \qquad t \geqslant 0,\ x, h \in H\,.\end{aligned} \tag{1.10}$$

Now the conclusion follows from the Itô formula. ■

Remark 1.3. In a similar way we can show that if $F \in C_b^k(H;H)$ for some $k > 3$ and $\varphi \in C_b^k(H)$ then $u(t,\cdot) \in C_b^k(H)$ for any $t \geqslant 0$.

We note that Proposition 1.2 has been proved by deterministic methods in [**8**] when det $C > 0$ and in [**7**] when $F = 0$.

We define now a semigroup of linear bounded operators in $B_b(H)$ by setting

$$P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \qquad \varphi \in B_b(H), \qquad t \geqslant 0\,, \tag{1.11}$$

this definition is meaningful since φ is bounded and Borel and X is continuous.

It is easy to see that P_t has the *Feller* property, that is the following implication holds:

$$\varphi \in C_b(H),\ t \geqslant 0 \ \Rightarrow\ P_t \varphi \in C_b(H)\,. \tag{1.12}$$

Consequently the restriction of P_t, $t \geqslant 0$ to $C_b(H)$ is a semigroup of linear bounded operators in $C_b(H)$ (not strongly continuous in general, see [**1**] and [**9**]).

The goal of this paper is to find sufficient conditions such that $P_t \varphi$ is differentiable in x for all $t > 0$ and for all $\varphi \in C_b(H)$. We are also interested in the behaviour of the derivative $DP_t \varphi$ for t close to 0, arriving to estimates such as

$$|DP_t \varphi(x)| \leqslant ct^{-k/2} \|\varphi\|_0, \tag{1.13}$$

for some $k \in \mathbb{N}$.

We believe that it would possible to find estimates, under suitable additional assumption, also for higher derivatives of φ. These estimates could be useful to prove Schauder estimates for the elliptic equation

$$\lambda\varphi - N\phi = f\,, \tag{1.14}$$

see [**7**]. However we shall only prove (1.13) for $k = 1$. As a consequence we will find that the transition semigroup P_t enjoys the strong Feller property, that is:

$$\varphi \in B_b(H),\ t > 0 \ \Rightarrow\ P_t \varphi \in C_b(H)\,. \tag{1.15}$$

Strong Feller property is important to study uniqueness of invariant measures, see [**4**].

When $F = 0$ there is a complete answer to the above problems that we recall in § **2**. In § **3** we consider a perturbation of the linear case. Finally in § **4** we give an example.

2 - The case when $F = 0$

We assume here $F = 0$. Then P_t is given, as well known, by the following *Mehler* formula:

$$P_t \varphi(x) = \int_H \varphi(e^{tA} x + y)\, \mathcal{N}(0, Q_t)(dy), \qquad \varphi \in B_b(H), \tag{2.1}$$

where

$$Q_t = \int_0^t e^{sA} C e^{sA^*} ds, \tag{2.2}$$

and $\mathcal{N}(0, Q_t)$ is the gaussian measure having mean 0 and covariance operator Q_t.

The following result is also well known, see e.g. [3],

Proposition 2.1. *The following statements are equivalent*:

(i) $\det Q_t > 0$, *for all* $t > 0$.

(ii) *For all* $\varphi \in B_b(H)$ *and for any* $t > 0$ *we have* $P_t \varphi \in C_b^\infty(H)$. *Moreover, if* (i) *holds we have*

$$\|D^k P_t \varphi(x)\| \leq \|\Lambda(t)\|^k \|\varphi\|_0, \qquad t > 0,\ k \in \mathbb{N}, \tag{2.3}$$

where

$$\Lambda(t) = Q_t^{-1/2} e^{tA}, \qquad t > 0. \tag{2.4}$$

We recall that when $C = I$ and $\|e^{tA}\| \leq e^{\omega t}$, $t \geq 0$, we have

