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The dual BBGKY hierarchy for the evolution of observables (**)

1 - Introduction

In this paper we develop a rigorous formalism for the description of the evolution of observables in classical systems of particles, [**1**].

As it is known, the description of evolution of many-particle systems as evolution of states is based on the BBGKY hierarchy for the infinite sequence of distribution functions. The solution to the initial value problem for the hierarchy can be constructed using the techniques of the theory of semigroups of operators. In fact, in the space L^1 of infinite sequences of summable functions, there exists a well defined C_0-group of bounded operators, $U(t)$, [**2**].

An equivalent picture can be proposed introducing the hierarchy formally conjugated to the BBGKY one. Such a «dual» hierarchy can describe the evolution of the observables associated to the state of the system, provided that it admits solution in a space correctly chosen from the physical point of view. We recall that the possibility of two equivalent representations for the description of the evolution of finitely many-particle systems has been formulated more explicitly for quantum systems through the Heisenberg and the Schrödinger evolution representations. The dual BBGKY hierarchy, which involves the observables, if settled in a physically suitable functional space, can be seen as the classical analogous one to the Heisenberg representation.

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It is well known that general theorems, [3], [4], ensure the existence of the conjugate semigroup, that we denote here explicitly as $U^*_{L^\infty}(t)$, in the dual space to L^1, i.e. in the space L^∞ and give a new perspective for the solution of the above stated problem. The conjugate operators form the group $U^*_{L^\infty}(t)$ of bounded operators in the space L^∞, but the limit of $U^*_{L^\infty}(t)\, f$, for $t \to 0$ and $\forall f \in L^\infty$, tends to f only in weak-$*$ sense. If $\mathcal{G}$ is the infinitesimal generator of the group $U(t)$, then the conjugate operator $\mathcal{G}^*_{L^\infty}$ is the weak infinitesimal generator of the group $U^*_{L^\infty}(t)$.

Nevertheless, actual examples of observables do not belong to L^∞ and, hence, it is really interesting to find a more general space, physically meaningful for observables, where it is possible to obtain the strongly continuous C_0-semigroup generated by the dual BBGKY hierarchy and which will be introduced in Section **3**.

In the present paper, Section **2** is devoted to the presentation of the problem and to a discussion of its physical motivations. In Section **3** we introduce the function spaces and give the rigorous formulation of the problem. Successively, in Section **4** we explicitly define the semigroup conjugate to the one generated by the BBGKY hierarchy and investigate its properties, giving also the proofs of the main results. In Section **5** we prove the existence theorem for the dual BBGKY hierarchy and construct explicitly the solution to the Cauchy problem for the hierarchy. Finally, in Section **6**, we introduce a different type of semigroup generated by a dual BBGKY hierarchy describing many particles systems with non-symmetric Hamiltonians.

Our present results can hopefully be extended to infinite particle systems, provided that it is possible to give sense to the expression of the average of observables (2.9). We believe that this is possible, at least in case of short range potentials, if the initial correlation functions in the sequence $F(0)$ are bounded. Bounded correlation functions correspond to probability measures describing a different approach to evolution of infinite particle systems, like in the classical papers by Lanford III [**5**], [**6**] and, among others, [**7**] and more recently [**8**].

2 - Formulation of the problem

The BBGKY hierarchy can be written as follows, [**2**]:

$$\frac{d}{dt} F(t) = -\mathcal{L} F(t) + [\mathcal{L}, \mathcal{A}]\, F(t)\,, \tag{2.1}$$

where $F(t)$ is an infinite sequence of functions

$$F(t) = (1, F_1(t, x_1), \dots, F_n(t, x_1, \dots, x_n), \dots), \quad x_i \equiv (q_i, p_i) \in \mathbb{R}^\nu \times \mathbb{R}^\nu,$$

$$\nu = 1, 2, 3, \ i = 1, 2, \dots,$$

representing probability density distributions, i.e. the state of a system.

