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Uniform polynomial approximation to solutions of the Cauchy problem for O.D.E. (**)

1 - Introduction

A wide literature is dedicated to the methods of numerical integration for initial value problems (shortly IVP) relative to ordinary differential equations.

The solution is obtained on a grid at a constant step or, in the case of adaptive methods, at a variable step. We deal, in every case, with methods that approximate the solution by a sequence of points belonging to the domain where the differential equation satisfies the Lipschitz condition.

Much less extension has the work dedicated to the uniform approximation of the solution in a suitable neighborhood of the initial point.

The prototype of these methods is that of Chaplygin and some variants of it, that can be found in classical books of I. S. Berezin and N. P. Zhidkov [**2**] and S. G. Mikhlin and K. L. Smolitskiy [**4**]. In both, after having approximated the solution of the IVP by a sequence of linear problems, it is excluded the possibility of the analytical computation of a certain sequence of functions uniformly approaching the solution itself, since the solutions of the linear problems can not be expressed in terms of elementary functions.

Subsequently it has been recognized by many authors (see e.g. [**6**]) that Chaplygin's method essentially coincides, under the Functional Analysis point of view, for a suitable choice of the involved spaces, with Newton's method.

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The motivation for which the analytical methods have been almost never considered by the authors who have been devoted to the integration of the IVP is mainly due to the fact that initially the computers, generally programmed in Fortran, allowed only the numerical treatment of the data.

The appearing of new programs of Computer Algebra, such as *MAPLE*©, *MATHEMATICA*© [7], etc., has ensured the analytical resolution of many problems and seems, also in this field, to permit, by a suitable adaptation of the techniques described in the above books [2], [4], the uniform approximation of the solution of an IVP.

A paper by O. Aramă [1] considered a technique of computation of the solution of an IVP using the translation of the problem to a Volterra integral equation, whose solution, by Peano-Picard's method, is approximated making use of Bernstein polynomials.

In a recent paper [5] the same idea has been exploited to obtain a sequence of polynomials uniformly approximating the solution in an interval, using Chaplygin's method, which guarantees a rate of convergence much more rapid than Peano-Picard's method.

In this note the approximation of the solution of an IVP by Bernstein polynomials is applied to a technique of computation, exposed in [2], which relaxes the hypotheses necessary for the application of Chaplygin's method.

Methods of quasi linearization with quadratic rate of convergence, essentially different from those considered in this article, can be found in a book by V. Lakshmikantham and A. S. Vatsala [3].

2 - Chaplygin's method

In this section we briefly recall the main ideas connected with Chaplygin's method and give information about rate of convergence, proved by N. N. Luzin.

Consider the Cauchy problem

$$y' = f(x, y); \qquad y(x_0) = y_0 \tag{2.1}$$

where $f(x, y)$ satisfies the Lipschitz condition in $A' \times B$, ($A' := \{x : |x - x_0| \leq a'\}$; $B := \{y : |y - y_0| \leq b\}$):

$$|f(x, y_2) - f(x, y_1)| \leq K |y_2 - y_1|, \qquad (x \in A', \; y_i \in B, \; (i = 1, 2))$$

(K independent of x), and suppose we have unique solution $y = y(x)$ defined in $A \subseteq A'$, ($A := \{x : |x - x_0| \leq a\}$), such that $y(x) \in B$.

Denoting by $S := \max_{A' \times B} |f(x, y)|$, then $a := \min(a', b/S)$.

Theorem 2.1 (Chaplygin). *Let $U(x, y)$, $V(x, y)$ be Lipschitz functions such that, in $A \times B$:*

$$U(x, y) \leqslant f(x, y) \leqslant V(x, y) \tag{2.2}$$

then the solutions $u(x)$, $v(x)$ of the Cauchy problems

$$y' = U(x, y); \qquad u(x_0) = y_0$$

$$y' = V(x, y); \qquad v(x_0) = y_0$$

satisfy

$$u(x) \leqslant y(x) \leqslant v(x), \tag{2.3}$$

and if at least one of inequalities (2.2) strictly holds in a point $x_1 \geqslant x_0$, then the corresponding inequality in (2.3) strictly holds $\forall x \geqslant x_1$.

Suppose now that the function $f(x, y)$ verifies the hypothesis:

$$\partial^2 f/\partial^2 y < 0 \qquad \text{in } A \times B. \tag{2.4}$$

Then, $\forall x \in A$, the surface $z = f(x, y)$ is concave in D, so that considering functions:

$$U_0(x, y) := M_0(x) y + N_0(x); \qquad V_0(x, y) := \widetilde{M}_0(x)\, y + \widetilde{N}_0(x),$$

where

$$M_0(x) := \frac{f(x, y_0 + b) - f(x, y_0 - b)}{2b},$$

$$N_0(x) := f(x, y_0 - b) - \frac{f(x, y_0 + b) - f(x, y_0 - b)}{2b} (y_0 - b)$$

and

$$\widetilde{M}_0(x) := f_y(x, y_0 - b),$$

$$\widetilde{N}_0(x) := f(x, y_0 - b) - f_y(x, y_0 - b)(y_0 - b)$$

conditions (2.3) hold true.

