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On the geometry of a pair of oriented planes (**)

1 - Introduction

In the papers [**5**] of 1991 and [**2**] to appear, concerning the bisectional curvature of a manifold, the notation of *related bases* for a pair of oriented planes plays an essential role.

Even though some geometrical properties of a pair of planes are known since a long time (see for example [**6**]), we think that it is worth translating them into a modern form and adding some more results, useful we hope, for further research.

In Section **3** we recall the definition of related bases for a pair of oriented 2-dimensional subspaces (planes) of a real vector space V, endowed with an inner product g. Then we prove the existence of related bases for any pair of oriented planes of V (Proposition 1).

In Section **4** we show that, given two oriented planes of V, in general there exists essentially only one pair of related bases (Remark 2). The special cases when we have ∞^1 or ∞^2 pairs of related bases (isoclinic planes) are also discussed.

A geometric property of related bases is evidenced in Proposition 2 of Section **5**.

Section **6** studies the special case of the isoclinic planes (Proposition 3, Remarks 3 and 4).

The problem of the existence of pairs of strictly orthogonal planes, transversal with respect to a given pair p, q of oriented planes of V is considered in Section **7**. Proposition 4 and Remark 5 give an exhaustive answer to the problem, showing also that, when $p \cap q = \{0\}$, the solution is strictly connected with the related bases of p, q.

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Last section studies the special case when the vector space V possesses a Hermitian structure. Remarks 6 and 7 show interesting examples of isoclinic planes, depending on the structure.

2 - Preliminaries

Let V be an m-dimensional real vector space and g an inner product on V. In the sequel, the 1-dimensional and the 2-dimensional subspace of V are called *lines* and *planes*, respectively.

Let p, q be two oriented planes of V. Let $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, $\overline{W}$ be oriented orthonormal bases of p and of q. It is well known that we can define

$$\cos pq = \det \begin{pmatrix} g(\overline{X}, \overline{Z}) & g(\overline{X}, \overline{W}) \\ g(\overline{Y}, \overline{Z}) & g(\overline{Y}, \overline{W}) \end{pmatrix} \tag{1}$$

([**1**], p. 9; [**3**], p. 149). So the angle of the planes p, q results to be uniquely determined in the closed interval $[0, \pi]$.

It is worth now recalling some basic facts about orthogonality.

The planes p, q are *orthogonal* if there exists in p (in q) a line, orthogonal to q (to p). In particular, p, q are *strictly orthogonal,* if any line of p (of q) is orthogonal to q (to p). It is easy to prove that if p, q are orthogonal, then we have $\cos pq = 0$; and conversely. In particular, if p,q are strictly orthogonal, then the rank of the matrix in (1) is zero; and conversely.

Let p, q be orthogonal and let $\overline{X}\,(\overline{Z})$ be a unit vector on the line of p (of q) orthogonal to q (to p). There exists in p (in q) only one vector $\overline{Y}\,(\overline{W})$, such that $\overline{X}$, $\overline{Y}$ ($\overline{Z}$, $\overline{W}$) is an oriented orthonormal basis of p (of q). Since $\overline{X}\,(\overline{Z})$ is orthogonal to q (to p), i.e. to any line of q (of p), we have $g(\overline{X}, \overline{Z}) = g(\overline{X}, \overline{W}) = 0$ ($g(\overline{Z}, \overline{X}) = g(\overline{Z}, \overline{Y}) = 0$) and consequently $\cos pq = 0$.

Conversely, $\cos pq = 0$ implies that the rows (the columns) of the matrix are linearly dependent. So there exist real numbers λ, μ (σ, τ), not all of which are zero, such that

$$\lambda\, g(\overline{X}, \overline{Z}) + \mu\, g(\overline{Y}, \overline{Z}) = 0 \qquad \lambda\, g(\overline{X}, \overline{W}) + \mu\, g(\overline{Y}, \overline{W}) = 0$$
$$(\sigma g(\overline{Z}, \overline{X}) + \tau\, g(\overline{W}, \overline{X}) = 0 \qquad \sigma g(\overline{Z}, \overline{Y}) + \tau\, g(\overline{W}, \overline{Y}) = 0)\,.$$

In other words, the non-zero vector $\lambda \overline{X} + \mu \overline{Y}$ of p ($\sigma \overline{Z} + \tau \overline{W}$ of q) results to be orthogonal to the vectors $\overline{Z}, \overline{W}$ of q ($\overline{X}, \overline{Y}$ of p), i.e. orthogonal to q (to p).