The position coordinates and the conjugate momenta of the i-th particle are, respectively, q_i and p_i. In Eq. (2.1) the symbol $[.,.]$ denotes the commutator between the Liouville operator $\mathcal{L}$ and the operator $\mathcal{A}$, defined as follows

$$(\mathcal{A}f)_n(x_1, \dots, x_n) = \int_{\mathbb{R}^\nu \times \mathbb{R}^\nu} dx_{n+1} f_{n+1}(x_1, \dots, x_n, x_{n+1}), \ n = 1, 2, \dots, \tag{2.2}$$

$$(\mathcal{L}f)_n(x_1, \dots, x_n) = \{f_n, H_n\}, \ n = 1, 2, \dots, \tag{2.3}$$

where $\{.,.\}$ denotes the Poisson bracket and

$$H_n = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{1=i<j}^{n} \Phi(q_i - q_j) \tag{2.4}$$

is the Hamiltonian of a n-particle system subject to an interaction potential Φ.

It has be proven, [2], that in a L_1-type space of infinite sequence of functions (with norm $\|F(t)\| = 1 + \sum_{n=1}^{\infty} \int_{\mathbb{R}^{\nu n}} \int_{\mathbb{R}^{\nu n}} dx_1 \dots dx_n \, |F_n(t, x_1, \dots, x_n)|$), i.e. the space of states, the hierarchy of equations (2.1) generates the following one-parameter group

$$U(t): t \to F(t), \ t \in \mathbb{R}^1, \qquad U(t) = e^{\mathcal{A}} S(-t)\, e^{-\mathcal{A}}, \tag{2.5}$$

where the mapping $S(-t)$ is a one-parameter group of operators generated by the Liouville operator $\mathcal{L}$, defined by (2.3). The generator of $U(t)$ is denoted by $\mathcal{G}$.

Let us now introduce the hierarchy of equations formally conjugated to (2.1), i.e. the so-called dual BBGKY hierarchy:

$$\frac{d}{dt} G(t) = \mathcal{L} G(t) + [\mathcal{L}, \mathcal{A}^*]\, G(t), \tag{2.6}$$

where we have used the equality $\mathcal{L}^* = -\mathcal{L}$ and, as a consequence, $[\mathcal{L}, \mathcal{A}]^* = [\mathcal{L}, \mathcal{A}^*]$. Moreover in (2.6) $G(t) = (G_0, G_1(t, x_1), \dots, G_n(t, x_1, \dots, x_n), \dots)$ is an infinite sequence of functions symmetric with respect to $x_i \in \mathbb{R}^\nu \times \mathbb{R}^\nu$ (G_0 is a number) and $\mathcal{A}^*$ is the formal adjoint of $\mathcal{A}$, defined by (2.2). From the definition of adjoint operator it can readily be deduced that the n-th component of $\mathcal{A}^*$ writes as follows:

$$(2.7)\qquad (\mathcal{A}^* f)_n(x_1, \dots, x_n) = \sum_{i=1}^{n} f_{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),\ n = 1, 2, \dots,$$

where $(\mathcal{A}^* f)_0 = 0$.

It is natural to consider the problem of characterizing the structure of the mapping generated by the dual BBGKY hierarchy (2.6). We shall prove that it comes out to be a semigroup (group) of operators and state its properties.