Then it is possible to construct sequences: $\{u_n(x)\}_{n\geq 0}$; $\{v_n(x)\}_{n\geq 0}$, defined by solving respectively Cauchy problem for linear equations:

$$u_n' = M_n(x)\, u_n + N_n(x)\,; \qquad u_n(x_0) = y_0$$

$$v_n' = \widetilde{M}_n(x)\, v_n + \widetilde{N}_n(x)\,; \qquad v_n(x_0) = y_0$$

where, for $n \geq 1$:

$$M_n(x) := \frac{f(x,\, v_{n-1}) - f(x,\, u_{n-1})}{v_{n-1} - u_{n-1}}$$

$$N_n(x) := f(x,\, u_{n-1}) - \frac{f(x,\, v_{n-1}) - f(x,\, u_{n-1})}{v_{n-1} - u_{n-1}}\, u_{n-1}$$

and

$$\widetilde{M}_n(x) := f_y(x,\, u_{n-1})$$

$$\widetilde{N}_n(x) := f(x,\, u_{n-1}) - f_y(x,\, u_{n-1})\, u_{n-1}\,.$$

This corresponds to applying the initial method to a sequence of sets $B_n(x) := \{u_{n-1}(x) \leq y \leq v_{n-1}(x)\}$ which are contained in B, $\forall x \in [x_0,\, x_0 + a]$.

The sequences $\{u_n(x)\}_{n\geq 0}$; $\{v_n(x)\}_{n\geq 0}$ converge to the exact solution $y(x)$ of the original Cauchy problem (2.1). Luzin's Theorem expresses the rate of convergence in A of the above mentioned process, which is very rapid, being expressed by the inequality

$$v_n(x) - u_n(x) < \frac{C}{2^{2^n}}\,, \tag{2.5}$$

where C is a constant independent of n and x.

Similar results hold if the partial derivative in (2.4) is strictly positive in $A \times B$. In this case the corresponding surface is convex, and all the machinery holds true, simply reversing the inequalities in the preceding formulas.

As it is well known, the Cauchy problem for linear equation:

$$y' = m(x) y + n(x)\,; \qquad y(x_0) = y_0 \tag{2.6}$$

is given by the explicit formula:

$$y(x) = e^{\int_{x_0}^{x} m(\xi)\, d\xi} \left[\int_{x_0}^{x} n(\xi)\, e^{-\int_{x_0}^{\xi} m(t)\, dt}\, d\xi + y_0 \right], \tag{2.7}$$

however, even in very simple cases (see e.g. [**4**], pp. 10-12), the primitive appearing in the above formula is not expressible in terms of elementary functions. Nevertheless, in [**5**], we showed how to construct the lower (and upper) polynomial approximations of the solution of problem (2.6), i.e. those satisfying Chaplygin's conditions (2.3). Consequently, applying Chaplygin's algorithm, in [**5**], we obtained, step by step, polynomial sequences uniformly approximating, in a right neighborhood of the initial point x_0, the solution of the original problem (2.1).

3 - A method for approximating solutions of O.D.E.

We show now another method for finding upper and lower approximations $u_n(x)$ and $v_n(x)$, starting from $u_0(x)$ and $v_0(x)$ for which $u_0'(x) - f(x, u_0), f(x, v_0) - v_0'(x)$ are not positive functions in $\widehat{A} := [x_0, x_0 + a]$ and such that $u_0(x_0) = v_0(x_0) = y_0$; this method does NOT require the sign of $\partial^2 f / \partial^2 y$ to be constant (see eq. (2.4)). Let K be the Lipschitz constant, one introduces sequences $\{u_n(x)\}_{n \geq 0}$; $\{v_n(x)\}_{n \geq 0}$ defined by

$$u_n(x) = u_{n-1}(x) + \int_{x_0}^{x} e^{-K(x-t)} [f(t, u_{n-1}(t)) - u_{n-1}'(t)]\, dt\,,$$

$$v_n(x) = v_{n-1}(x) - \int_{x_0}^{x} e^{-K(x-t)} [v_{n-1}'(t) - f(t, v_{n-1}(t))]\, dt\,.$$

Theorem 3.1. $\forall n \geq 1$,

$$u_{n-1}(x) \leq u_n(x) \leq \ldots \leq y(x) \leq \ldots \leq v_n(x) \leq v_{n-1}(x)\,.$$

Theorem 3.2. *The sequences* $\{u_n(x)\}$; $\{v_n(x)\}$ *uniformly approach* $y(x)$ *in* $\widehat{A}$ *as* $n \to \infty$.

