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MOHAMED AKKOUCHI (*)

Inequalities for real random variables connected with Jensen's inequality and applications (**)

1 - Introduction

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space. For all real number $r \geqslant 1$, we denote $L^r(P)$ the usual (real) Lebesgue space endowed with its norm $\|.\|_r$, and for each $Z \in L^1(P)$, the expected value of Z is denoted by $E[Z]$. We recall (Jensen's inequality) that

$$\phi(E[Z]) \leqslant (\geqslant)\, E[\phi(Z)]\,, \tag{1}$$

for all $Z \in L^1(P)$ and all convex (concave) function ϕ defined on an interval containing the range of Z, such that $\phi(Z) \in L^1(P)$. The purpose of this paper is to give some inequalities related to this inequality for a wide class of sequences of real integrable random variables on $(\Omega, \mathcal{F}, P)$. This class contains the sequences of independent and identically distributed real random variables.

This paper is organized as follows. In the section **2**, we state and prove our main results. In the section **3**, we give other results valid for the particular case of independent and identically distributed real random variables. In the sections **4** and **5**, we treat some examples and give some natural applications.

We notice that our results generalize and unify a great number of discrete inequalities established by S. S. Dragomir in several articles and bring some complements to them.

(*) Département de Mathématiques, Faculté des Sciences-Semlalia, Université Cadi Ayyad, Av. du prince My, Abdellah, B.P. 2390, Marrakech, Maroc.

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2 - Main results

Notations 2.1. In all this paper, $\mathbb{N}^*$ designates the set of nonzero integers, and $\mathcal{P}_f(\mathbb{N}^*)$ designates the collection of all finite and nonvoid subsets of $\mathbb{N}^*$. Let $\mathcal{X} := (X_n)_{n=1}^{\infty}$ be a sequence of real integrable random variables on $(\Omega, \mathcal{T}, P)$. For each $I \in \mathcal{P}_f(\mathbb{N}^*)$, we denote $Y_I := \dfrac{1}{|I|} \sum_{i \in I} X_i$, where $|I|$ is the cardinal of I. When $I = \{1, 2, \dots, n\}$, we use the notation Y_n instead of Y_I. We suppose that $X_n(\Omega) \subset \mathfrak{J}$, for all integer $n \geq 1$, where $\mathfrak{J}$ is a fixed interval of $\mathbb{R}$. We introduce the set $\Phi_{\mathcal{X}}(\mathfrak{J}, \mathbb{R})$ (resp. $\Lambda_{\mathcal{X}}(\mathfrak{J}, \mathbb{R})$) of all real convex (resp. concave) continuous functions ϕ defined on $\mathfrak{J}$, verifying the following properties:

(i) $\phi \circ Y_I$ is integrable for all $I \in \mathcal{P}_f(\mathbb{N}^*)$, and
(ii) $E[\phi \circ Y_I] = E[\phi \circ Y_J]$ for all $I, J \in \mathcal{P}_f(\mathbb{N}^*)$ having the same cardinal.

We remark that if $\mathcal{X}$ is a sequence of independent and identically distributed real random variables then $\Phi_{\mathcal{X}}(\mathfrak{J}, \mathbb{R})$ (resp. $\Lambda_{\mathcal{X}}(\mathfrak{J}, \mathbb{R})$) is nothing but the set of all real convex (resp. concave) continuous functions defined on $\mathfrak{J}$.

One of the main results of this paper is the following

Theorem 2.2. *Let $\phi \in \Phi_{\mathcal{X}}(\mathfrak{J}, \mathbb{R})$ (resp. $\phi \in \Lambda_{\mathcal{X}}(\mathfrak{J}, \mathbb{R})$). Then we have*

$$E[\phi(Y_{n+1})] \leq (\geq)\, E[\phi(Y_n)] \leq (\geq)\, E[\phi(X_1)] \quad \forall n \in \mathbb{N}^*. \tag{2}$$

Moreover, if one has $E[X_n] = E[X_1]$ for all $n \in \mathbb{N}^*$ then we have

$$\phi(E[X_1]) \leq (\geq)\, E[\phi(Y_{n+1})] \leq (\geq)\, E[\phi(Y_n)] \leq (\geq)\, E[\phi(X_1)] \quad \forall n \in \mathbb{N}^*. \tag{3}$$