Let us now give some elucidations about the physical motivations of the problem. If the Hamiltonian of the system is symmetric with respect to its arguments, the averages $\langle G \rangle(t)$ of the observables $G(t)$, ($G(t) = (G_0, G_1(t, x_1), \dots, G_n(t, x_1, \dots, x_n), \dots)$ is an infinite sequence of functions symmetric with respect to $x_i \in \mathbb{R}^\nu \times \mathbb{R}^\nu$ and G_0 is a number) can be written as

$$(2.8)\qquad \langle G \rangle(t) = (e^{\mathcal{A}} G(0)\, F(t))_0,$$

where $(.)_0$ denotes the 0-th component of the sequence $e^{\mathcal{A}} G(0)\, F(t)$, where $e^{\mathcal{A}} = \sum_{n=0}^{\infty} \frac{\mathcal{A}^n}{n!}$, and

$$G(0)\, F(t) = (G_0, G_1(0, x_1)\, F_1(t, x_1), \dots, G_n(0, x_1, \dots, x_n)\, F_n(t, x_1, \dots, x_n), \dots).$$

Hence

$$(e^{\mathcal{A}} G(0)\, F(t))_0$$

$$= G_0 + (\mathcal{A} G(0)\, F(t))_0 + \left(\frac{\mathcal{A}^2}{2} G(0)\, F(t) \right)_0 + \dots + \left(\frac{\mathcal{A}^n}{n!} G(0)\, F(t) \right)_0 + \dots .$$

From (2.8) it follows that there are two possible different methods to describe the evolution of a system of particles. One description is based on the BBGKY hierarchy (2.1) and the mapping $U(t)$ which determines the evolution of the probability density distributions

$$F(t) = U(t)\, F(0).$$

On the other hand, if we develop the definition of average (2.8), introducing the formally adjoint mapping $U^*(t)$, we obtain:

$$\langle G\rangle(t) = (e^{\alpha} G(0)\, F(t))_0 = (e^{\alpha} G(0)\, U(t)\, F(0))_0 = (e^{\alpha} U^*(t)\, G(0)\, F(0))_0 = (e^{\alpha} G(t) F(0))_0, \tag{2.9}$$

because $G(0)\, U(t)\, F(0) = U^*(t)\, G(0) F(0)$.

Eq. (2.9) suggests an alternative description of the evolution of a system in terms of the mapping $U^*(t): t \to G(t)$, $t \in \mathbb{R}^1$, which rules the evolution of the observables, i.e.

$$G(t) = U^*(t)\, G(0).$$

We shall give some example in the following Section. Finally we remark that for systems composed of a finite number of particles the group generated by the Liouville operator (2.3) is the mapping $S(-t)$. Thus, the group of operators conjugated to $S(-t)$ comes out to be $S(t)$, because $S(t) = (S(-t))^*$.

3 - Definitions and preliminary results

We now introduce the function spaces to be used for the description of the observables of a many particles system.

Let us consider the set of continuous functions, symmetric with respect to exchanges of their arguments x_i,

$$g_n(x_1, \ldots, x_n), \quad x_i \equiv (q_i, p_i) \in \mathbb{R}^\nu \times \mathbb{R}^\nu, \quad \nu = 1, 2, 3,$$

defined on the phase space $\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}$ of the n-particle system, such that the functions $\left(1 + \sum_{i=1}^{n} (|p_i|^2 + |q_i|^2)\right)^{-1} g_n(x_1, \ldots, x_n)$, are bounded with respect to $(x_1, \ldots, x_n) \in \mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}$. This means that functions from this space increase at infinity not faster than the polynomial $\left(1 + \sum_{i=1}^{n} (|p_i|^2 + |q_i|^2)\right)$. Then this set, equipped with the norm:

$$\|g_n\|_n = \max_{(x_1, \ldots, x_n) \in \mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}} \left(1 + \sum_{i=1}^{n} (|p_i|^2 + |q_i|^2)\right)^{-1} |g_n(x_1, \ldots, x_n)|$$

is a Banach space, which we denote by C_2. We now denote by $\mathcal{C}_2$ the Banach space of infinite sequences

$$g = (g_0, g_1(x_1), \ldots, g_n(x_1, \ldots, x_n), \ldots)$$

of functions $g_n \in C_2$ (g_0 is a number), with the norm:

$$\|g\| = \max_{n \geq 0} \frac{1}{n!} \|g_n\|_n . \tag{3.1}$$

We denote by $\mathcal{C}_{2,0}$ the subset of the Banach space $\mathcal{C}_2$ consisting of the finite sequences of continuously differentiable functions with compact support. Clearly, the set $\mathcal{C}_{2,0} \subset \mathcal{C}_2$ is everywhere dense in $\mathcal{C}_2$.

