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Classification of almost contact structures associated with a strongly pseudo-convex CR-structure (**)
also that in [4] D. Chinea and J. C. Marrero studied this classification on the viewpoint of conformal geometry.

Applying this classification to the CR-manifolds, we analyse in particular the properties for the gauge transformations under which it is possible to obtain different types of almost contact metric structures associated with the same CR-structure. Some conditions for remarkable structures are given.

As interesting examples, we consider our results on the unit tangent bundle of a Riemannian manifold of constant sectional curvature and on the Heisenberg group $H_3$; in $H_3$ we also construct the gauge transformations convenient to obtain different almost contact structures.

The outline of the paper is as follows. Sections 2 and 3 are devoted to general results on pseudo-convex CR-structures, gauge transformations and to the classification of almost contact structures respectively [3]. In section 4 we apply this classification to almost contact metric structures associated with a same strongly pseudo-convex CR-structure and finally in the last section we describe in detail the cited examples.

2 - Preliminaries

Let $M$ be an orientable $C^\infty$ m-dimensional manifold; a CR-structure $(M, H(M))$ on $M$, is defined by a complex vector subbundle $H(M)$ in the complexification $T^c M$ of the tangent bundle of $M$ so that:

(a) $A(M) \cap H(M) = \{0\}$ where $A(M) = H(M)$.

(b) $H(M)$ is complex involutive, i.e. for two $H(M)$-valued complex vector fields $Z, W$, the bracket $[Z, W]$ is $H(M)$-valued too.

Denoted now by $H(M)$ also the decomplexification of the complex subbundle, let $J$ be the operator on $H(M)$ corresponding to the multiplication by $i$; then the condition of complex involutivity can be expressed by:

\begin{align}
(2.1) \quad \left\{ \begin{array}{l}
(i) \ [X, Y] - [JX, JY] \in \Gamma(H(M)) \\
(ii) \ N_J(X, Y) = [JX, JY] - [X, Y] - J[[JX, Y] + [X, JY]] = 0
\end{array} \right.
\end{align}

for every $X, Y$ belonging to $\Gamma(H(M))$, $\Gamma(H(M))$ being the $C^\infty(M)$-module of cross-sections on $H(M)$.

From now on, we shall suppose that $\dim M = 2n + 1$, $\text{codim} H(M) = 1$ and that the Levi form of $(M, H(M))$ is nondegenerate, i.e. we shall consider only pseudo-convex CR-structures of hypersurface type. Then, if we denote by $\eta$ the local 1-form having $H(M)$ as null bundle, the property of pseudoconvexity of
(\(M, H(M)\)) assures that \(\eta \wedge (d\eta)^n \neq 0\) and \(\eta\) is a contact form on \(M\). Notice that, if we consider \(M\) globally oriented, then \(\eta\) is globally defined.

Then for a pseudo-convex structure we have \(TM = \text{span}\{\xi\} \oplus H(M)\), where \(\xi\) is the Reeb vector field defined by \(\eta(\xi) = 1\), \(i_\xi d\eta = 0\); moreover if \(\phi\) is the \((1,1)\)-tensor field given by

\[
\phi X = J(X - \eta(X)\xi), \quad \forall X \in \chi(M)
\]

the following relations hold

\[
\eta \circ \phi = 0, \quad \phi \xi = 0, \quad \phi^* = -I + \eta \otimes \xi;
\]

hence \((\phi, \xi, \eta)\) defines an almost contact structure on \(M\) which is called associated with the pseudo-convex CR-structure \((M, H(M))\) (see [2], [11]).

Consider now the new 1-form \(\tilde{\eta} = e^\sigma \eta\), where \(\sigma \in C^\infty(M)\) and \(e = \pm 1\); it is trivial that \(\tilde{\eta}\) defines the same distribution \(H(M)\) as \(\eta\). Examining the relations between the associated almost contact structures \((\phi, \xi, \eta)\) and \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta})\) respectively induced by \(\eta\) and \(\tilde{\eta}\) the following proposition follows

**Proposition 1** [10]. *Two almost contact structures \((\phi, \xi, \eta), (\tilde{\phi}, \tilde{\xi}, \tilde{\eta})\) are subordinated to the same pseudoconvex CR-structure if and only if there exists a function \(\sigma \in C^\infty(M)\) such that:

\[
\begin{align*}
\tilde{\eta} &= e^{\sigma} \eta, & d\tilde{\eta} &= e^{\sigma} (d\eta + d\sigma \wedge \eta) \\
\tilde{\xi} &= e^{-\sigma} (\xi + \phi A), & \tilde{\phi} &= \phi + \eta \otimes A
\end{align*}
\]

where, assuming \(e = 1\) and denoting by \(h\) the projection operator on \(H(M), A\) is a vector field defined by the conditions:

\[
\eta(A) = 0, \quad d\eta(\phi A, X) = d\sigma(hX) = hX(\sigma).
\]

It is an important geometric property that the complex involutivity is invariant under gauge transformations [7].

