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A construction of k-gonal curves with certain scrollar invariants (**)

1 - Introduction

Let X be a smooth projective curve of genus $g \geq 3$ and $R \in \mathrm{Pic}^k(X)$ with $h^0(X, R) = 2$ and R spanned. Using the pencil R one can produce $k-1$ integers e_i, $l \leq i \leq k-1$, (or $e_i(R)$ if there is any danger of misunderstanding) with $e_1 \geq \ldots \geq e_{k-1} \geq 0$ and $e_1 + \ldots + e_{k-1} = g - k + 1$. These invariants are called the scrollar invariants of R; see [**6**], Th. 2.5, for their geometric interpretation in terms of a certain scroll containing the canonical model of X. There is another interpretation of these invariants. Let $f : X \to \boldsymbol{P}^1$ be the degree k morphism induced by the complete linear system associated to R. We have $f_*(\boldsymbol{O}_X) \cong \boldsymbol{O}_{\boldsymbol{P}^1} \oplus E$ with E vector bundle on X with $\mathrm{rank}(E) = r-1$. Since $\chi(\boldsymbol{O}_X) = 1 - g = \chi(f_*(\boldsymbol{O}_X)) = k + \deg(E)$ (Riemann-Roch), we have $\deg(E) = 1 - k - g$. Every vector bundle on $\boldsymbol{P}^1$ is the direct sum of line bundles and this decomposition is essentially unique (Krull-Schmidt-Remak theorem). Hence there are uniquely determined integers $a_1, \ldots, a_{k-1}$ with $a_1 \geq \ldots \geq a_{k-1}$ such that $E \cong \boldsymbol{O}_{\boldsymbol{P}^1}(a_1) \oplus \ldots \oplus \boldsymbol{O}_{\boldsymbol{P}^1}(a_{k-1})$. Duality theory tell us that $e_i := -a_{k-1-i} - 2$. Alternatinatively, take this formula as our working definition of scrollar invariants. In particular we have $a_1 + \ldots + a_{k-1} = 1 - g - k$ (or see Remark 2.1). Since X is connected, we have $h^0(\boldsymbol{P}^1, f_*(\boldsymbol{O}_X)) = h^0(X, \boldsymbol{O}_X) = 1$. Hence $a_1 < 0$. This interpretation of the scrollar invariants of the morphism f works assuming only R spanned and $h^0(X, R) \geq 2$, just taking a

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base point free pencil $V \subseteq H^0(X, R)$ to obtain a degree k morphism $f: X \to \boldsymbol{P}^1$ with $R \cong f^*(\boldsymbol{O}_{\boldsymbol{P}^1}(1))$. By the projection formula we have $h^0(X, R) = h^0(\boldsymbol{P}^1, \boldsymbol{O}_{\boldsymbol{P}^1}(1)) + h^0(\boldsymbol{P}^1, E(1))$. Thus $h^0(X, R) = 2$ if and only if $a_1 \leq -2$. In this paper we work over an arbitrary algebraically closed field $\boldsymbol{K}$ with char$(\boldsymbol{K}) \neq 2$. In Section 2 we will prove the following result.

Theorem 1.1. *Fix an integer* $t \geq 2$ *and integers* d_j, $t \leq j \leq 2t-1$ *with* $d_i \geq d_j$ *if* $i \leq j$. *Then there exists an integer* x_0 *such that for all integers* $x \geq x_0$ *there exist a smooth connected projective curve* X *and a morphism* $f: X \to \boldsymbol{P}^1$ *with* $\deg(f) = 2t$ *such that, setting* $R \cong f^*(\boldsymbol{O}_{\boldsymbol{P}^1}(1))$ *and calling* $a_1, \ldots, a_{2t-1}$ *the associated invariants of f, we have* $h^0(X, R) = 2$ *and* $a_j = d_j - x$ *for every j with* $t \leq j \leq 2t-1$.

For other constructions of smooth curves with certain scrollar invariants, see [**3**] and [**1**].

2 - Proof of 1.1

In this section we prove Theorem 1.1 and give a variation of Theorem 1.1 for k odd but not prime (see Remark 2.3).

