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# Right ideals in transformation nearrings (**)

## Introduction

Let $(G, +)$ be a group with identity 0, but not necessarily abelian. The set of functions $M(G) = \{f: G \to G\}$ is a (right) nearring under pointwise addition and composition. For general results about nearrings see [**1**], [**4**], or [**5**]. It is well known that $M(G)$ is a simple nearring if $|G| \geqslant 3$ ([**4**], 1.43), hence studying two-sided ideals is trivial. Heatherly [**2**] and Johnson [**3**] studied left ideals in the simple nearring $M_0(G) = \{f: G \to G \mid f(0) = 0\}$. In this paper we assume that $G$ is a finite group and find all right ideals in $M_0(G)$ and all right ideals in $M(G)$ when $G$ is abelian.

For $f \in M(G)$ we let $\langle f \rangle_R$ denote the right ideal generated by $f$ in $M(G)$. The identical notation will be used when generating right ideals in $M_0(G)$, but the nearring will be clear from the context.

## Main Results

Let $G$ be a finite group and let $H$ be a normal subgroup of $G$. We define

$$(H : G) = \{f \in M(G) \mid f(g) \in H \text{ for all } g \in G\}$$

and

$$(H : G)_0 = \{f \in M_0(G) \mid f(g) \in H \text{ for all } g \in G\}.$$

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It is straightforward to check that $(H:G)$ is a right ideal of $M(G)$ and $(H:G)_0$ is a right ideal of $M_0(G)$.

We first determine all right ideals in $M_0(G)$.

Theorem 1. *If $G$ is a finite group, then every right ideal of $M_0(G)$ is of the form $(H:G)_0$ for some normal subgroup $H$ of $G$. Furthermore, every right ideal of $M_0(G)$ is principal.*

Proof. Let $f \in M_0(G)$ and let $N$ be the normal subgroup of $G$ generated by Im$f$. We claim that $\langle f \rangle_R = (N:G)_0$. Since Im$f \subseteq N$, it is clear that $\langle f \rangle_R \subseteq (N:G)_0$. Let $G = \{0 = x_0, x_1, \ldots, x_n\}$. Choose arbitrary elements $y_j \in \mathrm{Im} f$, say $f(x_j) = y_j$, and $x_k \in G$, and define the function $h_k(x) = \begin{cases} x_j & \text{if } x = x_k \\ 0 & \text{if } x \neq x_k \end{cases}$. Then $(f \circ h_k)(x) = \begin{cases} y_j & \text{if } x = x_k \\ 0 & \text{if } x \neq x_k \end{cases}$. So $f_{k,j} = f \circ h_k \in \langle f \rangle_R$. Since every function in $(N:G)_0$ can be generated by the functions $f_{k,j}$, we conclude that $\langle f \rangle_R = (N:G)_0$.

Let $I$ be a right ideal of $M_0(G)$ and let $f_j \in I$. Then $\langle f_j \rangle_R = (H_j : G)_0$ for some normal subgroup $H_j$ of $G$. So for $I = \{f_1, f_2, \ldots, f_m\}$, we have

$$I = \sum_{j=1}^{m} \langle f_j \rangle_R = \sum_{j=1}^{m} (H_j : G)_0 = \left( \sum_{j=1}^{m} H_j : G \right)_0 .$$

Letting $H = \sum_{j=1}^{m} H_j$ yields the result. The function $f(x) = \begin{cases} x & \text{if x} \in \text{H} \\ 0 & \text{if } x \notin H \end{cases}$ gives $\langle f \rangle_R = (H:G)_0$ and $(H:G)_0$ is principal. ■

Now we focus our attention on the non-zerosymmetric nearring $M(G)$. For the remainder of the paper, we assume that $G$ is a finite abelian group.

Let $K$ be a subgroup of $G$ and define $C_K = \{c_k \mid k \in K\}$ where $c_k : G \to G$ with $c_k(g) = k$ for all $g \in G$. It is straightforward to verify that $C_K$ is a right ideal of $M(G)$. We note that $G$ being abelian is not a necessary condition for $C_K$ to be a right ideal. Letting $K$ be a subgroup of the center of $G$ is sufficient. Notice, however, that if $K$ is not a subgroup of the center of $G$, then $C_K$ is not a right ideal of $M(G)$. Therefore, we must impose a commutativity condition on $K$ or $G$ in order to ensure that $C_K$ is a right ideal.