**Remark 2.** We shall consider from now on \(e = 1\) only, the case where \(e = -1\) being completely similar.

If we suppose the CR-structure strongly pseudo-convex, then the metric \(g\) defined for all \(X, Y \in T\ell H(M)\) by the equations

\[
g(X, Y) = d\eta(X, \phi Y), \quad g(X, \xi) = \eta(X)
\]

is positively defined and satisfies the following compatibility conditions with re-
pect to \((\phi, \xi, \eta)\)

\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \]

In the sequel, note that 
\[ d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta[X, Y]. \]

After a gauge transformation, imposing the compatibility conditions with respect to the new structure \((\bar{\phi}, \bar{\xi}, \bar{\eta})\), we obtain from \(g\) a new Riemannian metric \(\bar{g}\) on \(M\) which generally doesn't satisfy the equation 
\[ \bar{g}(X, Y) = d\bar{\eta}(X, \phi Y), \]
with \(X, Y \in \Gamma(H(M))\). But, if we require that the restrictions of \(g\) and \(\bar{g}\) are related by a conformal transformation on \(H(M)\), then we get the following relation between \(g\) and \(\bar{g}\) (see also [12])

\[
\tag{2.4}
\left\{ \begin{array}{l}
\tilde{g}(X, Y) = e^{2\sigma} \{ g(X, Y) - \eta(X) \, g(\phi A, Y) - \eta(Y) \, g(\phi A, X) \\
+ g(A, A) \, \eta(X) \, \eta(Y) \} \quad \forall X, Y \in \chi(M);
\end{array} \right.
\]

and the equality

\[ \tilde{g}(X, Y) = e^\sigma d\bar{\eta}(X, \phi Y) \]

holds for all \(X, Y \in \Gamma(H(M))\).

\section{The 12 classes}

It is known that the existence of an almost contact metric structure on \(M\) is equivalent to the existence of a reduction of the structural group \(\mathfrak{c}(2n+1)\) to \(\mathfrak{u}(n) \times 1\). If we denote by \(\Phi\) the fundamental 2-form of \((M, \phi, \xi, \eta, g)\) defined by \(\Phi(X, Y) = g(X, \phi Y)\) and by \(\nabla\) the Riemannian connection of \(g\), the covariant derivative \(\nabla\Phi\) is a covariant tensor of degree 3 which has various symmetry properties.

Let \(V\) be a real vector space of dimension \(2n+1\) endowed with an almost contact structure \((\phi, \xi, \eta)\) and a compatible inner product \((,\)\) and \(\mathcal{C}(V)\) the vector space of 3-forms on \(V\) having the same symmetries of \(\nabla\Phi\), i.e.

\[ \mathcal{C}(V) = \{ \alpha \in \bigotimes^3 V | \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, \phi y, \phi z) \]

\[ + \eta(y) \alpha(x, \xi, z) + \eta(z) \alpha(x, y, \xi) \}. \]

In [3] the authors have been obtained the following decomposition of \(\mathcal{C}(V)\) into twelve components \(\mathcal{C}_i(V)\) which are mutually orthogonal, irreducible and inva-
variant subspaces under the action of \( \mathfrak{u}(n) \times 1 \):

\[
\mathcal{C}(V) = \bigoplus_{i=1,\ldots,12} \mathcal{C}_i(V),
\]

where

\[
\mathcal{C}_1(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, x, y) = \alpha(x, y, \xi) = 0 \},
\]

\[
\mathcal{C}_2(V) = \{ \alpha \in \mathcal{C}(V) \mid \sum_{(x, y, z)} \alpha(x, y, z) = 0, \alpha(x, y, \xi) = 0 \},
\]

\[
\mathcal{C}_3(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) - \alpha(\phi x, \phi y, z) = 0, c_{12} \alpha = 0 \},
\]

\[
\mathcal{C}_4(V) = \left\{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = \frac{1}{2(n-1)} [\langle x, y \rangle \eta(x) \eta(y) c_{12} \alpha(z) - \langle x, z \rangle - \eta(x) \eta(z) c_{12} \alpha(y) - \langle x, \phi y \rangle c_{12} \alpha(\phi z) + \langle x, \phi z \rangle c_{12} \alpha(\phi y) \}, \quad c_{12} \alpha(\xi) = 0 \right\},
\]

\[
\mathcal{C}_5(V) = \left\{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = \frac{1}{2n} [\langle x, \phi y \rangle \eta(y) \bar{c}_{12} \alpha(\xi) - \langle x, \phi y \rangle \eta(y) \bar{c}_{12} \alpha(\xi) \right\},
\]

\[
\mathcal{C}_6(V) = \left\{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = \frac{1}{2n} [\langle x, y \rangle \eta(z) c_{12} \alpha(\xi) - \langle x, z \rangle \eta(y) c_{12} \alpha(\xi) \right\},
\]