Remark 2.1. Let D be a smooth projective connected curve, X an integral projective curve and $f: X \to D$ a degree k finite morphism. Since f is flat ([**4**], II.9.7), $f_*(\boldsymbol{O}_X)$ is locally free ([**4**], II.9.2 (e)). We have an inclusion j of $\boldsymbol{O}_D$ into $f_*(\boldsymbol{O}_X)$ with locally free cokernel. Set $E := f_*(\boldsymbol{O}_X)/\boldsymbol{O}_D$. Hence E is a rank $k-1$ vector bundle on D. If either char$(\boldsymbol{K}) = 0$ or char$(\boldsymbol{K}) > k$, then the trace map shows that $j_*(\boldsymbol{O}_D)$ is a direct summand of $f_*(\boldsymbol{O}_X)$. If $D = \boldsymbol{P}^1$, then this is true in arbitrary characteristic for the following reason. There are uniquely determined integers $a_1, \ldots, a_{k-1}$ with $a_1 \geq \ldots \geq a_{k-1}$ such that $E \cong \boldsymbol{O}_{\boldsymbol{P}^1}(a_1) \oplus \ldots \oplus \boldsymbol{O}_{\boldsymbol{P}^1}(a_{k-1})$. We saw in the introduction that $a_1 < 0$ and hence $a_i < 0$ for every i. Since $h^1(\boldsymbol{P}^1, \boldsymbol{O}_{\boldsymbol{P}^1}(t)) = 0$ for every integer $t > 0$, every extension of E by $\boldsymbol{O}_{\boldsymbol{P}^1}$ splits and in particular we have $f_*(\boldsymbol{O}_X) \cong \boldsymbol{O}_{\boldsymbol{P}^1} \oplus E$.

Proof of 1.1. Set $k = 2t$. Fix integers b_j, $t \leq j \leq 2t-1$, with $b_i \leq b_j$ for $i \leq j$. Set $B := \boldsymbol{O}_{\boldsymbol{P}^1}(b_t) \oplus \ldots \boldsymbol{O}_{\boldsymbol{P}^1}(b_{2t-1})$. Hence B is a rank t vector bundle on $\boldsymbol{P}^1$. By [**5**], Th. 4.2, there is a smooth connected projective curve Y, a degree t morphism $h: Y \to \boldsymbol{P}^1$ and $L \in \mathrm{Pic}(Y)$ such that $B \cong u_*(L)$. Fix $P \in \boldsymbol{P}^1$ and take any integer u such that $\deg(L^*(ah^{-1}(P))) > 0$ and the linear system $|L^*(ah^{-1}(P))^{\otimes 2}|$ has an effective divisor, D, with only simple zeroes. For instance, it is sufficient to take any integer a such that $2p_a(Y) + 1 \leq at - \deg(L)$. Since char$(\boldsymbol{K}) \neq 2$, the pair

$(L^*(ah^{-1}(P)), D)$ determines uniquely a double covering $u: X \to Y$. The curve X is smooth because D has only simple zeroes. Since $\deg(L^*(ah^{-1}(P))) > 0$, we have $D \neq \phi$. The curve X is connected because $D \neq \phi$. Set $f := h \circ u$. We have $u_*(\boldsymbol{O}_X) \cong \boldsymbol{O}_Y \oplus L(-ah^{-1}(P))$ and hence $f_*(\boldsymbol{O}_X) \cong h_*(\boldsymbol{O}_Y) \oplus h_*(L(-ah^{-1}(P)) \cong u_*(\boldsymbol{O}_X) \oplus B(-aP)$ (projection formula). A priori the relation $f_*(\boldsymbol{O}_X) \cong h_*(\boldsymbol{O}_Y) \oplus B(-aP)$ only gives that t of the $2t-1$ associated integers are the integers we want, the remaing ones coming from a decomposition of $h_*(\boldsymbol{O}_Y)$. However, if for a fixed pair (Y, u) we take $a \gg 0$, say $a \geqslant a_0$, then in this way we obtain that the lowest t associated integers of f are the ones coming from $B(-aP)$. Notice that in this way we fill in all integers $x \gg 0$, say $x \geqslant x_0$, without any gap and with x_0 depending only on the pair (Y, u). We have $h^0(X, R) = 2$ if and only if $h^0(\boldsymbol{P}^1, h_*(\boldsymbol{O}_Y)(P)) + h^0(\boldsymbol{P}^1, B(-aP)(P)) = 2$, i.e. if and only if $h^0(\boldsymbol{P}^1, h_*(\boldsymbol{O}_Y)(P)) \leqslant 2$ and $h^0(\boldsymbol{P}^1, B(-aP)(P)) = 0$ (see the discussion before the statement of 1.1). Take x_0 such that $d_1 - x_0 \leqslant -2$. Since $x \geqslant x_0$ we have $h^0(\boldsymbol{P}^1, B(-aP)(P)) = 0$. Call c_i, $1 \leqslant i \leqslant t-1$, the scrollar invariants of the morphism $h: Y \to \boldsymbol{P}^1$. We have $h^0(\boldsymbol{P}^1, h_*(\boldsymbol{O}_Y)(P)) \leqslant 2$ if and only if the morphism $h: Y \to \boldsymbol{P}^1$ is associated to a complete linear system on Y, i.e. if and only if $c_1 \leqslant 2$ (see the discussion before the statement of 1.1). When $x \gg 0$ this can be checked as in [**5**]: the connectedness of Y is equivalent to the condition $c_1 < 0$.