Theorem 2. *Let $H$ be a proper subgroup of $G$. Then the set $(H:G) + C_G$ is a proper right ideal of $M(G)$.*

Proof. Since the sum of right ideals is a right ideal ([4], 1.30), then $(H:G) + C_G$ is a right ideal of $M(G)$.

Consider $id_G \in M(G)$ and suppose that $id_G \in (H:G) + C_G$. Then $id_G = f + c_k$ for some constant function $c_k \in C_G$ and $f \in (H:G)$. So $f = id_G - c_k$. In particular,

$f(0) = 0 - k = -k \in H$. So $k \in H$. Since $H$ is a proper subgroup of $G$, we can choose an element $g \in G \backslash H$. Hence $f(g) = g - k \in H$. Since $k \in H$, then $g \in H$, a contradiction. So $id_G \in M(G) \backslash [(H : G) + C_G]$ and $(H : G) + C_G$ is a proper right ideal of $M(G)$. ■

Corollary 3. *Let $H$ and $K$ be proper subgroups of $G$. Then the set $(H : G) + C_K$ is a proper right ideal of $M(G)$.*

Proof. Since $(H : G) + C_K \subseteq (H : G) + C_G \neq M(G)$, the result follows immediately from the theorem. ■

If $G$ is nonabelian, then the previous corollary does not necessarily hold. As an example, consider $G = S_3$. Then for $(1\ 2) \in G$, $\langle c_{(1\ 2)} \rangle_R = M(G)$.

Theorem 4. *For each $f \in M(G)$, $\langle f \rangle_R = (H : G) + C_K$ for some subgroups $H$ and $K$ of $G$.*

Proof. Let $f \in M(G)$ and assume $\operatorname{Im} f = \{x_1, x_2, \ldots, x_n\}$. Fix $x_t \in \operatorname{Im} f$ and let $K$ be the cyclic subgroup of $G$ generated by $x_t$. Since $c_{x_t} \in \langle f \rangle_R$, then $C_K \subseteq \langle f \rangle_R$. Let $f_t = f - c_{x_t}$. Then $f_t \in \langle f \rangle_R$. Since $x_t \in \operatorname{Im} f$, then there exists an element $y \in G$ such that $f(y) = x_t$. Then $f_t(y) = f(y) - c_{x_t}(y) = x_t - x_t = 0$. Using a proof similar to that in Theorem 1, we get $\langle f_t \rangle_R = (H : G)$ where $H$ is the subgroup of $G$ generated by $\operatorname{Im} f_t$. So $(H : G) = \langle f_t \rangle_R \subseteq \langle f \rangle_R$ and $(H : G) + C_K \subseteq \langle f \rangle_R$. But $f = f_t + c_{x_t} \in (H : G) + C_K$ implies that $\langle f \rangle_R \subseteq (H : G) + C_K$. The result now follows. ■

Since the subgroup generated by $\operatorname{Im} f_t = \{x_1 - x_t, x_2 - x_t, \ldots, x_n - x_t\}$ is the same for each $x_t$, then the subgroup $H$ does not depend upon the choice of $x_t$ in the above proof. On the other hand, the subgroup $K$ could also have been defined as the subgroup of $G$ generated by $\operatorname{Im} f$. Therefore, the representation of $\langle f \rangle_R = (H : G) + C_K$ is not unique.

We now reach the main result for right ideals in $M(G)$.

Theorem 5. *If $G$ is a finite abelian group, then every right ideal of $M(G)$ is of the form $(H : G) + C_K$ for some subgroups $H$ and $K$ of $G$.*

Proof. Let $I$ be a right ideal of $M(G)$. Then for every $f_i \in I$, $i = 1, 2, \ldots, n$, $\langle f_i \rangle_R = (H_i : G) + C_{K_i}$ for some subgroups $H_i$ and $K_i$ of $G$ by Theorem 4. So

$$I = \sum_{i=1}^{n} \langle f_i \rangle_R = \sum_{i=1}^{n} [(H_i : G) + C_{K_i}] = \left( \sum_{i=1}^{n} H_i : G \right) + C_{[\sum_{i=1}^{n} K_i]}.$$

Letting $H = \sum_{i=1}^{n} H_i$ and $K = \sum_{i=1}^{n} K_i$ yields the result. ■

Corollary 6. *Every maximal right ideal of* $M(G)$ *is of the form* $(N:G)+C_G$ *for some maximal subgroup* $N$ *of* $G$.