\[
\mathcal{C}_7(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = \eta(z) \alpha(y, x, \xi) - \eta(y) \alpha(\phi x, \phi z, \xi), \quad c_{12} \alpha(\xi) = 0 \},
\]

\[
\mathcal{C}_8(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = -\eta(z) \alpha(y, x, \xi) - \eta(y) \alpha(\phi x, \phi z, \xi), \quad \bar{c}_{12} \alpha(\xi) = 0 \},
\]

\[
\mathcal{C}_9(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = \eta(z) \alpha(y, x, \xi) + \eta(y) \alpha(\phi x, \phi z, \xi) \},
\]

\[
\mathcal{C}_{10}(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = -\eta(z) \alpha(y, x, \xi) + \eta(y) \alpha(\phi x, \phi z, \xi) \},
\]

\[
\mathcal{C}_{11}(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = -\eta(x) \alpha(\xi, \phi y, \phi z) \},
\]

\[
\mathcal{C}_{12}(V) = \{ \alpha \in \mathcal{C}(V) \mid \alpha(x, y, z) = \eta(x) \eta(y) \alpha(\xi, \eta, \xi, z) + \eta(x) \eta(z) \alpha(\xi, \eta, \xi, \eta) \}.
\]
Here, if \( \{ e_i \}, \ i = 1, 2, \ldots, 2n + 1 \) denotes an arbitrary orthonormal basis we have

\[

c_{12}a(x) = \sum a(e_i, e_i, x) \\
\overline{c}_{12}a(x) = \sum a(e_i, \phi e_i, x), \quad \text{for all } x \in V.
\]

Applying this algebraic decomposition to the geometry of almost contact structures, for each invariant subspace we obtain a different class of almost contact metric manifolds; more precisely, we shall say \( M \) of class \( C_k, k \geq 1, R, 2, n \) if, for every \( p \in M \), the 3-form \((\tilde{\Phi})_p\) of the vector space \((T_p M, \phi, \xi_p, \eta_p, g_p)\) belongs to \( C_k(T_p M)\).

For example, \( C_6 \) corresponds to the class of \( \alpha \)-Sasakian manifolds, \( C_9 \) to the class of almost cosymplectic manifolds, \( C_8 \) to that one of normal manifolds (for an extensive study of these structures see [3]).

4 - Classification of gauge transformations

Let \( M \) be an \((2n + 1)\)-dimensional manifold endowed with an almost contact metric structure associated with a pseudo-convex CR-structure \((M, H(M))\) of hypersurface type.

**Theorem 3.** \( M \) is of class \( C_6 \oplus C_9 \).

**Proof.** Following [3] we split the space \( \mathcal{C}(T_p M), p \in M \), into the direct sum

\[
\mathcal{C}(T_p M) = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3,
\]

where

\[
\mathcal{O}_1 = \{ \alpha \in \mathcal{C}(V) | \alpha(\xi, x, y) = \alpha(x, \xi, y) = 0 \} \\
\mathcal{O}_2 = \{ \alpha \in \mathcal{C}(V) | \alpha(x, y, z) = \eta(x) \alpha(\xi, y, z) + \eta(y) \alpha(x, \xi, z) + \eta(z) \alpha(x, y, \xi) \} \\
\mathcal{O}_3 = \{ \alpha \in \mathcal{C}(V) | \alpha(x, y, z) = \eta(x) \eta(y) \alpha(\xi, \xi, z) + \eta(x) \eta(z) \alpha(\xi, y, \xi) \}
\]

obtaining

\[
\mathcal{O}_1 = C_1 \oplus \ldots \oplus C_4 \\
\mathcal{O}_2 = C_5 \oplus \ldots \oplus C_{12} \\
\mathcal{O}_3 = C_{13}.
\]

As a consequence of (4.3) we can consider \((\nabla \Phi)_p\) as the sum of three compo-
nents \( \alpha_k \in \Omega_k, \; k = 1, 2, 3 \):

(4.4) \quad (\nabla\Phi)_\rho = \alpha_1 + \alpha_2 + \alpha_3.

On the other hand, a straightforward computation proves that, for all \( X, \; Y, \; Z \in \Gamma(H(M)) \), the involutivity conditions (2.1) imply the equations:

(4.5) \quad \begin{cases} 
(\nabla_X\Phi)(Y, Z) = \frac{1}{2} \eta([[\phi Z, \phi Y] - \phi(\phi Z, Y) - \phi[Z, \phi Y] - [Z, Y], X]) = \\
= \frac{1}{2} \eta([\mathcal{N}_\phi(Z, Y), X]) = 0, \\
\nabla_{\xi}\Phi = 0,
\end{cases}

(4.6)

(in the following, as in (4.5) and (4.6), to simplify the notations, we shall omit indicating the point \( p \)).