We remark that the sequences of functions identified with the observables of a many particles system should belong to the space $\mathcal{C}_2$, and, hence, our choice about the function space is quite natural. As a matter of fact, the observables of additive type for Hamiltonian systems, [**2**], are represented by sequences like

$$a = (a_0, a_1(x_1), \ldots, a_n(x_1, \ldots, x_n), \ldots),$$

where $a_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_1(x_i)$. It is the case, for example, of momentum, i.e. $a_1(x_i) = p_i$, kinetic energy, i.e. $a_1(x_i) = \frac{1}{2} p_i^2$, and number of particles, i.e. $a_1(x_i) = 1$ (then $a_n = n$).

Keeping in mind (2.8), or in the componentwise form:

$$g_n(x_1, \ldots, x_n) = \sum_{l=0}^{n} (-1)^{n-l} \sum_{1 = i_1 < \ldots < i_l}^{n} a_l(x_{i_1}, \ldots, x_{i_l}),$$

we can define the sequence g from the sequence a, through the equality

$$g = e^{-\mathcal{A}^*} a .$$

Hence, to observables of additive kind it corresponds the sequence

$$g = (0, a_1(x_1), 0, \ldots).$$

Another meaningful example of observables is furnished by observables of binary type, represented by sequences

$$b = (b_0, b_1(x_1), \ldots, b_n(x_1, \ldots, x_n), \ldots),$$

where the 0-th and 1-st components, b_0 and b_1, are equal to zero and $b_n(x_1, \ldots, x_n) = \sum_{1 = i < j}^{n} b_2(x_i, x_j)$, for $n \geq 2$. In this case we have

$$g = (0, 0, b_2(x_1, x_2), 0, \ldots)$$

and, consequently, the energy, or Hamiltonian (2.4), is represented by the sequence

$$g = \left(0,\ \frac{p_1^2}{2},\ \Phi(q_1 - q_2),\ 0,\ \dots\right)$$

which belongs to the space $\mathcal{C}_2$ under suitable assumptions on the function Φ.

In what follows we shall study a system of particles interacting through the potential Φ, satisfying the assumptions:

$$\Phi \in C^2, \tag{3.2a}$$

$$\left| \operatorname*{grad}_{(q_1, \dots, q_n)} \sum_{i<j=1}^{n} \Phi(q_i - q_j) \right| \leqslant c\left(1 + \sum_{i=1}^{n} q_i^2\right)^{1/2}, \tag{3.2b}$$

where $c < \infty$ is a suitable constant and C^2 denotes the space of twice continuously differentiable functions on $\mathbb{R}^\nu$.

It is known, [**9**], that, if the conditions (3.2) are fulfilled, there exist the global in time solutions of the Hamilton equations; i.e. for an arbitrary $t \in \mathbb{R}^1$ and initial data $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}$ the phase trajectory $\boldsymbol{X}(t, \boldsymbol{x}) \equiv \{X_i(t, x_1, \dots, x_n)\}_{i=1}^n$ is a unique well defined function.