$$\|\Lambda(t)\| \leq \frac{e^{\omega t}}{t^{1/2}}, \tag{2.5}$$

whereas if $\det C = 0$, but $\det Q_t > 0$, $t > 0$, there exists $k \in \{3, 5, \ldots, 2d-1\}$, and a positive constant c_k such that

$$\|\Lambda(t)\| \leq c_k \frac{e^{\omega t}}{t^{k/2}}. \tag{2.6}$$

We recall that assumption (i) of Proposition 2.1 is equivalent to the Hörmader condition, see [5], ensuring hypoellipticity of N, and also to the controllability of the deterministic system

$$\xi' = A\xi + C^{1/2}\eta, \qquad \xi(0) = \xi_0, \tag{2.7}$$

where ξ is the state and η the control. In fact, given $T > 0$ and $\xi_0 \in H$, the control

$$\eta(s) = -C^{1/2} e^{(T-s)A^*} Q_T^{-1} e^{TA} \xi_0, \qquad s \in [0, T],$$

drives system (2.7) to 0 in time T.

3 - The case when $F \neq 0$

Let $\varphi \in C_b^2(H)$ and let u be the strict solution of (1.1). First we are going to prove an estimate for $Du(t, x)$ depending on $\|\varphi\|_0$ but not on $\|D\varphi\|_0$. To do this, we shall use a generalization of a well known method due to Bernstein.

We set

$$z(t, x) = u^2(t, x) + \langle G(t)\, Du(t, x), Du(t, x)\rangle, \qquad x \in H, \ t \geq 0, \tag{3.1}$$

where $G(t)$, $t \geq 0$ are symmetric positive matrices, to be specified later, whose matrix elements will be denoted by $(G_{i,j}(t))$.

We will need the following identities involving the differential operator N, that can be easily checked.

$$N(\varphi\psi) = \varphi N\psi + \psi N\varphi + \langle CD\varphi, D\phi\rangle, \qquad \varphi, \psi \in C_b^2(H), \tag{3.2}$$

and

$$D_i N\varphi = ND_i\varphi + \langle D_i F(x), D\varphi\rangle, \qquad \varphi, \psi \in C_b^2(H), \qquad i = 1, 2, \ldots, d. \tag{3.3}$$

Lemma 3.1. *Let $\varphi \in C_b^3(H)$, $u(t, \cdot) = P_t\varphi$, and let $G \in C^1([0, +\infty)); L(H))$ with $G(t)$ symmetric for all $t \geq 0$. Then the following identity holds:*

$$\begin{aligned} D_t z(t, x) &= Nz(t, \cdot)(x) + \langle G'(t)\, Du(t, x), Du(t, x)\rangle \\ &+ 2\big\langle (A + DF(x))\, G(t)\, Du(t, x), Du(t, x)\big\rangle - \langle C\, Du(t, x), Du(t, x)\rangle \\ &- \mathrm{Tr}\,[CD^2 u(t, x)\, G(t)\, D^2 u(t, x)]. \end{aligned} \tag{3.4}$$

Proof. We first notice that by (3.2) we have

$$D_t(u^2) = 2u\, D_t u = 2uNu = N(u^2) - |C^{1/2} Du|^2. \tag{3.5}$$

Let us compute $D_t(D_i u D_j u)$. Taking into account (3.3) we have

$$D_t(D_i u D_j u) = D_i Nu\, D_j u + D_i u\, D_j Nu$$

$$= ND_i u\, D_j u + ND_j u\, D_i u + \langle D_i F, Du\rangle\, D_j u + \langle D_j F, Du\rangle\, D_i u\,.$$

By (3.2) it follows

$$D_t(D_i u\, D_j u) = N(D_i u D_j u) - \langle CDD_i u, DD_j u\rangle$$
$$+\langle D_i F, Du\rangle\, D_j u + \langle D_j F, Du\rangle\, D_i u\,.$$