From (4.5) and (4.6) we deduce that \( \nabla\Phi \) has not component in \( \Omega_1 = C_1 \oplus \ldots \oplus C_4 \) as well as in \( \Omega_0 = C_{12} \); therefore \( \nabla\Phi \) reduces to the only component \( \alpha_2 \in \Omega_2 \).

Now comparing the equalities

(4.7) \quad \begin{cases} 
\tau_{12}(\nabla\Phi)(\xi) = 0, \\
\tau_{12}(\nabla\Phi)(\xi) = n,
\end{cases}

with (3.1) we immediately obtain that \( \nabla\Phi \) has not component in \( C_5 \) too, and that the component in \( C_6 \) is different from zero.

The non-existence of components for \( \nabla\Phi \) in \( C_7 \oplus C_8 \) follows from the relation

(4.8) \quad (\nabla_X\Phi)(\xi, Z) = - (\nabla_{\phi X}\Phi)(\xi, \phi Z) - g(X, Z)

true for all \( X, \; Z \in \Gamma(H(M)) \).

Computing now directly from (3.1) the components of \( \nabla\Phi \) in \( C_5 \oplus C_{10} \), applying (2.1), we find that \( \nabla\Phi \) has a component different from zero in \( C_5 \); for \( X, \; Z \in \Gamma(H(M)) \) and \( Y = \xi \) its expression is:

\[ \frac{1}{2} g((\mathcal{L}_\xi\phi) Z, X), \]

where \( \mathcal{L}_\xi\phi \) is the Lie derivative of \( \phi \) with respect to \( \xi \).

Finally, a simple computation proves that the component in \( C_{11} \) vanishes.

This completes the proof.

According to [3] we obtain
Corollary 4. $M$ is of class $C_6$ if and only if the almost contact structure $(\phi, \xi, \eta)$ is normal.

Proof. From the previous theorem we have that the component in $C_9$ is zero iff $L_j f = 0$, and this relation is always satisfied when the almost contact structure is normal, i.e. when the (1,2)-tensor field $N$ given by

$$N = N_\phi + d\eta \otimes \xi$$

vanishes.

On the other hand, in [7] it has been also proved that if $(M, H(M))$ satisfies the involutivity conditions and $L_j f = 0$, then the almost contact structure $(\phi, \xi, \eta)$ is normal. $\blacksquare$

Let $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be now the new almost contact metric structure on $M$ obtained from $(\phi, \xi, \eta, g)$ by a gauge transformation (2.3) and (2.4); this means that both almost contact structures are associated to the same strongly pseudo-convex $CR$-structure $(M, H(M))$ of $M$.

If $\bar{\nabla}$ and $\bar{\Phi}$ denote the Levi-Civita connection and the fundamental 2-form of $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ respectively, taking into account (2.3) and (2.4), an easy computation gives

$$\bar{\Phi}(X, Y) = e^{2\eta} \{ \Phi(X, Y) - \eta(X) g(A, Y) + \eta(Y) g(A, X) \} \quad \text{for all } X, Y \in \chi(M);$$

furthermore it will be useful for us to remark that the following formula holds:

$$L_\xi \bar{\phi}(X) = e^{-\eta} \{ L_\xi \phi(X) + (\phi X(\sigma) + \eta(X) A(\sigma))(\xi + \phi A) + [\phi A, \phi X]$$

$$- \phi[\phi A, X] + hX(\sigma) A + \eta(X)[\xi + \phi A, A] \}.$$

Theorem 5. If dimension of $M$ is $2n + 1$, $n \geq 2$, $(M, \phi, \xi, \eta, g)$ is of class $C_7 \oplus C_5 \oplus C_6 \oplus C_9$. When $n = 1$ then $M$ has dimension 3 and $(M, \phi, \xi, \eta, g)$ is of class $C_5 \oplus C_6 \oplus C_9$.

Proof. Taking into account previous formulas and definitions, after lengthy straightforward computation, it is possible to prove the following relations bet-
ween $\tilde{\nabla} \Phi$ and $\nabla \Phi$

$$
(\tilde{\nabla}_X \Phi)(Y, Z) = e^{2\sigma}(\nabla_X \Phi)(Y, Z) + 
$$

$$
+ \frac{e^{2\sigma}}{2} \left\{ Z(\sigma)g(X, \varphi Y) - Y(\sigma)g(X, \varphi Z) + \varphi Z(\sigma)g(X, Y) - \varphi Y(\sigma)g(X, Z) \right\},
$$

(4.11)

$$
(\tilde{\nabla}_X \Phi)(\xi, Z) = e^{\sigma}(\nabla_X \Phi)(\xi, Z) - \frac{e^{\sigma}}{2} \left\{ \xi(\sigma)g(X, \varphi Z) - \varphi Z(\sigma)g(\varphi A, X)
$$

$$
- g([\varphi A, \varphi Z], X) - g([\varphi A, Z], \varphi X) - Z(\sigma)g(A, X) \right\},
$$

(4.12)

$$
\tilde{\nabla}_X \Phi = 0.
$$

(4.13)

Suppose $n \geq 2$ and, as above, consider $\tilde{\nabla} \Phi$ as the sum of three components $\alpha_k \in \Omega_k$, $k = 1, 2, 3$:

$$
\tilde{\nabla} \Phi = \alpha_1 + \alpha_2 + \alpha_3.
$$

(4.14)

The vanishing of $\alpha_3$ follows easily from the equations (4.6) and (4.13); as a consequence $\tilde{\nabla} \Phi$ has no component in $C_{12}$.