However the rate of convergence is less than the rate given by Luzin's Theorem.

Consider now the sequence $\{v_n(x)\}_{n\geqslant 0}$ defined by

$$v_n(x) = v_{n-1}(x) - \int_{x_0}^{x} e^{-K(x-t)} h_{n-1}(t)\, dt\,,$$

where $h_{n-1}(t) := v'_{n-1}(t) - f(t, v_{n-1}(t))$.

Our purpose in Section 4 is to construct a sequence of polynomials $\{q_n(x)\}_{n\geqslant 0}$ uniformly approximating the sequence $\{v_n(x)\}_{n\geqslant 0}$ (the approximation will depend on f and $\widehat{A}$, but not on x). The case of finding a sequence of polynomials $\{p_n(x)\}_{n\geqslant 0}$ uniformly approximating the sequence $\{u_n(x)\}_{n\geqslant 0}$ can be obtained in a similar way.

Let $M := \max\limits_{\widehat{A}\times B} f(x, y)$, $v(x) = Mx + (y_0 - Mx_0)$ is the solution of the Cauchy problem

$$v' = M\,; \qquad v(x_0) = y_0\,.$$

Choose $v_0(x) := v(x)$.

4 - Uniform polynomial approximations

Define $q_0(x) := v_0(x) = Mx + (y_0 - Mx_0)$. Compute $h_0^*(x) := q_0'(x) - f(x, q_0(x))$. Approximate the function $h_0^*(x)$, in $\widehat{A}$, by a Bernstein polynomial $q_0^*(x)$. Approximate the function $\int_{x_0}^{x} e^{-K(x-t)} q_0^*(t)\, dt$ by a Bernstein polynomial $\tilde{q}_0(x)$. Compute $q_1(x) := q_0(x) - \tilde{q}_0(x)$.

Perform $\forall n > 1$ the steps

i) Compute $h_{n-1}^*(x) := q'_{n-1}(x) - f(x, q_{n-1}(x))$.

ii) Approximate the function $h_{n-1}^*(x)$, in $\widehat{A}$, by a Bernstein polynomial $q_{n-1}^*(x)$.

iii) Approximate the function $\int_{x_0}^{x} e^{-K(x-t)} q_{n-1}^*(t)\, dt$ by a Bernstein polynomial $\tilde{q}_{n-1}(x)$.

iv) Compute $q_n(x) := q_{n-1}(x) - \tilde{q}_{n-1}(x)$.

Theorem 4.1 (Uniform approximation). $|q_n(x) - y(x)| = o(1)$ as $n \to \infty$ in $\widehat{A}$.

Proof. We observe that $|q_1(x) - v_1(x)| < \varepsilon$ uniformly in $\widehat{A}$.

Let $|q_{n-1}(x) - v_{n-1}(x)| < \varepsilon$ uniformly in $\widehat{A}$ as an inductive hypothesis.

$$\left| \tilde{q}_{n-1}(x) - \int_{x_0}^{x} e^{-K(x-t)} q_{n-1}^*(t)\, dt \right| < \varepsilon \tag{4.1}$$

holds by construction.

$$|h^*_{n-1}(x) - h_{n-1}(x)| < \varepsilon \tag{4.2}$$

holds.

Indeed, one can prove it by using the Lipschitz condition, the approximation property of derivatives preserved by Bernstein polynomials and an inductive procedure which shows that $|q'_{n-1}(x) - v'_{n-1}(x)| < \varepsilon$.

Moreover one can easily see that

$$\int_{x_0}^{x} e^{-K(x-t)}\,dt < \frac{1}{K}, \tag{4.3}$$

and

$$|q^*_{n-1}(x) - h^*_{n-1}(x)| < \varepsilon \tag{4.4}$$

is true by construction.

We have

$$|q_n(x) - y(x)| \leq |q_n(x) - v_n(x)| + |y(x) - v_n(x)|,$$

therefore, by Theorem (3.2), it is enough to show that $|q_n(x) - v_n(x)| = o(1)$ as $n \to \infty$.