If p, q are strictly orthogonal, then the vectors $\overline{X}$ and $\overline{Y}$ ($\overline{Z}$ and $\overline{W}$) are orthogonal to q (to p), i.e. to any vector of q (of p). It follows

$$g(\overline{X}, \overline{Z}) = g(\overline{X}, \overline{W}) = g(\overline{Y}, \overline{Z}) = g(\overline{Y}, \overline{W}) = 0\,. \tag{2}$$

Thus the rank of the matrix in (1) is zero.

Conversely, if the rank is zero, i.e. if (2) is true, since any vector of p (of q) can be written in the form $\lambda\overline{X} + \mu\overline{Y}$ $(\sigma\overline{Z} + \tau\overline{W})$, we find that any vector of p (of q) results to be orthogonal to $\overline{Z}$ and to $\overline{W}$ (to $\overline{X}$ and to $\overline{Y}$), i.e. orthogonal to q (to p).

We complete the section with a remark. Let A be a vector and q an oriented plane of V. Denote by A_q the vector obtained by *orthogonal projection* of A on q. Then, if $\overline{Z}, \overline{W}$ is an oriented orthonormal basis of q, we have

$$A_q = g(A, \overline{Z})\,\overline{Z} + g(A, \overline{W})\,\overline{W}\,. \tag{3}$$

To prove this fact, just check that the vector $A - A_q$ results to be orthogonal to $\overline{Z}$ and to $\overline{W}$, i.e. to q. Note also that A_q does not depend on the orientation of q.

3 - Related bases

We come now to the definition of related bases for a pair p, q of oriented planes. Two oriented orthonormal bases X, Y and Z, W of p and of q respectively are said to be *related bases,* if we have

$$g(X, W) = g(Y, Z) = 0\,. \tag{4}$$

Proposition 1. *For any pair p, q of oriented planes of V there exist always related bases.*

Let $\overline{X}, \overline{Y}$ and $\overline{Z}, \overline{W}$ be oriented orthonormal bases of p and of q, respectively. Then any other pair X, Y and Z, W of oriented orthonormal bases of p, q is given by

$$\begin{aligned} X &= \overline{X}\cos\phi + \overline{Y}\sin\phi \qquad & Z &= \overline{Z}\cos\psi + \overline{W}\sin\psi \\ Y &= -\overline{X}\sin\phi + \overline{Y}\cos\phi \qquad & W &= -\overline{Z}\sin\psi + \overline{W}\cos\psi\,. \end{aligned} \tag{5}$$

Remark now that condition (4) can be written in equivalent form as

$$
\begin{aligned}
&-g(\overline{X}, \overline{Z}) \cos\phi \sin\psi + g(\overline{X}, \overline{W}) \cos\phi \cos\psi \\
&-g(\overline{Y}, \overline{Z}) \sin\phi \sin\psi + g(\overline{Y}, \overline{W}) \sin\phi \cos\psi = 0 \\
&-g(\overline{X}, \overline{Z}) \sin\phi \cos\psi - g(\overline{X}, \overline{W}) \sin\phi \sin\psi \\
&+g(\overline{Y}, \overline{Z}) \cos\phi \cos\psi + g(\overline{Y}, \overline{W}) \cos\phi \sin\psi = 0 .
\end{aligned}
$$

By sum and difference we get the equivalent conditions

$$-(g(\overline{X}, \overline{Z}) - g(\overline{Y}, \overline{W})) \sin(\phi + \psi) + (g(\overline{X}, \overline{W}) + g(\overline{Y}, \overline{Z})) \cos(\phi + \psi) = 0$$

$$(g(\overline{X}, \overline{Z}) + g(\overline{Y}, \overline{W})) \sin(\phi - \psi) + (g(\overline{X}, \overline{W}) - g(\overline{Y}, \overline{Z})) \cos(\phi - \psi) = 0$$

that can be written in the equivalent form

$$(6) \qquad tg(\phi + \psi) = \frac{g(\overline{X}, \overline{W}) + g(\overline{Y}, \overline{Z})}{g(\overline{X}, \overline{Z}) - g(\overline{Y}, \overline{W})} \qquad tg(\phi - \psi) = -\frac{g(\overline{X}, \overline{W}) - g(\overline{Y}, \overline{Z})}{g(\overline{X}, \overline{Z}) + g(\overline{Y}, \overline{W})} .$$

Since there exist always ϕ and ψ satisfying (6), Proposition 1 is proved.