The mapping in the phase space $\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}$:

$$T_t \boldsymbol{x} = \boldsymbol{X}(t, \boldsymbol{x}), \qquad \forall t \in \mathbb{R}^1$$

induces the following mapping in the space $\mathcal{C}_2$:

$$\begin{aligned}(S(t)\, g)_n(x_1, \dots, x_n) &= g_n(T_t(x_1, \dots, x_n)) \\ &= g_n(X_1(t, x_1, \dots, x_n), \dots, X_n(t, x_1, \dots, x_n)),\ n = 1, 2, \dots .\end{aligned} \tag{3.3}$$

As we can see from definition (3.3), the one-parameter family of operators $S(t)$ is defined in the space $\mathcal{C}_2$ and forms a strongly continuous group. Moreover, this group is quasi-bounded in $\mathcal{C}_2$, as we prove in the following

Lemma 3.1. *The group $S(t)$, defined in* (3.3), *is a strongly continuous, quasi-bounded group in $\mathcal{C}_2$, satisfying the following estimate:*

$$\|S(t)\| \leqslant e^{(c+1)t}, \tag{3.4}$$

where c is the constant introduced in the assumption (3.2*b*).

Proof. Before proving the estimate (3.4) on $\|S(t)\|$ we shall state an auxiliary inequality.

Let us introduce the function

$$E(t) = 1 + \sum_{i=1}^{n} (P_i^2(t) + Q_i^2(t)),$$

where $X_i(t) = (Q_i(t), P_i(t))$, $i = 1, \ldots, n$, are solutions of the Hamilton equations and, for convenience, we use $X_i(t) \equiv X_i(t, x_1, \ldots, x_n)$. Then, using the Schwarz inequality, together with the condition (3.2b), we have:

$$\frac{d}{dt}E(t) = 2\sum_{i=1}^{n}\left(\left\langle P_i(t), \frac{d}{dt}P_i(t)\right\rangle + \left\langle Q_i(t), \frac{d}{dt}Q_i(t)\right\rangle\right)$$

$$\leqslant 2\left(\left(\sum_{i=1}^{n}P_i^2(t)\right)^{1/2}\left(\sum_{j=1}^{n}\left(\frac{d}{dt}P_j(t)\right)^2\right)^{1/2} + \left(\sum_{i=1}^{n}Q_i^2(t)\right)^{1/2}\left(\sum_{j=1}^{n}\left(\frac{d}{dt}Q_j(t)\right)^2\right)^{1/2}\right)$$

$$\leqslant 2\left(\sum_{i=1}^{n}P_i^2(t)\right)^{1/2}\left(c\left(\sum_{i=1}^{n}Q_i^2(t)+1\right)^{1/2} + \left(\sum_{i=1}^{n}Q_i^2(t)\right)^{1/2}\right)$$

$$\leqslant 2(c+1)\left(\sum_{i=1}^{n}P_i^2(t)\right)^{1/2}\left(\sum_{i=1}^{n}Q_i^2(t)+1\right)^{1/2}$$

$$\leqslant (c+1)\left(1+\sum_{i=1}^{n}(P_i^2(t)+Q_i^2(t))\right) \equiv (c+1)\,E(t).$$

Hence, for arbitrary t one holds:

$$\frac{E(t)}{E(0)} \leqslant e^{(c+1)t}, \tag{3.4$'$}$$

where $E(0) = 1 + \sum_{i=1}^{n}(p_i^2 + q_i^2)$. According to definition (3.3) and the estimate (3.4$'$), we have

$$\|S(t)\,g\| = \max_{n\geqslant 0}\frac{1}{n!}\max_{x_1,\ldots,x_n}\left(1+\sum_{i=1}^{n}(p_i^2+q_i^2)\right)^{-1}$$

$$\cdot\left|g_n(X_1(t, x_1, \ldots, x_n), \ldots, X_n(t, x_1, \ldots, x_n))\right|$$

$$\leqslant \max_{n\geqslant 0}\frac{1}{n!}\max_{x_1,\ldots,x_n}\frac{E(t)}{E(0)}\max_{X_1(t),\ldots,X_n(t)}E^{-1}(t)\left|g_n(X_1(t), \ldots, X_n(t))\right| \leqslant e^{(c+1)t}\|g\|.$$