Proof. (a) Let $\phi \in \Phi(\mathfrak{J}, \mathbb{R})$ and let $n \in \mathbb{N}^*$. For each $i \in I_{n+1} := \{1, 2, \dots, n+1\}$, we set $I(i) := I_{n+1} \setminus \{i\}$. By using the convexity of ϕ, we get for almost all $\omega \in \Omega$,

$$\phi \circ Y_{I_{n+1}}(\omega) = \phi\left[\frac{1}{n+1} \sum_{i \in I_{n+1}} X_i(\omega)\right] \leq \phi\left[\frac{1}{n+1} \sum_{i \in I_{n+1}} \frac{1}{n}\left[\sum_{j \in I(i)} X_j(\omega)\right]\right]$$

$$\leq \frac{1}{n+1} \sum_{i \in I_{n+1}} \phi\left[\frac{1}{n} \sum_{j \in I(i)} X_j(\omega)\right] = \frac{1}{n+1} \sum_{i \in I_{n+1}} \phi \circ Y_{I(i)}(\omega).$$

By integrating all members of these inequalities and using (ii), we obtain (2).

(b) Now, suppose that all elements of the sequence $(X_n)_{n=1}^{\infty}$ have the same expected value. Then by using Jensen's inequality, we obtain for all integer $n \in \mathbb{N}^*$,

$$\phi(E[X_1]) = \phi\left(\int_\Omega X_1(\omega)\, dP(\omega)\right) = \phi\left(\frac{1}{n}\sum_{i=1}^{i=n}\int_\Omega X_i(\omega)\, dP(\omega)\right)$$

$$= \phi\left(\int_\Omega Y_n(\omega)\, dP(\omega)\right) \leqslant \int_\Omega \phi \circ Y_n(\omega)\, dP(\omega) = E[\phi \circ Y_n]$$

$$\leqslant \frac{1}{n}\sum_{i=1}^{i=n}\int_\Omega \phi \circ X_i\, dP(\omega) = E[\phi \circ X_1].$$

This proves (3). The case where ϕ is concave is treated in a similar manner. ■

Some improvements to Theorem 2.2 are given by the following

Theorem 2.3. *We suppose that* $E[X_n] = E[X_1]$ *for all* $n \in \mathbb{N}^*$ *and let* $\phi \in \Phi_{\mathcal{X}}(\mathfrak{I}, \mathbb{R})$. *Then for all integer* $n \in \mathbb{N}^*$, *the following inequalities hold*:

$$E[\phi(X_1)] - \phi(E[X_1]) \geqslant E[\phi(X_1)] - E[\phi \circ Y_n] \tag{4}$$
$$\geqslant \left| \frac{1}{n} E\left[\left| \sum_{i=1}^{n} \phi \circ X_i \right|\right] - E[\,|\phi \circ Y_n|\,] \right| \geqslant 0\,.$$

Proof. The first inequality of (4) is true. We have only to prove the second inequality. By using the convexity of ϕ, we obtain for all $n \in \mathbb{N}^*$ and almost all $\omega \in \Omega$,

$$\frac{1}{n}\sum_{i=1}^{n} \phi \circ X_i(\omega) - \phi \circ Y_n(\omega) = \left| \frac{1}{n}\sum_{i=1}^{n} \phi \circ X_i(\omega) - \phi \circ Y_n(\omega) \right|$$

$$\geqslant \left| \left| \frac{1}{n}\sum_{i=1}^{n} \phi \circ X_i(\omega) \right| - |\phi \circ Y_n(\omega)| \right| \geqslant 0\,.$$

By integrating all members of these inequalities, we get

$$E[\phi(X_1)] - E[\phi \circ Y_n] \geqslant E\left[\left| \frac{1}{n} \sum_{i=1}^{n} \phi \circ X_i - \phi \circ Y_n \right| \right]$$
$$\geqslant \left| \frac{1}{n} E\left[\left| \sum_{i=1}^{n} \phi \circ X_i \right| \right] - E[\,|\phi \circ Y_n|\,] \right| \geqslant 0\,.$$

This achieves the proof of Theorem 2.3. ■

Remark 2.4. The inequalities obtained in the previous theorems may be generalized. Indeed, let $(q_n)_{n=1}^{\infty}$ be a sequence of positive real numbers such that $Q_n := \sum_{i=1}^{n} q_i > 0$ for all $n \in \mathbb{N}^*$, and set

$$Z_n := \frac{1}{Q_n} \sum_{i=1}^{n} q_i X_i \qquad \forall n \in \mathbb{N}^*.$$