The special cases when $tg(\phi + \psi)$ or $tg(\phi - \psi)$ takes the interminate form $\frac{0}{0}$ will be examined in the next section.

4 - Some remarks

The aim of the present section is to give some information about the pairs of related bases, concerning two given oriented planes p, q.

Remark 1. If X, Y and Z, W are related bases of p, q, then

$$
\begin{array}{llll}
X, \; Y \text{ and } Z, \; W & \qquad & Y, -X \text{ and } W, -Z \\
-X, -Y \text{ and } Z, \; W & & -Y, \; X \text{ and } W, -Z \\
X, \; Y \text{ and } -Z, -W & & Y, -X \text{ and } -W, \; Z \\
-X, -Y \text{ and } -Z, -W & & -Y, \; X \text{ and } -W, \; Z
\end{array}
$$

are related bases for p, q. These eight pairs will be considered as *equivalent* in the sequel.

The proof follows immediately from (4).

Assume now that in (6) $tg(\phi + \psi)$ and $tg(\phi - \psi)$ do not take the form $\frac{0}{0}$. We have

$$\phi + \psi = \lambda + r\pi \qquad \phi - \psi = \mu + s\pi$$

where λ, μ are real numbers and r, s vary in $\mathbb{Z}$. It follows

$$\phi = \frac{1}{2}(\lambda + \mu) + \frac{1}{2}(r + s), \qquad \psi = \frac{1}{2}(\lambda - \mu) + \frac{1}{2}(r - s).$$

Since the angles ϕ and ψ must be regarded mod 2π, we can consider for $r + s$ and $r - s$ only the values 0, 1, 2, 3. On the other hand $r + s$ and $r - s$ are both even or both odd. Consequently, the possible cases for the pair $(r + s, r - s)$ are

$$(0, 0),\ (1, 1),\ (2, 2),\ (3, 3),\ (0, 2),\ (2, 0),\ (1, 3),\ (3, 1).$$

Note that these pairs lead to *equivalent* related bases in the sense defined in Remark 1.

If $tg(\phi + \psi)$ takes the form $\frac{0}{0}$, but $tg(\phi - \psi)$ is not indeterminate, we have $\phi = \psi + \mu + s\pi$ where ψ can vary in $[0, 2\pi)$. So we have ∞^1 pairs of related bases for p, q. Starting from any pair, we can obtain all other non-equivalent pairs by simultaneous rotations of a same angle and in the same sense of the bases of p and of q. Similarly in the case when $tg(\phi - \psi)$ takes the form $\frac{0}{0}$, the rotations now having opposite sense. Note that the senses of rotations on p and on q can be actually compared, since p and q are oriented planes of V.

Last, if $tg(\phi + \psi)$ and $tg(\phi - \psi)$ take the form $\frac{0}{0}$, then (6) implies (2). So $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, $\overline{W}$ satisfy condition (4). Consequently, any oriented orthonormal basis of p and any oriented orthonormal basis of q form a pair of related bases for p, q. In conclusion, there exist ∞^2 non-equivalent related bases for p, q.

Finally, taking into account the equivalence relation of Remark 1, we are now able to summarize the previous results as follows

Remark 2. For a pair of oriented planes p, q, in general, there exists, essentially, only one pair of related bases. There are however special cases when the pairs of related bases are ∞^1 or ∞^2. In the first case, starting from one of these pairs, you obtain, essentially, all the ∞^1 pairs of related bases of p, q by equal or opposite rotations of the bases of p and of q. Similarly, in the second case, independent rotations of the bases of p and of q lead, essentially, to all ∞^2 pairs.

In Section **6** we will see that, when the mentioned special cases occur, then the pair p, q of oriented planes of V enjoys specific geometric properties (Proposition 3).

5 - A geometric property

Given two oriented planes p, q we denote by α $\left(0 \leqslant \alpha \leqslant \frac{\pi}{2}\right)$ the angle that a line of p forms with the plane q and by α_M, α_m the maximum, minimum value of α, as the line varies in p.