With regard to $\alpha_2$, Theorem 3, (4.10) and (4.12) imply that we have only three components different from zero in $C_{15}$, $C_{3}$ and $C_{9}$ given respectively by

$$
- \frac{1}{2} e^{\sigma} \xi(\sigma)g(X, \varphi Z), \quad - \frac{1}{2} e^{\sigma}g(X, Z), \quad \frac{1}{2} \tilde{g}(([\xi_2] \Phi) Z, X),
$$

(4.15)

for every $X, Z \in \Gamma(H(M))$ and $Y = \xi$. 

Supposing at the end $X, Y, Z \in \Gamma(H(M))$ we can compute the component in $C_{15}$. As the restriction to $H(M)$ of our structure reduces to an almost Hermitian structure, applying [6] and comparing with (4.5) and (4.11) we find for $(\tilde{\nabla}_X \Phi)(Y, Z)$ the only following component in $C_{15}$:

$$
\left\{ \begin{array}{l}
\frac{1}{2} e^{\sigma}(Z(\sigma)g(X, \varphi Y) - Y(\sigma)g(X, \varphi Z)) + \\
+ \frac{1}{2} e^{\sigma}((\varphi Z)(\sigma)g(X, \varphi Y) - (\varphi Y)(\sigma)g(X, \varphi Z)).
\end{array} \right.
$$

(4.16)

The case $n = 1$ directly follows from [3] and the above considerations.

$\blacksquare$
Corollary 6. Supposing $M$ of dimension $2n + 1 \geq 5$, we have:

(i) $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is of class $C_4 \oplus C_6 \oplus C_9$ iff $\xi(\alpha) = 0$;
(ii) $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is of class $C_5 \oplus C_6 \oplus C_9$ iff $X(\alpha) = 0, \forall X \in \Gamma(H(M))$;
(iii) $(M, \phi, \xi, \eta, \tilde{g})$ is of class $C_5 \oplus C_6 \oplus C_9$ iff $(\phi, \xi, \eta)$ is normal, i.e. iff (4.14) holds.

Remark 7. From Corollary 4 and Corollary 6, (iii), we deduce that the normality of the structure is preserved iff

$[\phi A, \phi X] - \phi[\phi A, X] = -\phi X(\alpha)(\xi + \phi A) + hX(\alpha)A$.

Then we can state

Corollary 8. If $(M, \phi, \xi, \eta, g)$ is Sasakian and $\dim M = 3$ then $(M, \phi, \xi, \eta, \tilde{g})$ obtained by (2.3) with $\alpha$ not constant is Sasakian iff

(a) $\xi(\alpha) = 0$;
(b) $[\phi A, A] = -A(\alpha)(\xi + \phi A)$.

5 - Examples

The unit tangent bundle

Let $(M, g)$ be an $(n + 1)$-dimensional Riemannian manifold, $n \geq 2$; we denote by $TM$ the tangent bundle of the manifold $M$ and by $\pi: TM \to M$ the canonical projection. If $(x^1, \ldots, x^{n+1})$ are local coordinates on $M$, then $(x^1, \ldots, x^{n+1})$ and the fibre coordinates $(y^1, \ldots, y^{n+1})$ define together a system of local coordinates on $TM$. The Levi-Civita connection $D$ of $g$ determines a decomposition of $TTM$ in the direct sum of the vertical distribution $VTM$ and the horizontal distribution $HTM$, i.e. $TTM = VTM \oplus HTM$. Then the well known almost complex structure on $TM$ is defined by:

$$JX^H = X^V, \quad JX^V = -X^H \quad X \in \chi(M)$$

where $X^H, X^V$ are the horizontal and vertical lifts of $X$ with respect to $D$ respectively. Furthermore the Sasaki metric $\hat{g}$ on $TM$ is given by

$$\hat{g}(X^V, Y^V) = g(X, Y), \quad \hat{g}(X^H, Y^H) = g(X, Y), \quad \hat{g}(X^V, Y^H) = 0 \quad X, Y \in \chi(M).$$

Let $T_1M$ be the unit tangent bundle of $M$; then, we have $v \in T_1M$ iff $v \in TM$ and $g(v, v) = 1$. If $v = y^i \frac{\partial}{\partial x^i}$, we conclude that the unit tangent bundle
\( \pi : T_1M \to M \) is a hypersurface in \( TM \), given in the local coordinates by the equation:

\[
g_{ij}(x) y^i y^j - 1 = 0.
\]

It is possible to prove that, as hypersurface of the almost Kaehlerian manifold \((TM, J, \hat{g})\), \( T_1M \) has a natural almost contact metric structure which defines a pseudo-convex CR-structure \((T_1M, H(T_1M))\) iff the base manifold \( M \) has constant sectional curvature \( c \) (see [8], [9], [13]).