Then, with these notations, we have the following

Theorem 2.5. *We suppose that* $E[X_n] = E[X_1]$ *for all* $n \in \mathbb{N}^*$. *Let* $\phi \in \Phi_{\mathcal{X}}(\mathfrak{I}, \mathbb{R})$ (*resp.* $\phi \in \Lambda_{\mathcal{X}}(\mathfrak{I}, \mathbb{R})$). *Then we have*

$$\phi(E[X_1]) \leqslant (\geqslant)\, E[\phi(Z_n)] \leqslant (\geqslant)\, E[\phi(X_1)] \qquad \forall n \in \mathbb{N}^*. \tag{5}$$

Proof. (a) Let $\phi \in \Phi_{\mathcal{X}}(\mathfrak{I}, \mathbb{R})$. Then, by using Jensen's inequality, we obtain for all integer $n \in \mathbb{N}^*$,

$$\phi(E[X_1]) = \phi\left(\int_{\Omega} X_1(\omega)\, dP(\omega) \right) = \phi\left(\frac{1}{Q_n} \sum_{i=1}^{n} \int_{\Omega} q_i X_i(\omega)\, dP(\omega) \right)$$
$$= \phi\left(\int_{\Omega} Z_n(\omega)\, dP(\omega) \right) \leqslant \int_{\Omega} \phi \circ Z_n(\omega)\, dP(\omega) = E[\phi \circ Z_n]$$
$$\leqslant \frac{1}{Q_n} \sum_{i=1}^{n} q_i E[\phi \circ X_i] = E[\phi \circ X_1]\,.$$

(b) In a similar manner we treat the case where $\phi \in \Lambda_{\mathcal{X}}(\mathfrak{I}, \mathbb{R})$. Thus our theorem is proved. ■

Some improvements to Theorem 2.5 are given by the following

Theorem 2.6. *We suppose that* $E[X_n] = E[X_1]$ *for all* $n \in \mathbb{N}^*$, *and let* ϕ

$\in \Phi_{\mathcal{X}}(\mathfrak{I}, \mathbb{R})$. *Then for all integer* $n \in \mathbb{N}^*$, *the following inequalitites hold*:

$$\begin{aligned} &E[\phi(X_1)] - \phi(E[X_1]) \geqslant E[\phi(X_1)] - E[\phi \circ Z_n] \\ (6) \qquad &\geqslant \left| \frac{1}{Q_n} E\left[\left| \sum_{i=1}^{n} q_i \phi \circ X_i \right| \right] - E[\,|\phi \circ Z_n|\,] \right| \geqslant 0 \,. \end{aligned}$$

This theorem is proved by arguments analogous to those used in the proof of Theorem 2.3.

3 - The case of i.i.d. real random variables

This section deals with the particular case of independent and identically distributed real random variables.

Theorem 3.1. *Let* $(X_n)_{n=1}^{\infty}$ *be a sequence of independent and identically distributed real integrable r.v.s on* $(\Omega, \mathcal{F}, P)$. *Let* $\mathfrak{I}$ *be an interval of* $\mathbb{R}$ *containing* $X_1(\Omega)$, *and let* ϕ *be any arbitrary convex* (*resp. concave*) *continuous function on* $\mathfrak{I}$. *Then the following inequalities hold*:

$$(7) \qquad \phi(E[X_1]) \leqslant (\geqslant)\, E[\phi(Y_n)] \leqslant (\geqslant)\, E[\phi(Z_n)] \leqslant (\geqslant)\, E[\phi(X_1)] \qquad \forall n \in \mathbb{N}^*.$$

Proof. We need to prove only the second inequality in (7). To this end, for all integer $n \in \mathbb{N}^*$, we introduce the following random variables defined on Ω by

$$U_1 := \frac{1}{Q_n}(q_1 X_1 + q_2 X_2 + \ldots + q_{n-1} X_{n-1} + q_n X_n)$$

$$U_2 := \frac{1}{Q_n}(q_n X_1 + q_1 X_2 + \ldots + q_{n-2} X_{n-1} + q_{n-1} X_n)$$

....