Now, let X, Y and Z, W be a pair of related bases of p, q. If

$$A = X \cos \xi + Y \sin \xi \tag{7}$$

is a unit vector on the line, then, taking account of (3), (4), we have

$$A_q = g(X, Z)\, Z \cos \xi + g(Y, W) \sin \xi\,.$$

Since we have $g(A, A_q) = g(A_q, A_q)$, we find

$$\cos^2 \alpha = g(A_q, A_q) = (g(X, Z))^2 \cos^2 \xi + (g(Y, W))^2 \sin^2 \xi$$

and

$$\frac{d \cos^2 \alpha}{d\xi} = -[(g(X, Z))^2 - (g(Y, W))^2] \sin 2\xi\,. \tag{8}$$

Assume first

$$(g(X, Z))^2 \neq (g(Y, W))^2. \tag{9}$$

Then the extreme values α_m, α_M of α are attained when $\xi = 0, \frac{\pi}{2}$, that is when we consider the lines of p defined by X and by Y.

More explicity, if we have

$$|g(X, Z)| > |g(Y, W)| \tag{10}$$

then

$$\cos \alpha_m = |g(X, Z)| \qquad \cos \alpha_M = |g(Y, W)|\,. \tag{11}$$

If we replace (10) with the opposite inequality, then α_m and α_M interchange in (11).

Consider now a line of q and denote by β $\left(0 \leqslant \beta \leqslant \frac{\pi}{2}\right)$ the angle that this line forms with the plane p. Then the extreme values β_m, β_M of β correspond to the lines of q defined by the vectors Z, W. Moreover we have $\beta_m = \alpha_m$ and $\beta_M = \alpha_M$.

We can conclude with

Proposition 2. *Let X, Y and Z, W be related bases of p, q, satisfying the inequality* (9). *Denote by α, β $\left(0 \leqslant \alpha, \beta \leqslant \dfrac{\pi}{2}\right)$ the angle that a line of p, q forms with the plane q, p, respectively. Then the extreme values α_m, α_M of α, β_m, β_M of β are attained in correspondence with the lines of p, of q, defined by the vectors X, Y of p, Z, W of q, respectively. Moreover we have $\alpha_m = \beta_m$ and $\alpha_M = \beta_M$.*

In a different form, the present result can be found in [**6**] (p. 72).

6 - Isoclinic planes

Let $\overline{X}, \overline{Y}$ and $\overline{Z}, \overline{W}$ be a pair of oriented orthonormal bases of p, q, satisfying the condition

$$(12') \qquad g(\overline{X}, \overline{W}) = -g(\overline{Y}, \overline{Z}) \qquad g(\overline{X}, \overline{Z}) = g(\overline{Y}, \overline{W}).$$

By using (5), we can immediately check that any other pair of oriented orthonormal bases of p, q satisfies condition (12′).

In order to evidence the geometric meaning of condition (12′), we consider a pair X, Y and Z, W of related base of p, q. Then condition (12′) reduces to

$$(13') \qquad g(X, Z) = g(Y, W)$$

and (8) implies that the angle α, defined in Sec. **5**, is a constant, i.e. $\alpha = \alpha_*$. Similarly, the angle β, defined in the same section, results to be constant, i.e. $\beta = \beta_*$. Moreover we have $\alpha_* = \beta_*$.

Further, it is easy to check that, when condition

$$(12'') \qquad g(\overline{X}, \overline{W}) = g(\overline{Y}, \overline{Z}) \qquad g(\overline{X}, \overline{Z}) = -g(\overline{Y}, \overline{W}).$$

replaces (12′) and consequently

$$(13'') \qquad g(X, Z) = -g(Y, W)$$

replaces (13′), we arrive to the same conclusion.

In particular, if (12′) and (12″) hold true, then we have (2). So p, q are strictly orthogonal (Sec. 2) and we have $\alpha_* = \beta_* = \frac{\pi}{2}$.

We are now able to state

Proposition 3. *Let X, Y and Z, W be related bases of p, q, satisfying the condition*

$$(g(X, Z))^2 = (g(Y, W))^2. \tag{13}$$

Then any line of p (of q) forms the same angle α_* (β_*) with the plane q (p) and we have $\alpha_* = \beta_*$. In particular, when $\alpha_* = \beta_* = \frac{\pi}{2}$, the planes p, q are strictly orthogonal.

When the pair p, q enjoys the geometrical property of Proposition 3, we say that p and q are *isoclinic planes.*

Remark 3. If p, q are isoclinic planes, then there exist ∞^1 pairs of related bases for p, q and conversely. In particular, if p, q are strictly orthogonal, then there exist ∞^2 pairs of related bases for p, q and conversely.