Let us compute $D_t(\langle G(t)\, Du, Du\rangle)$,

$$D_t(\langle G(t)\, Du, Du\rangle) = D_t(\langle G'(t)\, Du, Du\rangle) + \sum_{i,j=1}^{d} G_{i,j}(t)\, D_t(D_i u\, D_j u)$$

$$= D_t(\langle G'(t)\, Du, Du\rangle) + \sum_{i,j=1}^{d} N((D_i u\, D_j u)) - \sum_{i,j=1}^{d} G_{i,j}(t)\langle CDD_i u, DD_j u\rangle \tag{3.6}$$

$$+2\sum_{i,j=1}^{d} G_{i,j}(t)\langle D_i F, Du\rangle\, D_j u\,.$$

From (3.5) and (3.6) the conclusion follows. ■

We prove now the main result of the paper. In its formulation we set

$$G(t) = [\Lambda(t)^* \Lambda(t)]^{-1} = \int_0^t e^{-sA} C e^{-sA^*}\, ds, \qquad t \geq 0\,. \tag{3.7}$$

Moreover we recall that $P_t\varphi$ is defined by (1.11).

Theorem 3.2. *Assume, besides Hypothesis* 1.1, *that* $\det Q_t > 0$ *for* $t > 0$, *and*

$$\langle DF(x)\, G(t)\, \xi, \xi\rangle \leq \kappa\langle G(t)\, \xi, \xi\rangle, \qquad t > 0,\ \ \xi \in H\,, \tag{3.8}$$

for some $\kappa \in \mathbb{R}$.

Then for any $\varphi \in C_b(H)$ *and any* $t > 0$ *we have* $P_t\varphi \in C_b^1(H)$. *Moreover the following estimate holds*

$$|DP_t\varphi(x)| \leq e^{\kappa t/2}\, \|\Lambda(t)\|\, \|\varphi\|_0, \qquad t > 0,\ \ x \in H\,. \tag{3.9}$$

Finally if $\varphi \in B_b(H)$, *and* $t > 0$, *then* $P_t \varphi$ *is Lipschitz continuous, so that* P_t *is strong Feller.*

Proof. We first prove the assertion for $\varphi \in C_b^3(H)$. For this purpose we use Lemma 3.1 taking

$$G(t) = [\Lambda(t)^* \Lambda(t)]^{-1} = \int_0^t e^{-sA} C e^{-sA^*} ds .$$

By a straightforward computation we find

$$G'(t) + AG(t) + G(t) A^* - C = 0 , \qquad t \geq 0 ,$$

so that, taking into account that

$$\mathrm{Tr}[CD^2 u(t, x)\, G(t)\, Du^2(t, x)] > 0 ,$$

(3.4) yields the following

$$D_t z(t, x) = Nz(t, \cdot)(x) + \langle DF(x)\, G(t)\, Du(t, x), Du(t, x)\rangle$$

$$- \mathrm{Tr}[CD^2 u(t, x)\, G(t)\, D^2 u(t, x)]$$

$$\leq Nz(t, \cdot)(x) + \kappa \langle G(t)\, Du(t, x), Du(t, x)\rangle .$$

Therefore

$$\langle G(t)\, Du(t, x), Du(t, x)\rangle \leq z(t, x)$$

$$\leq P_t(\varphi^2(x)) + \kappa \int_0^t P_{t-s}(\langle G(s)\, Du(s, \cdot), Du(s, \cdot)\rangle)(x)\, ds ,$$

and consequently

$$\sup_{x \in H} \langle G(t)\, Du(t, x), Du(t, x)\rangle$$

$$\leq \|\varphi\|_0^2 + \kappa \int_0^t \sup_{x \in H} \langle G(s)\, Du(s, x), Du(s, x)\rangle\, ds .$$

By the Gronwall lemma it follows

$$\langle G(t)\, Du(t, x), Du(t, x)\rangle \leq e^{\kappa t} P_t(\varphi^2) , \qquad t \geq 0 .$$

Finally we have

$$|Du(t, x)|^2 \leq \|\Lambda(t)\|^2 \langle G(t)\, Du(t, x), Du(t, x)\rangle \leq e^{\kappa t}\|\varphi\|_0^2,$$

and (3.9) is proved when $\varphi \in C_b^2(H)$.