$$U_{n-1} := \frac{1}{Q_n}(q_3 X_1 + q_4 X_2 + \ldots + q_1 X_{n-1} + q_2 X_n)$$

$$U_n := \frac{1}{Q_n}(q_2 X_1 + q_3 X_2 + \ldots + q_n X_{n-1} + q_1 X_n)\,.$$

A simple computation will give the equality

$$\frac{U_1 + U_2 + \ldots + U_{n-1} + U_n}{n} = \frac{X_1 + X_2 + \ldots + X_{n-1} + X_n}{n},$$

from which we derive the following inequality,

$$\phi\left(\frac{U_1 + U_2 + \ldots + U_n}{n}\right) \leqslant (\geqslant) \frac{1}{n}(\phi(U_1) + \phi(U_2) + \ldots + \phi(U_n)),$$

which implies

$$\begin{aligned} &\phi\left(\frac{X_1 + X_2 + \ldots + X_{n-1} + X_n}{n}\right) \leqslant (\geqslant) \\ &\frac{1}{n}\phi\left(\frac{1}{Q_n}(q_1 X_1 + q_2 X_2 + \ldots + q_{n-1} X_{n-1} + q_n X_n)\right) + \ldots \\ &\ldots + \frac{1}{n}\phi\left(\frac{1}{Q_n}(q_2 X_1 + q_3 X_2 + \ldots + q_n X_{n-1} + q_1 X_n)\right). \end{aligned}$$

By integrating the members of the last inequality we obtain $E[\phi(Y_n)] \leqslant (\geqslant) E[\phi(Z_n)]$. ■

In the next theorem, we discuss the convergence of the sequence $(E[\phi(Y_n)])_{n \geqslant 1}$.

Theorem 3.2. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent and identically distributed real integrable r.v.s on $(\Omega, \mathcal{F}, P)$. We suppose that $X_n(\Omega) \subset \mathfrak{I}$ for all $n \in \mathbb{N}^*$, where $\mathfrak{I}$ is a bounded interval of $\mathbb{R}$. Let ϕ be a any arbitrary convex (concave) continuous function on $\mathfrak{I}$. Then the following holds*:

$$\phi(E[X_1]) = \lim_{n \to \infty} E[\phi(Y_n)] = \inf_{n \in \mathbb{N}^*} E[\phi(Y_n)] \left(\mathrm{resp} \sup_{\mathrm{n} \in \mathbb{N}^*} \mathrm{E}[\phi(\mathrm{Y_n})]\right). \tag{8}$$

Proof. The strong law of large numbers and the continuity of ϕ, ensure pointwise convergence of the sequence of functions $(\phi(Y_n))_n$ to $\phi(E[X_1])$ on Ω. We apply then Lebesgue's dominated convergence in order to get the first equality in (8). The remainder is a consequence of Theorem 2.2. Thus our result is proved. ■

In the next proposition, we prove the convergence of the sequence $(E[\phi(Y_n)])_{n \geqslant 1}$ in a more general setting. Indeed, here the variables $X_n, (n \geqslant 1)$

need not to be identically distributed and the function ϕ needs not to be convex (resp. concave). More precisely, we have

Proposition 3.3. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent real integrable r.v.s on $(\Omega, \mathcal{F}, P)$. We suppose that: $E(X_n) = E(X_1)$, and $X_n(\Omega) \subset \mathfrak{I}$ for all $n \in \mathbb{N}^*$, where $\mathfrak{I}$ is an interval of $\mathbb{R}$. We suppose also that $E(X_n^4)$ is bounded. Let ϕ be a continuous function on $\mathfrak{I}$ such that $\sup_{n,\omega} |\phi \circ Y_n(\omega)| < \infty$. Then the following holds*:

$$\phi(E[X_1]) = \lim_{n \to \infty} E[\phi(Y_n)]. \tag{9}$$

Proof. Cantelli's theorem ensures that $Y_n \to E(X_1)$ with probability one. The continuity of ϕ and Lebesgue's dominated convergence theorem will imply the result we want. ∎

4 - Applications and examples

Let $r \geqslant 1$. In this section, $(X_n)_{n=1}^{\infty}$ designates a sequence of independent and identically distributed real integrable r.v.s on $(\Omega, \mathcal{F}, P)$, and $\mathfrak{I}$ denotes an interval of $\mathbb{R}$ containing $X_1(\Omega)$.