Note first that, if condition (12′) holds true, then in (6) $tg(\phi + \psi)$ takes the form $\frac{0}{0}$; and conversely. Similarly for condition (12″) and $tg(\phi - \psi)$. The remarks of Sec. 4 lead now to the conclusion. When the dimension m of V is greater or equal to four, we can prove also

Remark 4. For any real number γ satisfying $0 \leq \gamma \leq \frac{\pi}{2}$, there exist isoclinic planes p, q such that $\alpha_* = \beta_* = \gamma$.

Let $E_1, \ldots, E_m$ be an orthonormal basis of V. Put

$$X = E_1 \qquad Z = E_1 \cos\gamma + \frac{\sqrt{2}}{2} E_3 \sin\gamma + \frac{\sqrt{2}}{2} E_4 \sin\gamma$$

$$X = E_2 \qquad W = E_2 \cos\gamma - \frac{\sqrt{2}}{2} E_3 \sin\gamma + \frac{\sqrt{2}}{2} E_4 \sin\gamma .$$

Let p, q be the oriented planes, defined by X, Y and by Z, W, respectively. It is immediate to check that X, Y and Z, W are related bases of p, q satisfying (13), i.e. that p, q are isoclinic planes. Consequently (11) becomes

$$\cos\alpha_* = \cos\beta_* = |g(X, Z)| = |g(Y, W)|$$

and, since we have $g(X, Z) = g(Y, W) = \cos\gamma$, we arrive to the conclusion. The section ends with some additional remarks.

Two planes p, q having one and only one line in common cannot be isoclinic. In effect, the line $p \cap q$ of p forms a zero angle with the plane q. On the contrary, if ν denotes the normal plane of p, q, then the line $p \cap \nu$ of p forms an angle different from zero with the plane q.

Let p' be the plane p with opposite orientation. If we have $q = p$ or $q = p'$, then the planes p, q are isoclinic and $\alpha_* = \beta_* = 0$; and conversely. This fact can be proved as follows. Let X, Y be an orthonormal basis of p. If we have $q = p$, $q = p'$, we choose $Z = X$, $W = \pm Y$, respectively, as basis of q and remark that X, Y and Z, W are related bases for p, q satisfying (13). So by Propositionn 3, p and q are isoclinic planes. Further, from (11) we derive $\alpha_* = \beta_* = 0$. The converse is obvious.

7 - Transversal planes

We have seen in Sec. 4 that the related bases of p, q can be divided into *equivalence classes*. In order to give a geometrical characterization of related bases we need another definition. A plane t is said to be *transversal* to the pair of oriented planes p, q, if t has a line in common with p and a line in common with q.

We are now able to prove

Proposition 4. *Let p, q be oriented planes of V without lines in common. Then, there exists a one-to-one correspondence between the equivalence classes of the related bases of p, q and the non-ordered pairs r, s of non-oriented planes, such that r and s are transversal to p, q and strictly orthogonal.*

In particular, when p, q are not isoclinic planes, there exists only one pair of planes, transversal to p, q and strictly orthogonal.

Let ε be an equivalence class and let X, Y and Z, W be a pair of related bases of p, q belonging to ε. Consider the planes r and s defined by the vectors X, Z and by the vectors Y, W, respectively. Then the planes r and s, that are transversal to p, q, result to be strictly orthogonal by virtue of (4).

To complete the direct part of the proof, we note that any pair of equivalent related bases of p, q listed in Remark 1 of Sec. 4 leads to the non-oriented planes r, s. The last part of the statement follows immediately from Remarks 2 and 3.

Conversely, let $X(Z)$ be a unit vector of the line $p \cap r$ (of the line $q \cap r$). We choose a unit vector $Y(W)$ of the line $p \cap s$ (of the line $q \cap s$) in such a way that X, Y (Z, W) be an oriented basis of p (of q). Then, since r and s are strictly orthogonal, the vectors X and Y (Z and W) are orthogonal and condition (4) is satisfied. So X, Y and Z, W are related bases of p, q.

We end the section with

Remarks 5. If p, q have one and only one line in common, then any plane transvesal to the pair p, q belongs to the 3-dimensional subspace of V, spanned by $p \cup q$. Consequently two planes transversal to p, q cannot be strictly orthogonal. However in this case there exists essentially one pair of related bases of p, q with $X = Z$ or $Y = W$. So one of the planes r, s, occurring in the proof of the direct part of Proposition 4, degenerates into the line $p \cap q$, the other plane being now the normal plane ν; obviously $p \cap q$ is orthogonal to ν. Finally it is easy to check that, if we have $q = p$ or $q = p'$, then there exist ∞^{2m-7} solutions of our problem.