Theorem 7. *If* $G$ *is a finite abelian group and* $N$ *is a maximal subgroup of* $G$, *then every maximal right ideal* $(N:G)+C_G$ *of* $M(G)$ *is principal.*

Proof. Let $g\in G$ such that $g\notin N$ and define $f(x)=\begin{cases} x & \text{if } x\in N \\ 0 & \text{if } x\notin N\end{cases}$. We claim that $\langle f+c_g\rangle_R=(N:G)+C_G$. Clearly $\langle f+c_g\rangle_R\subseteq(N:G)+C_G$. Since $(f+c_g)\circ c_g=c_g\in\langle f+c_g\rangle_R$, then $f\in\langle f+c_g\rangle_R$. As in Theorem 1, $\langle f\rangle_R=(N:G)\subseteq\langle f+c_g\rangle_R$. It follows that $C_N\subseteq\langle f+c_g\rangle_R$. Since $N$ is a maximal subgroup of $G$, then $\langle C_N, c_g\rangle_R=C_G\subseteq\langle f+c_g\rangle_R$. Therefore $(N:G)+C_G\subseteq\langle f+c_g\rangle_R$ and the result follows. ∎

## Applications to nearrings of polynomials

In this section we consider the nearring of polynomials $F[x]$ where $F$ is a finite field. We use the results of the previous section to find some right ideals in $F[x]$.

Let $(H,+)$ be a subgroup of $(F,+)$. We define

$$(H:F)_x=\{p(x)\in F[x]\mid p\circ a\in H \text{ for all } a\in F\}$$

and $P_H=\{p_h\mid h\in H\}$ where $p_h\in F[x]$ with $p_h\circ a=h$ for all $a\in F$. It is easy to check that $(H:F)_x$ and $P_H$ are right ideals of $F[x]$.

In a finite field $F$ with $q$ elements we have that $a^q=a$ for all $a\in F$. So any polynomial that is a multiple of $x^q-x$ acts as the zero function on $F$. Therefore, $(0:F)_x=\{p(x)\cdot(x^q-x)\mid p(x)\in F[x]\}$. Since elements in $P_H$ act as constant functions on $F$, then $P_H=\{h+p(x)\cdot(x^q-x)\mid h\in H \text{ and } p(x)\in F[x]\}$.

Theorem 8. *Let* $I$ *be a right ideal of* $F[x]$ *that contains* $(0:F)_x$. *Then for some subgroups* $H$ *and* $K$ *of* $F$, $I=(H:F)_x+P_K$.

Proof. Consider the function $\varphi: F[x]\to M(F)$ which assigns to each polynomial its corresponding function. It is straightforward to check that $\varphi$ is a nearring homomorphism with kernel $(0:F)_x$. Then $\varphi$ is an epimorphism ([**6**], 2.4). Hence by Theorem 1.31 of [**4**], there is a one-to-one correspondence between right ideals of $M(F)$ and right ideals of $F[x]$ containing $(0:F)_x$. In particular, the inverse image of each right ideal $(H:F)+C_K$ of $M(F)$ under $\varphi$ gives the corresponding right ideal in $F[x]$. But the inverse image of $(H:F)+C_K$ under $\varphi$ is $(H:F)_x+P_K$, and the result follows. ∎

## References

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## Abstract

*For a finite group G, all right ideals of the nearring* $M_0(G) = \{f : G \to G \mid f(0) = 0\}$ *are determined. If G is a finite abelian group, all right ideals of the nearring* $M(G) = \{f : G \to G\}$ *are determined. These results are used to find related right ideals in the nearring of polynomials* $F[x]$ *where* $F$ *is a finite field.*

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