4.1 - We assume here that $\mathfrak{I} := \mathbb{R}$, and we let $\phi : \mathbb{R} \mapsto \mathbb{R}$, to be given by $\phi(t) := |t|^r$. Then an application of the results of the sections **2** and **3**, gives (for all $n \in \mathbb{N}^*$) the following inequalities:

$$\|X\|_r \geqslant \left\| \frac{X_1 + X_2 + \ldots + X_n}{n} \right\|_r \geqslant \left\| \frac{X_1 + X_2 + \ldots + X_{n+1}}{n+1} \right\|_r \geqslant |E[X]| . \tag{4.1.1}$$

Moreover, one has the following:

$$\lim_{n \mapsto \infty} \left\| \frac{X_1 + X_2 + \ldots + X_n}{n} \right\|_r = |E[X]| . \tag{4.1.2}$$

4.2 - We assume here that $\mathfrak{I} := \left]0, \frac{1}{2}\right]$, and we let $\phi : \left]0, \frac{1}{2}\right] \mapsto [0, +\infty[$, to be given by $\phi(t) := \left[\frac{t}{1-t}\right]^r$, where $r \geqslant 1$. Then, ϕ is convex on $\left]0, \frac{1}{2}\right]$ and also logarithmically concave on this interval. Now, by application of the results of section two and three, we obtain (for all $n \in \mathbb{N}^*$) the following inequalities:

$$
\begin{aligned}
\left\| \frac{X}{1-X} \right\|_r &\geqslant \left\| \frac{X_1 + X_2 + \ldots + X_n}{n - X_1 - X_2 - \ldots - X_n} \right\|_r \\
&\geqslant \left\| \frac{X_1 + X_2 + \ldots + X_{n+1}}{n + 1 - X_1 - X_2 - \ldots - X_{n+1}} \right\|_r \\
&\geqslant \frac{E[X]}{1 - E[X]} \\
&\geqslant \exp\left(E\left[\log\left(\frac{X_1 + X_2 + \ldots + X_{n+1}}{n + 1 - X_1 - X_2 - \ldots - X_{n+1}} \right)\right]\right) \\
&\geqslant \exp\left(E\left[\log\left(\frac{X_1 + X_2 + \ldots + X_n}{n - X_1 - X_2 - \ldots - X_n} \right)\right]\right) \\
&\geqslant \exp\left(E\left[\log\left(\frac{X}{1-X} \right)\right]\right).
\end{aligned}
\tag{4.2.1}
$$

Moreover, one has the following:

$$
\lim_{n \mapsto \infty} \left\| \frac{X_1 + X_2 + \ldots + X_n}{n - X_1 - X_2 - \ldots - X_n} \right\|_r = \frac{E[X]}{1 - E[X]} . \tag{4.2.2}
$$

Remark 4.3. The relations written in (4.2.1) provide also refinements of C.-L. Wang's inequality (see [Wa]) and the well-known result of Ky Fan (see [Be, Be]). We invite the reader to see also the papers of S. S. Dragomir and their references for other applications given in case of discrete Jensen's inequality.

Remark 4.4. We suppose that $\mathfrak{I} = [a, b]$, and that the variables $X_n, (n \geqslant 1)$ have the same distribution law with density given by $\dfrac{1}{b-a} \chi_{[a,\, b]}(x)\, dx$, where χ is the characteristic function of the interval $[a, b]$. Then one has the following result which may be considered as a refinement of the well-known Jensen-Hadamard inequality:

Theorem 4.4.1. *Let $\phi : \mathfrak{I} \mapsto \mathbb{R}$ be a continuous convex (concave) function on the interval $\mathfrak{I}$ and let $a, b \in \mathfrak{I}$ $(a < b)$, $n \in \mathbb{N}^*$. Then one has the following* *inequalities*:

$$\phi\left(\frac{a+b}{2}\right) \leqslant (\geqslant) \frac{1}{(b-a)^{n+1}} \int_a^b \ldots \int_a^b \phi\left(\frac{1}{n+1} \sum_{i=1}^{n+1} x_i\right) dx_1 \ldots dx_{n+1}$$

$$\leqslant (\geqslant) \frac{1}{(b-a)^n} \int_a^b \ldots \int_a^b \phi\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) dx_1 \ldots dx_n$$

$$\leqslant (\geqslant) \frac{1}{(b-a)^2} \int_a^b \int_a^b \phi\left(\frac{x_1+x_2}{2}\right) dx_1\, dx_2$$

$$\leqslant (\geqslant) \frac{1}{(b-a)} \int_a^b \phi(x)\, dx \leqslant (\geqslant) \frac{\phi(a)+\phi(b)}{2}\, .$$