Adding and subtracting $\int_{x_0}^{x} e^{-K(x-t)} q^*_{n-1}(t)\,dt$ and $\int_{x_0}^{x} e^{-K(x-t)} h^*_{n-1}(t)\,dt$ in the right hand side of

$$|v_n(x) - q_n(x)| = \left| v_{n-1}(x) - \int_{x_0}^{x} e^{-K(x-t)} h_{n-1}(t)\,dt - (q_{n-1}(x) - \tilde{q}_{n-1}(x)) \right|,$$

thanks to inequalities (4.1)-(4.2)-(4.3)-(4.4), the proof is complete. ■

Remark 4.2. *The rate of convergence of the method we have proposed in this article is less than the original method that can be found in* [**2**]. *However it is possible to show that, increasing in a suitable way the maximum degree m_n of the Bernstein polynomials involved in the approximation of $q_n =: q_n^{(m_n)}$, we can obtain a rate of convergence of an arbitrary fixed order α (in particular more than quadratic).*

Recalling the rate of approximation of continuous functions by means of

Bernstein polynomials, we can write:

$$\big|\,|v_n - y| - |q_n^{(m_n)} - y|\,\big| = \mathcal{O}\left(\frac{1}{\sqrt{m_n}}\right).$$

Then, assuming $m_n = n^{2\alpha}$, *we get*

$$\big|\,|v_n - y| - |q_n^{(m_n)} - y|\,\big| = \mathcal{O}(n^{-\alpha}).$$

Working in the same way for lower polynomial approximations, and applying the above described algorithm, we are sure that the sequences of polynomial approximations $\{p_n(x)\}$; $\{q_n(x)\}$ uniformly approach $y(x)$ in $\widehat{A}$ as $n \to \infty$.

5 - Monotone polynomial approximations

We show that, under suitable conditions, we are able to construct monotone sequences.

Theorem 5.1 (Monotonicity). *There exists a sequence* $\{q_n(x)\}$ *such that in* $[x_0, x_0 + a]$

$$y(x) \leq \ldots \leq q_n(x) \leq q_{n-1}(x) \leq \ldots \leq q_0(x).$$

Proof. We observe that, since $h_0(x) \geq 0$, the functions $q_0^*(x)$ and, consequently, $\tilde{q}_0(x)$ are forced to be positive. Therefore $q_1(x) \leq q_0(x)$.

Now, suppose that in $[x_0, x_0 + a]$

1_0) $q_0^*(x) \leq h_0^*(x)$,

2_0) $\tilde{q}_0'(x) \leq -K\int_{x_0}^{x} e^{-K(x-t)} q_0^*(t)\, dt + q_0^*(x)$.

We explicitly note that that condition 2_0) implies, by Theorem 2.1, that $\tilde{q}_0(x) \leq \int_{x_0}^{x} e^{-K(x-t)} q_0^*(t)\, dt$.

We show, exploiting the above conditions and the lipshitzianity of f, that $q_1'(x) - f(x, q_1(x)) \geq 0$.

$$q_1'(x) - f(x, q_1(x)) = q_1'(x) - f(x, q_0(x)) + f(x, q_0(x)) - f(x, q_1(x)) =$$

$$= q_0'(x) - \tilde{q}_0'(x) - f(x, q_0(x)) + f(x, q_0(x)) - f(x, q_1(x)) =$$

$$= h_0^* - \tilde{q}_0'(x) + f(x, q_0(x)) - f(x, q_1(x))$$

$$\geqslant h_0^* + K\int_{x_0}^{x} e^{-K(x-t)} q_0^*(t)\,dt - q_0^*(x) + f(x, q_0(x)) - f(x, q_1(x))$$

$$\geqslant K\int_{x_0}^{x} e^{-K(x-t)} q_0^*(t)\,dt + f(x, q_0(x)) - f(x, q_1(x))$$

$$\geqslant K\tilde{q}_0(x) + f(x, q_0(x)) - f(x, q_1(x)) =$$

$$= K(q_0(x) - q_1(x)) + (f(x, q_0(x)) - f(x, q_1(x))) \geqslant 0\,.$$

Therefore, by Theorem 2.1, $y(x) \leqslant q_1(x)$.

Inductively, if in $[x_0, x_0 + a]$ the inequality $q_n'(x) - f(x, q_n(x)) \geqslant 0$ holds, straightforwardly we derive $q_{n+1}(x) \leqslant q_n(x)$. Moreover, if

$1_n)$ $q_n^*(x) \leqslant h_n^*(x)$,

$2_n)$ $\tilde{q}_n'(x) \leqslant -K\int_{x_0}^{x} e^{-K(x-t)} q_n^*(t)\,dt + q_n^*(x)$,

it is easy to show that $y(x) \leqslant q_{n+1}(x)$.

We stress that by adding suitable B-splines, it is always possible to construct an uniformly approximating sequence for which the above conditions hold. ■

The machinery is almost the same for lower polynomial approximations.

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Abstract

We show an analytical method, based on Chaplygin's differential inequalities theorem, for the construction of sequences of polynomials uniformly approaching the solution of the Cauchy problem for ordinary differential equations in the real line $y' = f(x, y)$; $y(x_0) = y_0$, *in a suitable neighborhood of the initial point* x_0.

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