Look at the proof of Theorem 1.1 just given. By the Riemann-Hurwitz formula we have $p_a(X) = 2p_a(Y) - 2 - \deg(L(-ah^{-1}(P))) = 2p_a(Y) - 2 + \deg(L) + at$. Hence we see that there is at least one congruence class modulo t, say associated to an integer e with $0 \leqslant e < t$, and an integer c such that if $a \geqslant c$, we find a solution (X, f) with $p_a(X) = e + at$. Thus we have proved the following result.

Proposition 2.2. *Fix an integer $t \geqslant 2$ and integers d_j, $t \leqslant j \leqslant 2t-1$, with $d_i \geqslant d_j$ if $i \geqslant j$. Then there exist an integer e with $0 \leqslant e < t$ and an integer c such that for all integers $a \geqslant c$ there exist a smooth connected projective curve X of genus $e + at$, a morphism $f: X \to \boldsymbol{P}^1$ with* $\deg(f) = 2t$ *and an integer x such that, calling $a_1, \ldots, a_{2t-1}$ the associated invariants of f, we have $a_j = d_j - x$ for every j with $t \leqslant j \leqslant 2t-1$.*

Remark 2.3. Fix an integer $k \geqslant 4$ which is not prime, say $k = ct$ with $2 \leqslant a \leqslant t$. We will use simple cyclic coverings of degree c in the sense of [**2**], Example 1.1, to obtain pairs (X, R) with $R \in \operatorname{Pic}(X)$, R spanned, $h^0(X, R) = 2$ and such that we prescribe, up to a twists, the scrollar invariants $e_1, \ldots, e_t$. The proof of Theorem 1.1 is just the case $c = 2$. Here we assume either $\operatorname{char}(\boldsymbol{K}) = 0$ or $\operatorname{char}(\boldsymbol{K}) > c$. Fix integers b_j, $(c-1)t \leqslant j \leqslant ct-1$, with $b_i \leqslant b_j$ for $i \leqslant j$. Set $B := \boldsymbol{O}_{\boldsymbol{P}^1}(b_{(c-1)t}) \oplus \ldots \boldsymbol{O}_{\boldsymbol{P}^1}(b_{ct-1})$. Hence B is a rank t vector bundle on $\boldsymbol{P}^1$. By [**5**], Th. 4.2, there is a smooth connected projective curve Y, a degree t morphism $h : Y \to \boldsymbol{P}^1$ and $L \in \mathrm{Pic}(Y)$ such that $B \cong u_*(L)$. From L we obtain $M \in \mathrm{Pic}(X)$, $M \cong L^*(ah^{-1}(P))$ for some large integer a, such that $\deg(M) > 0$ and the linear system $|M^{\otimes c}|$ has an effective divisor with only simple zeroes. The pair (M, D) uniquely determines a simple cyclic covering $u : X \to Y$ with $\deg(u) = c$. The curve X is smooth because D has only simple zeroes. Since $\deg(M) > 0$, we have $D \neq \phi$. The curve X is connected because $D \neq \phi$. Set $f := u \circ h$. We have $u_*(\boldsymbol{O}_X) \cong \boldsymbol{O}_Y \oplus M^* \oplus \ldots \oplus M^{*\otimes(c-1)}$.

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Abstract

Let X be a smooth projective curve of genus $g \geq 3$ and $R \in \mathrm{Pic}^k(X)$ with $h^0(X, R) = 2$ and R spanned. There are $k-1$ integers e_i, $1 \leq i \leq k-1$, with $e_1 \geq \ldots \geq e_{k-1} \geq 0$ and $e_1 + \ldots + e_{k-1} = g - k + 1$ associated to R (the so-called scrollar invariants of R). Here if k is even we construct a pair (X, R) such that the first $k/2$ scrollar invariants of R are $k/2$ prescribed integers c_i, $1 \leq i \leq k/2$, up to a twist, i.e. $e_i = c_i + x$ for $1 \leq i \leq k/2$ and any $x \in \boldsymbol{Z}$, $x \gg 0$.

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