Moreover one has the following:

$$\text{(4.4.1.1.)} \qquad \lim_{n \mapsto +\infty} \frac{1}{(b-a)^n} \int_a^b \ldots \int_a^b \phi\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) dx_1 \ldots dx_n = \phi\left(\frac{a+b}{2}\right).$$

Remarks 4.4.2. The first part of this result was proved in the paper [Dr, Pe, Sá]. For an application of the second part of this result, we take $\psi(t) := (\log \Gamma(t))'$, where Γ and ψ are the Euler gamma function and digamma functions, respectively. It is well known that ψ is concave on $]0, \infty[$. Clearly, we have for all $0 < a < b < \infty$ and all $n \in \mathbb{N}^*$,

$$\int_a^b \psi\left(\frac{x_1+\ldots+x_n+x_{n+1}}{n+1}\right) dx_{n+1} = (n+1) \log\left[\frac{\Gamma\left(\dfrac{b+x_1+\ldots+x_n}{n+1}\right)}{\Gamma\left(\dfrac{a+x_1+\ldots+x_n}{n+1}\right)}\right],$$

so the equality (4.4.1.1) may be written in the form

$$\text{(4.4.2.1)} \qquad \lim_{n \mapsto +\infty} \frac{n+1}{(b-a)^{n+1}} \int_a^b \ldots \int_a^b \log\left[\frac{\Gamma\left(\dfrac{b+x_1+\ldots+x_n}{n+1}\right)}{\Gamma\left(\dfrac{a+x_1+\ldots+x_n}{n+1}\right)}\right] dx_1 \ldots dx_n = \psi\left(\frac{a+b}{2}\right).$$

Our results can be applied to prove the following generalization of Theorem 4.4.1.

Theorem 4.4.3. *Let μ be a probability measure on $\mathbb{R}$ having a support contained in the segment $[a, b]$. Let $\mathfrak{I}$ be an open interval containing this segment and let $\phi : \mathfrak{I} \mapsto \mathbb{R}$ be a continuous function. We set for all integer $n \in \mathbb{N}^*$,*

$$u_n := \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) d\mu(x_1) \dots d\mu(x_n).$$

Then, the sequence $(u_n)_n$ converges and one has the following:

$$\lim_{n \to +\infty} u_n = \phi\left(\int_{\mathbb{R}} x d\mu(x)\right).$$

Moreover, if the function ϕ is convex (resp. concave) on $\mathfrak{I}$ then the sequence $(u_n)_{n=1}^{\infty}$ is decreasing (resp. increasing), and one has the following inequalities:

$$\phi\left(\int_{\mathbb{R}} x d\mu(x)\right) \leqslant (\geqslant)\, u_{n+1} \leqslant (\geqslant)\, u_n \leqslant (\geqslant) \int_{\mathbb{R}} \phi(x)\, d\mu(x) \qquad \forall n \in \mathbb{N}^*. \tag{4.4.3.1}$$

5 - A result of approximation

We end this paper by deriving a result of approximation connected to Bernstein's Theorem. Before stating the result, let us introduce some notations.

5.1 - Let m be a fixed nonzero integer and let $T_m = \{x \in [0, 1]^m : x_1 + x_2 + \dots + x_m \leqslant 1\}$ be a simplex in $\mathbb{R}^m$. We denote by $C(T_m)$ the Banach space of real continuous functions defined on T_m equiped with the norm $\|f\|_{C(T_m)} := \sup\{|f(x)| : x \in T_m\}$. Let $\langle . | . \rangle$ be the usual inner product of $\mathbb{R}^m$, defined for all $x = (x_1, \dots x_m)$ and all $y = (y_1, \dots y_m)$ by setting $\langle x | y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_m y_m$, and denote $| . |_m$ the associated norm. Let $C([0, 1])$ be the Banach space of real continuous functions defined on the real segment $[0, 1]$ equipped with its usual norm $\| . \|_{C([0,1])}$. To all $z \in [0, 1]^m$ and all $\phi \in C([0, 1])$, we associate the function $A_z \phi$ defined on T_m by

$$A_z \phi(x) := \phi(\langle x | z \rangle) \qquad \forall x \in T_m,$$

and a sequence $(A_z^n(\phi))_{n\geq m}$ of functions defined for all $x=(x_1, \dots x_m)\in T_m$ by

