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On hypergroupoid actions (**)

1 - Introduction

We recall some notions of hyperstructure theory. A hypergroupoid $(H; \otimes)$ is a set H endowed with a binary multivalued operation (hyperoperation) i.e a function $< \otimes >$ from $H \times H$ to $\varrho(H)$ the non empty set of subsets of H. A quasi-hypergroup is a hypergroupoid such that $\forall x \in H$, $x \otimes H = H \otimes x$,(the reproduction axiom), where $H \otimes x = \bigcup_{h \in H} h \otimes x$. A hypergroup is a quasi-hypergroup such that for all $(x, y, z) \in H^3$, we have $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ i.e $\bigcup_{u \in x \otimes y} u \otimes z = \bigcup_{v \in y \otimes z} x \otimes v$. For more information about hyperstructure theory, see [**1**], [**2**], [**3**], [**4**] and [**5**].

2 - Results

Now we introduce the following definitions:

Definition 2.1. If $(H, \otimes)$ and $(H', *)$ are two hypergroupoids, then a function $\phi H \to H'$ is called a good homomorphism iff $\phi(x \otimes y) = \phi(x) * \phi(y)$, $\forall (x,y) \in H^2$.

Definition 2.2. Let (G, O) be a hypergroupoid. The action of (G, O) on a non empty set A is a map $\bullet : G \times A \to \varrho(A)$ such that for all $(g_1, g_2) \in G \times G$, $a \in A$: (i) $\bigcup_{t \in g_1 \circ g_2} t \bullet a = \bigcup_{s \in g_2 \bullet a} g_1 \bullet s$, (ii) $\exists e \in G$ such that $a \in e \bullet a$.

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Proposition 2.3. *Let $A^{\varrho(A)}$ be the set of all functions from A to $\varrho(A)$, endowed with the composition operation o, then $\phi : G \to A^{\varrho(A)}$ defined by $\phi(g)(a) = g \bullet a$ is a homomorphism.*

Proof. $(\phi(g_1 O g_2))(a) = \bigcup_{t \in g_1 O g_2} t \bullet a$. From Definition 2.2 one obtains

$$\bigcup_{t \in g_1 O g_2} t \bullet a \bigcup_{s \in g_2 \bullet a} g_1 \bullet s = g_1 \bullet (g_2 \bullet a) = \phi(g_1)(\phi(g_2)(a)) = (\phi(g_1) \circ \phi(g_2))(a).$$

The homomoiphism $\phi : G \to A^{\varrho(A)}$ is called a hyperrepresentation associated with the hypergroupoid action. This process is reversible in the sense that if $\phi : G \to A^{\varrho(A)}$ is any homomorphism then the map from $G \times A \to \varrho(A)$ defined by $g \bullet a = \phi(g)(a)$ satisfies the properties of a hypergroupoid action of G on A.

Definition 2.4. Let $(H, \odot), (K, O)$ be two hypergroupoids and let $\phi : K \to H^{\varrho(H)}$ be a hyperrepresentation determined by the hypergroup action $\bullet$ of K on H. Let G be the set of ordered pairs (h, k) where $(h, k) \in H \times K$ and define the following multiplication on G: $(h_1, k_1) * (h_2, k_2) = (h_1 \odot \phi(k_1)(h_2), k_1 O k_2)$. This multiplication makes G into a hypergroupoid which is denoted by $H \int_\phi K = (H \times K, *)_\phi$.

Remark 2.5. $h_1 \odot \phi(k_1)(h_2) = \bigcup_{t \in \phi(k_l)(h_2)} h_1 \odot t$ where $\phi(k_1)(h_2) = k_1 \bullet h_2$.

Definition 2.6. In the above definition ϕ is called closed if $h \in \phi(k)(h)$ $\forall h \in H$.

Lemma 2.7. *$H \int_\phi K =$ is a semihypergroup.*

Proof. The associative hyperoperation is verified as follows:

$$((h_1, k_1) * (h_2, k_2)) * (h_3, k_3) = (h_1 \odot \phi(k_1)(h_2), k_1 O k_2) * (h_3, k_3)$$

$$= ((h_1 \odot \phi(k_1)(h_2)) \odot \phi(k_1 O k_2)(h_3), (k_1 O k_2) O k_3)$$

$$= ((h_1 \odot \phi(k_1)(h_2) \odot \phi(k_1)(\phi(k_2)(h_3)), (k_1 O k_2) O k_3)$$

$$= ((h_1 \odot \phi(k_1)(h_2)) \odot \phi(k_2)(h_3)), (k_1 O(k_2 O k_3))$$

$$= (h_1, k_1) * ((h_2, \phi(k_2)(h_3), k_2 O k_3)$$

$$= (h_1, k_1) * ((h_2, k_2) * (h_3, k_3)) \ \forall (h_1, k_1), (h_2, k_2), (h_3, k_3) \in H \int_\phi K.$$