Moreover, if we consider a generator system for \( H(T_1M) \) given by the following vector fields:

\[
Y_i = (\delta^i_j - g_{\omega j} y^j) \frac{\partial}{\partial y^i} \quad \text{and} \quad X_i = (\delta^i_j - g_{\omega j} y^j) \frac{\partial}{\partial x^j},
\]

where \( g_{\omega} = g_{\omega k} y^k \), and we still denote by \( \hat{g} \) the metric induced from \( TM \) on \( T_1M \), the almost contact structure \((\phi, J, \eta, \hat{g})\) associated with the CR-structure \((T_1M, H(T_1M))\) satisfies the following relations:

\[
\begin{align*}
\delta = \eta dx^i, \\
\xi = y^i \frac{\partial}{\partial x^i}, \\
\phi X_i = Y_i, \\
\phi Y_i = -X_i, \\
\phi \xi = 0, \\
i, j = 1, \ldots, n + 1,
\end{align*}
\]

where \( \delta \frac{\partial}{\partial x^i} = (\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} - \Gamma_{\omega i}^{j} \frac{\partial}{\partial y^j}, \Gamma_{\omega i}^{j} = y^k \Gamma_{\omega i k}, \) where \( \Gamma_{\omega i k} \) are the Christoffel symbols corresponding to the connection \( D \).

Computing now the Levi-Civita connection \( \hat{\nabla} \) of the metric \( \hat{g} \) on the vector fields \( Y_i, X_i, \xi \) we find:

\[
\begin{align*}
\hat{\nabla}_i Y_j = & -g_{\omega} Y_i, \\
\hat{\nabla}_i X_j = & (\Gamma_{\omega i}^{k} - g_{\omega j} \Gamma_{\omega j k}) Y_k + \frac{c}{2} h_{ij} \xi \\
\hat{\nabla}_i X_j = & (\Gamma_{\omega i}^{k} - g_{\omega j} \Gamma_{\omega j k}) X_k \\
\hat{\nabla}_i Y_j = & -\frac{c}{2} h_{ij} \xi, \\
\hat{\nabla}_i X_j = & -\frac{c}{2} Y_i, \\
\hat{\nabla}_i \xi = & 0
\end{align*}
\]

and

\[
\begin{align*}
\hat{\nabla}_i X_j = & \frac{c}{2} X_i, \\
\hat{\nabla}_i Y_j = & \frac{c}{2} Y_i, \\
i, j, k = 1, \ldots, n + 1,
\end{align*}
\]

where

\[
h_{ij} = g_{i \omega} - g_{\omega} g_{\omega j}.
\]
Then we easily can write the expressions of the following Lie brackets:

\[
\begin{align*}
[Y_i, Y_j] &= g_{\theta\theta} Y_i - g_{\phi\phi} Y_j, \quad [X_i, X_j] = (g_{\theta\theta} \Gamma^k_{\theta\theta} - g_{\phi\phi} \Gamma^k_{\phi\phi}) X_k \\
[Y_i, X_j] &= -g_{\theta\theta} X_i - (\Gamma^k_{\theta\phi} - g_{\phi\phi} \Gamma^k_{\phi\phi}) Y_k - h_{ij} \xi \\
[Y_i, \xi] &= X_i - \Gamma^k_{\theta\phi} Y_k, \quad [X_i, \xi] = -e Y_i - \Gamma^k_{\phi\phi} X_k.
\end{align*}
\]

(5.6)

From the previous formulas, we obtain that the covariant derivative \(\hat{\nabla} \Phi\) of the fundamental 2-form \(\Phi(X, Y) = \hat{g}(X, \phi Y) = -d\eta(X, Y)\) of \((\phi, \xi, \eta, \hat{g})\) is not vanishing only in the following cases

\[
\begin{align*}
(\hat{\nabla}_Y \Phi)(Y_j, \xi) &= -(\hat{\nabla}_Y \Phi)(\xi, Y_j) = \frac{c - 2}{2} h_{ij} \\
(\hat{\nabla}_X \Phi)(X_j, \xi) &= -(\hat{\nabla}_X \Phi)(\xi, X_j) = -\frac{c}{2} h_{ij},
\end{align*}
\]

(5.7)

and finally, from formulas (5.6), we have that the following equations hold

\[
(\mathcal{L}_\xi \phi) X_i = (c - 1) X_i, \quad (\mathcal{L}_\xi \phi) Y_i = (1 - c) Y_i, \quad i = 1, \ldots, n + 1.
\]

(5.8)

As a consequence, taking into account Theorem 3 and Corollary 4, we can state

**Proposition 9.** \((T_1M, \phi, \xi, \eta, \hat{g})\) is of class \(C_6 \oplus C_9\). In particular, \((T_1M, \phi, \xi, \eta, \hat{g})\) belongs to \(C_6\) iff \(c = 1\).