$$A_z^n(\phi)(x)$$

$$:=\sum_{\alpha\in N_m}\frac{n!}{(n-|\alpha|)!\,\alpha_1!\,\alpha_2!\dots\alpha_m!}(1-\langle x|u\rangle)^{n-|\alpha|}x_1^{\alpha_1}x_2^{\alpha_2}\dots x_m^{\alpha_m}\phi\left(\frac{1}{n}\langle\alpha|z\rangle\right),$$

where $N_m := \{\alpha=(\alpha_1, \alpha_2, \dots, \alpha_m)\in\mathbb{N}^m\colon |\alpha| := \alpha_1+\alpha_2+\dots+\alpha_m\leq n\}$, and u is the vector $(1, 1, \dots, 1)$. We remark that if $m=2$ and $z=(0, 1)$ then for all $x\in[0, 1]$ we have $A_z\phi(0, x)=\phi(x)$ and the expressions $A_z^n(\phi)(0, x)$ for all integer $n\geq 2$ are reduced to be

$$A_z^n(\phi)(0, x) := \sum_{\alpha=0}^{n}\frac{n!}{(n-\alpha)!\,\alpha!}\phi\left(\frac{\alpha}{n}\right)x^{\alpha}(1-x)^{n-\alpha},$$

which are nothing but the usual Bernstein's polynomials associated to the function ϕ, and we know that these sequence of polynomials approximate uniformly the function ϕ on the segment $[0, 1]$. In the next theorem, we prove the uniform convergence of the sequence $(A_z^n(\phi))_{n\geq m}$ to the function $A_z\phi$ on the simplex T_m.

Theorem 5.2. *Let ϕ be a convex (concave) continuous function on the segment $[0, 1]$ and let $z=(z_1, z_2, \dots, z_m)$ be a fixed element in $[0, 1]^m$ having distinct coordinates such that $0<\mathrm{Min}(z_1, z_2, \dots, z_m)$. Then $(A_z^n(\phi))_{n\geq m}$ is a decreasing (resp. increasing) sequence of continuous functions on the compact T_m, converging uniformly to the function $A_z\phi$ on the simplex T_m.*

Proof. Let $n\geq m$ and let $x=(x_1, \dots x_m)\in T_m$. We know (see for example [Bi]) that there exists at least a sequence $(X_k)_{k=1}^{\infty}$ of independent and identically distributed random variables on a some probability space $(\Omega, \mathcal{F}, P)$, such that $X_n(\Omega)=\{0, z_1, z_2, \dots, z_m\}$, and verifying:

$$P(X_k=0)=1-\sum_{j=1}^{m}x_j, \qquad P(X_k=z_j)=x_j, \qquad \forall j\colon 1\leq j\leq m, \text{ and } \forall k\in\mathbb{N}^*.$$

We see that each random variable X_k is discrete and the expected value of X_k is given by $E[X_k]=\langle x|z\rangle$.

Now, we put $Y_n := \frac{1}{n} \sum_{i=1}^{n} X_i$. Then a short calculation will give

$$P\left(Y_n = \frac{1}{n}\langle\alpha|z\rangle\right) = \frac{n!}{(n-|\alpha|)!\,\alpha_1!\,\alpha_2!\ldots\alpha_m!}(1-\langle x|u\rangle)^{n-|\alpha|}\,x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_m^{\alpha_m}.$$

Therefore, the expected value of the random variable $\phi \circ Y_n$ is given by the following formula:

$$E[\phi \circ Y_n] = A_z^n(\phi)(x).$$

The strong law of large numbers and the continuity of ϕ ensure pointwise convergence of $(A_z^n(\phi))_{n \geqslant m}$ to $A_z \phi$ on T_m. Since T_m is a compact subset of $\mathbb{R}^m$ and since the sequence $(A_z^n(\phi))_{n \geqslant m}$ is decreasing (increasing) then Dini's Theorem will ensure the uniform convergence of the sequence $(A_z^n(\phi))_{n \geqslant m}$ to $A_z \phi$ on T_m. ■

Remarks 5.3. Theorem 5.2 is valid for all continuous function ϕ on the segment $[0, 1]$, and for all $z = (z_1, z_2, \ldots, z_m) \in [0, 1]^m$ without any condition on the coordinates.

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Abstract

We establish some inequalities, connected to the well known Jensen's integral inequality, for a class of sequences of integrable real random variables on a probability space. These inequalities are valid for independent and identically distributed real random variables. The results obtained here are generalizations of those obtained by S. S. Dragomir in the discrete case. We bring also some complements to Dragomir's work. We treat examples and give some natural applications.

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