Let now $\varphi \in C_b(H)$, and let $\{\varphi_n\} \subset C_b^3(H)$ such that $\varphi_n \to \varphi$ in $C_b(H)$. Set

$$u_n(t, x) = P_t \varphi_n(x), \qquad x \in H, \qquad t \geq 0 .$$

Then by (3.9) it follows that, for any $m, n \in \mathbb{N}$,

$$|Du_n(t, x) - Du_m(t, x)| \leq e^{\kappa t/2}\|\Lambda(t)\|\,\|\varphi_n - \varphi_m\|_0.$$

This implies that $u(t, \cdot) \in C_b^1(H)$ and (3.9) holds.

Let finally $t > 0$ be fixed and $\varphi \in B_b(H)$. Let $\{\varphi_n\} \subset C_b^1(H)$ such that $\varphi_n(x) \to \varphi(x)$ almost everywhere and $\|\varphi_n\|_0 \leq \|\varphi\|_0$. Then for any $n \in \mathbb{N}$, we have, by the first part of the proof,

$$|u_n(t, x) - u_n(t, y)| \leq e^{\kappa_1 t/2}\|\Lambda(t)\|\,|\varphi\|_0\,|x - y| ,$$

for all $x, y \in H$. Consequently, by the Ascoli–Arzelà lemma, there exists a subsequence $\{u_{n_k}\}$ such that

$$\lim_{k \to \infty} u_{n_k}(t, x) \to u(t, x), \qquad \text{uniformly on the compact subsets of } H ,$$

where $u(t, x) = P_t \varphi(x)$. Therefore $P_t \varphi$ is continuous as required. ∎

3.1 - *A generalization*

We assume here that Hypothesis 1.1–(i) holds, but we replace Hypothesis 1.1–(ii) by the following

Hypothesis 3.3. *F is locally Lipschitz continuous, and there exists $\eta \in \mathbb{R}$ such that*

$$\langle F(x) - F(y), x - y\rangle \leq \eta |x - y|^2, \qquad x, y \in H . \tag{3.10}$$

Under these assumptions the differential stochastic equation (1.4) can be solved by monotonicity methods, see [**6**] and [**3**], Chapter 5. Then we can still define the transition semigroup

$$u(t, x) = P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \qquad t \geq 0,\ x \in H ,$$

for all $\varphi \in B_b(H)$. However if $\varphi \in C_b^2(H)$, we cannot conclude that u is a strict solution to (1.1). In fact we do not know whether $X(t, x)$ is twice differentiable, and so we cannot uses formulas (1.9) and (1.10). We shall say that u is a *generalized* solution of (1.1).

We prove finally the following result.

Theorem 3.4. *Assume, besides Hypotheses* 1.1–(i) *and Hypotheses* 3.3, *that* $\det Q_t > 0$ *for* $t > 0$, *and that* (3.8) *holds.*

Then for any $\varphi \in C_b(H)$ *and any* $t > 0$ *we have* $P_t \varphi \in C_b^1(H)$. *Moreover the following estimate holds*

$$|DP_t \varphi(x)| \leq e^{\kappa t/2} \|\Lambda(t)\| \|\varphi\|_0, \qquad t > 0, \ x \in H. \tag{3.11}$$

Finally if $\varphi \in B_b(H)$, *and* $t > 0$, *then* $P_t \varphi$ *is Lipschitz continuous, so that* P_t *is strong Feller.*

Proof. There exists a sequence $\{F_n\}$ in $C_b^2(H; H)$ such that

(i) We have

$$\lim_{n \to \infty} F_n(x) = F(x), \qquad x \in H, \ n \in \mathbb{N},$$

uniformly on the bounded subset of H.