8 - The Hermitian case

From now on we assume that the dimension of the vector space V is even ($m = 2n$), that there exists in V an isomorphism J with the property $J^2 = -1$ and that for any pair X, Y of vectors of V relation

$$g(X, Y) = g(JX, JY) \tag{14}$$

is satisfied. In other words, in the previous section V could be considered as a Riemannian manifold; in the present section V can be regarded as a Hermitian manifold.

We recall first that an oriented plane h is called *holomorphic* iff $Jh = h$. In particular, we say that h is canonically oriented iff X, JX is an oriented orthonormal basis of h. A plane a is called *anti-holomorphic* iff a is orthogonal to Ja. If δ_p denotes the *holomorphic deviation* of the oriented plane p (See for example [4], p. 179), we have $\delta_p = 0, \pi$ when p is holomorphic, and conversely. In particular, we have $\delta_p = 0$, when p is canonically oriented, and conversely. We have $\delta_p = \frac{\pi}{2}$ when p is an anti-holomorphic plane, and conversely.

We are now able to state some results.

Remark 6. Let p be an oriented plane of V. Then p and Jp are isoclinic planes. We have $\alpha_* = \beta_* = \delta_p$ when $0 \leq \delta_p \leq \frac{\pi}{2}$ and $\alpha_* = \beta_* = \pi - \delta_p$ when $\frac{\pi}{2} \leq \delta_p \leq \pi$. Consequently p is holomorphic, anti-holomorphic if $\alpha_* = \beta_* = 0$, $\alpha_* = \beta_* = \frac{\pi}{2}$, respectively.

Remark 7. Let h_1, h_2 be two canonically oriented holomorphic planes. Then, h_1, h_2 are isoclinic planes and the planes r, s of Proposition 4 are anti-holomorphic. We have $\alpha_* = \beta_* = 0$, $\alpha_* = \beta_* = \dfrac{\pi}{2}$ when h_1 and h_2 coincide, are orthogonal, respectively.

To prove Remark 6, we consider an oriented orthonormal basis X, Y of p. Then JX, JY and $JY, -JX$ are oriented orthonormal bases of Jp. Moreover, since by (14) we have $g(X, -JX) = g(Y, JY) = 0$, we find that X, Y and $JY, -JX$ are related bases of p, Jp. Note that (13′) is satisfied. Therefore p and Jp are isoclinic planes and we have

$$\alpha_m = \alpha_M = \alpha_* = \beta_* = \beta_M = \beta_m.$$

Taking into account (11), (14), we can write

$$\cos\alpha_* = \cos\beta_* = |g(X, JY)| = |g(JX, Y)| = |\cos\delta_p|.$$

The remaining part of the proof is immediate.

Finally, we prove Remark 7. Since h_1 and h_2 are canonically oriented, the oriented orthonormal bases of h_1 and of h_2 have the form $\overline{X}, J\overline{X}$ and $\overline{Z}, J\overline{Z}$, where $\overline{X}$ and $\overline{Z}$ are unit vectors of h_1 and h_2, respectively. On the other hand, by Proposition 1 we know that there exist always related bases for the pair of planes h_1, h_2. Thus, let X, JX and Z, JZ be related bases of h_1, h_2. then (4) becomes

$$g(X, JZ) = g(JX, Z) = 0$$

and shows that the planes r, s of Proposition 4, now defined by X, Z and by JX, JZ, are anti-holomorphic. Moreover, taking account of (14), we see immediately that condition (13′) is satisfied. So h_1 and h_2 are isoclinic planes and from (11) we derive

$$\cos\alpha_* = \cos\beta_* = |g(X, Z)|.$$

Finally, if we have $h_1 = h_2$, we can take $Z = X$ and we find $\alpha_* = \beta_* = 0$. The last part of the statement follows immediately by remarking that (1), (14) imply $\cos h_1 h_2 = (g(X, Z))^2$.

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Abstract

The notion of related bases permits to evidence some geometrical properties, concerning the pairs of planes of a real vector space V, endowed with an inner product. Further results are obtained in the special case when V possesses a Hermitian structure.

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