Apply now the gauge transformation (2.3) to \((\phi, \xi, \eta)\), obtaining \(\tilde{\eta} = e^\sigma g_{\theta\theta} dx^i\); furthermore the vector field \(A \in H(M)\) can be expressed by means of \(\{Y_i, X_i\}\) as

\[
A = \lambda^i Y_i + \mu^i X_i, \quad \text{where } \lambda^i, \mu^i \in C^\infty(T_1M).
\]

(5.9)

Moreover, taking into account (2.4), we obtain for the new metric \(\tilde{g}\) the relations:

\[
\begin{align*}
\tilde{g}(Y_i, Y_j) &= \tilde{g}(X_i, X_j) = e^{2\sigma} h_{ij}, \quad \tilde{g}(X_i, Y_j) = 0 \\
\tilde{g}(X_i, \xi) &= e^{2\sigma} Y_i(\alpha), \quad \tilde{g}(X_i, \xi) = -e^{2\sigma} X_i(\alpha) \\
\tilde{g}(\xi, \xi) &= e^{2\sigma} (1 + \|A\|^2), \quad \tilde{g}(\xi, \xi) = 1, \quad \tilde{g}(\xi, \xi) = e^\sigma,
\end{align*}
\]

(5.10)

where \(\|A\|^2 = \lambda^i Y_i(\sigma) + \mu^i X_i(\sigma)\).

Then, considering the covariant derivative \(\tilde{\nabla} \tilde{\Phi}\) of the fundamental 2-form
\( \Phi(X, Y) = \tilde{g}(X, \tilde{\phi} Y) \) of the new structure, we obtain:

\[
\begin{align*}
(\tilde{\nabla}_Y \Phi)(Y_j, Y_k) &= - (\tilde{\nabla}_Y \Phi)(X_j, X_k) = \\
&= (\tilde{\nabla}_X \Phi)(Y_j, X_k) = \frac{e^{2a}}{2} (X_j(\alpha) h_{ik} - X_k(\alpha) h_{ij})
\end{align*}
\]

\[
(\tilde{\nabla}_X \Phi)(Y_j, Y_k) = - (\tilde{\nabla}_X \Phi)(X_j, X_k) =
\]

\[
= - (\tilde{\nabla}_Y \Phi)(Y_j, X_k) = \frac{e^{2a}}{2} (Y_j(\alpha) h_{ik} - Y_k(\alpha) h_{ij})
\]

\[
(\tilde{\nabla}_Y \Phi)(Y_j, \tilde{\xi}) = \frac{e^a}{2} (c - 2) h_{ij} - \frac{e^a}{2} g_{ik} \lambda^k h_{ij} - \frac{e^a}{2} \mu^k (\Gamma^i_{jk} - g_{ik} \Gamma^i_{jk}) h_{ij} +
\]

\[
+ \frac{e^a}{2} (X_i(\alpha) X_j(\alpha) - Y_i(\alpha) Y_j(\alpha)) + \frac{e^a}{2} (Y_i(\lambda^k) h_{jk} - Y_j(\mu^k) h_{ik})
\]

\[
(\tilde{\nabla}_Y \Phi)(X_j, \tilde{\xi}) = e^a \xi(\alpha) h_{ij} - e^a g_{ik} \mu^k h_{ij} -
\]

\[
- \frac{e^a}{2} (X_i(\alpha) Y_j(\alpha) + Y_i(\alpha) X_j(\alpha)) + \frac{e^a}{2} (Y_i(\mu^k) h_{jk} + Y_j(\lambda^k) h_{ik})
\]

\[
(\tilde{\nabla}_X \Phi)(Y_j, \tilde{\xi}) = - e^a \xi(\alpha) h_{ij} + \frac{e^a}{2} \lambda^k \frac{\partial}{\partial y^k} (h_{ij}) -
\]

\[
- \frac{e^a}{2} (X_i(\alpha) Y_j(\alpha) + Y_i(\alpha) X_j(\alpha)) + \frac{e^a}{2} (X_i(\lambda^k) h_{jk} + X_j(\mu^k) h_{ik})
\]

\[
(\tilde{\nabla}_X \Phi)(X_j, \tilde{\xi}) = - \frac{e^a}{2} \chi_{ij} + \frac{e^a}{2} g_{ik} \lambda^k h_{ij} + \frac{e^a}{2} \mu^k (\Gamma^i_{jk} - g_{ik} \Gamma^i_{jk}) h_{ij} +
\]

\[
+ \frac{e^a}{2} (Y_i(\alpha) Y_j(\alpha) - X_i(\alpha) X_j(\alpha)) + \frac{e^a}{2} (X_i(\mu^k) h_{jk} - Y_j(\lambda^k) h_{ik})
\]

and, as in the general case, \( \tilde{\nabla}_Y \Phi = 0 \).