(ii) We have

$$\langle F_n(x) - F_n(y), x - y \rangle \leq \eta |x - y|^2, \qquad x, y \in H. \tag{3.12}$$

It is enough to set

$$F_n(x) = \int_H e^{-\frac{1}{2n}} F(e^{-\frac{1}{2n}} x + y)\, \mathcal{N}(0, (1 - e^{-\frac{1}{2n}}))(\mathrm{d}y).$$

Let $X_n(t, x)$ be the solution to the differential stochastic equation

$$\begin{cases} \mathrm{d}X_n(t) = (AX_n(t) + F_n(X_n(t))\, \mathrm{d}t + C^{1/2} \mathrm{d}W(t)) \\ X_n(0) = x, \end{cases} \tag{3.13}$$

and let P_t^n be the corresponding transition semigroup:

$$P_t^n \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \qquad \varphi \in B_b(H), \ t \geq 0, \ x \in H. \tag{3.14}$$

It is not difficult to see that $P_t^n \varphi(x) \to P_t \varphi(x)$ when $n \to \infty$ uniformly on the bounded subsets of H, see e.g. [2], Chapter 2. Now by Theorem 3.4 we have the estimate

$$|DP_t^N \varphi(x)| \leq e^{\kappa t/2} \|\Lambda(t)\| \|\varphi\|_0, \qquad t > 0, \ x \in H, \ N \in \mathbb{N}, \tag{3.15}$$

for any $\varphi \in B_b(H)$. Now the conclusion follows from standard arguments. ■

4 - An example

We consider here the evolution equation in $\mathbb{R}^2$,

$$\begin{cases} D_t u(t, x) = \dfrac{1}{2} D_1^2 u(t, x) + x_1 D_2 u(t, x) + F_1(x)\, D_1 u(t, x) \\ \qquad\qquad + F_2(x)\, D_2 u(t, x), \\ u(0, x) \quad \varphi(x), \quad x \in H . \end{cases} \tag{4.1}$$

It is a perturbation of a well known Kolmogorov equation.

In this case we have

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$G(t) = \frac{t}{6} \begin{pmatrix} 6 & 2 - 3t \\ -3t & 2t^2 \end{pmatrix}.$$

It is easy to see that det $Q_t > 0$ and

$$\|\Lambda(t)\| \leq ct^{-3/2}, \qquad t \geq 0 . \tag{4.2}$$

Lemma 4.1. *Let* $M = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$. *Then* $MG(t) \leq 0$ *for any* $t \geq 0$ *if and only if*

$$b = c = 0, \qquad a \geq 0, \qquad d \geq 0,$$

and

$$\frac{d}{3} \leq a \leq 3d .$$

Corollary 4.2. *Let* $M = \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix}$, *with* $a \geqslant 0$, $d \geqslant 0$. *Then we have*

$$MG(t) \leqslant \kappa_1 G(t), \qquad t \geqslant 0,$$

where

$$\kappa_1 = \sup\left\{ \frac{d-3a}{2}, \qquad \frac{a-3d}{2} \right\}.$$

Now by Theorem 3.2 it follows the result

Proposition 4.3. *Assume that* $F_1(x) = f_1(x_1)$, $F_2(x) = f_2(x_2)$, *with* $f_1, f_2 \in C_b^2(\mathbb{R})$, $f_1 \leqslant 0$, $f_2 \leqslant 0$, *and that there exists* $c_1 > 0$ *such that*

$$|D_1 f_1(x_1)| + |D_2 f_1(x_2)| \leqslant c_1.$$

Then for any $\varphi \in C_b(H)$ *and any* $t > 0$ *we have* $P_t \varphi \in C_b^1(H)$. *Moreover the following estimate holds*

$$|DP_t \varphi(x)| \leqslant c e^{c_1 t/2} t^{-3/2} \|\varphi\|_0, \qquad t > 0, \ x \in H. \tag{4.3}$$

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Abstract

We consider a degenerate parabolic equation fulfilling controllability conditions and prove differentiability of the solution.

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