Lemma 2.8. *Let* $(H, \odot)$, (K, O) *be two quasi-hypergroups, let* $\phi : K \to H^{\wp(H)}$ *be a hyperrepresentation determined by the hypergroup action* $\bullet$ *of* K *on* H, *then* $H \int_\phi K$ *is a quasi-hypergroup if* ϕ *is closed.*

Proof. The reproduction axiom for $H \int_\phi K$ is verified as follows:

$$(H \times K) * (x, y) = \Big(\bigcup_{k \in H, k \in K} (h, k) * (x, y)\Big) = \bigcup_{h \in H, k \in K} (h \odot \phi(k)(x), kOy)$$

$$= \Big(\bigcup_{k \in H, k \in K} h \odot \phi(k)(x) \bigcup_{h \in K} kOy\Big) = \Big(\bigcup_{t \in \phi(k)(x), h \in H, k \in K} h \odot t, \bigcup_{k \in K} yOk\Big)$$

(use the assumption that K is a quasi-hypergroup)

$$= \Big(\bigcup_{h \in h} t \odot h, \bigcup_{k \in K} yOk\Big) = \Big(\bigcup_{h \in H} x \odot h, \bigcup_{k \in K} yOk\Big)$$

(use the assumption that H is a quasi-hypergroup)

$$= \Big(\bigcup_{h \in H} x \odot (y)(h), \bigcup_{k \in K} yOk\Big)$$

(use the property that ϕ is closed)

$$= \bigcup_{h \in H, k \in K} (x \odot \phi(y)(h), yOk)$$

$$= (x, y) * \bigcup_{h \in H, k \in K} (h, k) = (x, y) * (H \times K) \ \forall (x, y) \in (H \times K).$$

Theorem 2.9. $H \int_\phi K$ *is a hypergroup if the hyperrepresentation* ϕ *is closed.*

Proof. From Lemma 2.7 and Lemma 2.8 the theorem is proved.

Proposition 2.10. *Let* $(H, \odot)$ *and* (K, O) *are two hypergroupoids and* K *is acting trivially on* H, *that is* $k \bullet h = e \bullet h$, *then* $H \int_\phi K$ *is a hypergroup.*

Proof. Since K is acting trivially on H, it results that $\phi(k)(h) = k \bullet h = e \bullet h$ and since $h \in e \bullet h$ this means that $h \in \phi(k)(h)$ for all $k \in K$. So ϕ is closed. From the previous theorem $H \int_\phi K$ is a hypergroup.

Definition 2.11. Let (K, O) be a hypergroupoid acting on the hypergroupoid $(H, \odot)$. K is called acting reversibly on H if the following implication holds: $a \in \phi(k)(b) \Rightarrow$ there exists $k' \in K$ such that $b \in \phi(k')(a)$.

Proposition 2.12. *Let* $(H, \odot), (K, O)$ *be two hypergroupoids. If* K *is acting reversibly on* H, *then the relation on* H *defined by* aRb *if and only if* $a \in \phi(k)(b)$ *for some* $k \in K$, *is an equivalence relation.*

Proof 2.13. R is symmetric for: if aRb, then $a \in \phi(k)(b)$ for some $k \in K$. Since K is reversibly acting on H, then $b \in \phi(k')(a)$ for some $k' \in K$ which means that bRa.

R is reflexive for: $a \in \phi(e)(a)$ for all $a \in H$. Finally if aRb and bRc then $a \in \phi(k_1)(b)$ and $b \in \phi(k_2)(c)$ for some $k_1, k_2 \in K$. So $a \in \phi(k_1)((\phi(k_2)(c)) = \phi(k_1 O k_2)(c) = \bigcup_{t \in k_1 O k_2} \phi(t)(c)$. Thus $a \in \phi(t_\propto)(c)$ for some $\propto \in k_1 O k_2$. Hence aRc and the relation is transitive.

Remark . 2.13. Let $C_a = \{\phi(k)(a) : k \in K\}$ denote the class of a and let $H_a = \{k \in K : a \in \phi(k)(a)\}$ denote the stabilizer of a in H, then we have.

Corollary 2.15. *The action of* K *on* H *is transitive iff* $C_a = \varrho(H)$ *and* H_a *has an identity* e.

Proof. This result is an immediate consequence of Proposition 2.12 and Remark 2.13.

References

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Summary

The aim of this paper is to introduce the notion of hypergroupoid action on hypergroupoid from which we are able to build a hypergroup and among other results it is proved that actions of hypergroupoids are associated with hyperrepresentations. Hypermatrix representations of multivalued structures were studied by T. Vougiouklis in [**6**]. *This study is more general.*

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