Finally, after a straightforward computation, we find that the new structure \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta})\) is not normal and Theorem 5 and Corollary 6 imply that \((T_1 M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) belongs to \( C_4 \oplus C_5 \oplus C_6 \oplus C_9 \).

Every component of \((T_1 M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) with respect to the basis \(\{X_i, Y_j, \tilde{\xi}\}\) can be explicitly written by means of (5.11).

The Heisenberg group

As it is well known (see for example [14]), the Heisenberg Lie group \(H_3\) is the
subgroup of $GL(3, \mathbb{R})$ given by

$$H_3 = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}; \ x, y, z \in \mathbb{R}$$

(5.12)

with the usual matrix multiplication.

Then it is easy to see that

$$ds^2 = dx^2 + dz^2 + (dy - xdz)^2$$

(5.13)

is a left invariant metric on $H_3$ as well as the following vector fields:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial y}.$$  

(5.14)

If we consider $H(H_3)$ generated by $X_1$ and $X_2$, we have that $(H_3, H(H_3))$ is a pseudo-convex $CR$-structure on the Heisenberg group with associated almost contact metric structure defined by the formulas:

$$\begin{cases} \eta = x \, dz - dy \\ \phi X_1 = X_2, \quad \phi X_2 = -X_1, \quad \phi \xi = 0, \end{cases}$$

(5.15)

while the equation (5.13) gives the associated metric $g$.

Let $\nabla$ be the Levi-Civita connection of $g$ and $\Phi$ the fundamental 2-form defined as usual. Then, the only cases where the covariant derivative is different from zero are the following:

$$(\nabla_{X_1} \Phi)(X_1, \xi) = (\nabla_{X_2} \Phi)(X_2, \xi) = \frac{1}{2},$$

and $(H_3, \phi, \eta, \xi, g) \in C_6$.

Put now $A = \mu X_1 + \lambda X_2$, $\lambda, \mu \in C^\infty(H_3)$; after the gauge transformation we have

$$\mu = -X_1(\sigma), \quad \lambda = -X_2(\sigma),$$

where $\sigma$ is a smooth function.
and the components of the new covariant derivative are:

\[
\begin{align*}
(\tilde{\nabla}_X, \bar{\Phi})(X_1, \tilde{\xi}) &= \frac{e^\sigma}{2} (X_1(\mu) - X_2(\lambda) - \lambda^2 + \mu^2 + 1) \\
(\tilde{\nabla}_X, \bar{\Phi})(X_2, \tilde{\xi}) &= \frac{e^\sigma}{2} (X_2(\lambda) - X_1(\mu) - \mu^2 + \lambda^2 + 1) \\
(\tilde{\nabla}_X, \bar{\Phi})(X_2, \tilde{\xi}) &= e^\sigma (-\tilde{\xi}(\sigma) + X_1(\lambda) + \mu\lambda) \\
(\tilde{\nabla}_X, \bar{\Phi})(X_1, \tilde{\xi}) &= e^\sigma (\tilde{\xi}(\sigma) + X_2(\mu) + \mu\lambda).
\end{align*}
\] (5.16)

Formulas (5.16) and Theorem 5 imply that \((H_3, \phi, \eta, \tilde{\xi}, \tilde{\gamma}) \in \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9\). In particular taking into account Corollary 8, after a straightforward computation, we can state

**Proposition 10.** \((H_3, \phi, \eta, \tilde{\xi}, \tilde{\gamma})\) is of class \(\mathcal{C}_6\) iff

\[\sigma(x, y, z) = -\ln((x - \alpha)^2 + (z - \beta)^2 + \gamma) + \epsilon,\]

with \(\alpha, \beta, \gamma, \epsilon \in \mathbb{R}\) and \(\gamma > 0\).

**Remark 11.** We remark that, from Corollary 8, for every \(\sigma = \sigma(y)\) a not constant function one obtains an almost contact metric structure associated with \((H_3, H(H_3))\) belonging to \(\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9\). We have also for

\[\sigma(x, y, z) = -\ln((x - \alpha)^2 + a(z - \beta)^2 + \gamma) + \epsilon\]

with \(\alpha, \beta, \gamma, \epsilon, a \in \mathbb{R}\) and \(\gamma > 0, a \neq 1\) an almost contact metric structure belonging to \(\mathcal{C}_6 \oplus \mathcal{C}_9\).

**References**


Abstract

In this paper gauge transformations of almost contact metric structures associated with strongly pseudo-convex CR-structures are studied from an algebraic point of view and some examples are given.