As a result, the Lemma is proven. ■

From (3.4) it follows that the infinitesimal generator $\mathcal{L}$ of the group $S(t)$ exists, it is closed and $\mathcal{L}S(t) = S(t)\mathcal{L}$. On the subset $\mathcal{C}_{2,0} \subset \mathcal{C}_2$ the generator $\mathcal{L}$ is defined as follows:

$$(\mathcal{L}g)_n(x_1, \ldots, x_n) = \{g_n, H_n\}$$

$$\equiv \sum_{i=1}^{n}\left(\left\langle p_i, \frac{\partial}{\partial q_i}\right\rangle - \left\langle\frac{\partial}{\partial q_i}\sum_{i\neq j=1}^{n}\Phi(q_i-q_j)\frac{\partial}{\partial p_i}\right\rangle\right)g_n(x_1, \ldots, x_n),\; n = 1, 2, \ldots, \tag{3.5}$$

where the symbol $\langle . , . \rangle$ denotes the inner product. We remark that the operator $\mathcal{A}^*$, as defined by (2.7) exists and is bounded in the space $\mathcal{C}_2$. Moreover:

$$\|\mathcal{A}\|^* \leqslant 1 .$$

As a consequence, the operators $e^{\pm \mathcal{A}^*}$ are defined in the space $\mathcal{C}_2$ and

$$\|e^{\pm \mathcal{A}^*}\| \leqslant e .$$

In particular, $e^{\mathcal{A}^*} e^{-\mathcal{A}^*} = I$, where I is the unit operator.

4 - Main results

Let us consider the following mapping in the space $\mathcal{C}_2$

$$\mathcal{U}^*(t) \stackrel{\text{def}}{=} e^{-\mathcal{A}^*} S(t)\, e^{\mathcal{A}^*}, \tag{4.1}$$

where the operators $\mathcal{A}^*$ and $S(t)$ are defined, respectively, by (2.7) and (3.3) (see also (2.5)).

In what follows we shall use also an alternative representation for the semigroup $\mathcal{U}^*(t)$, which is identical to (4.1):

$$\mathcal{U}^*(t) = \sum_{n=0}^{\infty} \frac{1}{n!} [\dots[S(t), \underbrace{\mathcal{A}^*], \dots], \mathcal{A}^*}_{\text{n-times}}], \tag{4.2}$$

where the bracket $[. , .]$ again denotes the commutator. We remark that the series (4.2) is actually a polynomial, as follows from definition (2.7) of the operator $\mathcal{A}^*$.

The mapping $\mathcal{U}^*(t)$ (as defined in (4.1) or in (4.2)) has the following componentwise form:

$$\begin{aligned} (\mathcal{U}^*(t)\, g)_n(x_1, \dots, x_n) = \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{(-1)^k}{k!(m-k)!} \\ \cdot \sum_{\substack{i_1, \dots, i_k = 1 \\ i_1 \neq \dots \neq i_k}}^{n} \sum_{\substack{j_1, \dots, j_{m-k} = 1; \\ j_1 \neq \dots \neq j_{n-k}; \{j_1, \dots, j_{m-k}\} \neq \{i_1, \dots, i_k\}}}^{n} S^{n-k}\Big(t, x_1, \dots, \overset{i_1}{\vee}, \dots, \overset{i_k}{\vee}, \dots, x_n\Big) \\ \cdot g_{n-m}\Big(x_1, \dots, \overset{i_1}{\vee}, \dots, \overset{i_k}{\vee}, \dots, \overset{j_1}{\vee}, \dots, \overset{j_{m-k}}{\vee}, \dots, x_n\Big), \end{aligned} \tag{4.3'}$$

where we have used the notation

$$\Big(x_1, \dots, \overset{i}{\vee}, \dots, x_n\Big) \stackrel{\text{def}}{=} (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

or in the nonsymmetrized form:

$$(\mathcal{U}^*(t)\,g)_n(x_1,\,\dots,\,x_n)$$

$$(4.3'')\quad = \sum_{m=0}^{n} \frac{n!}{(n-m)!} \sum_{k=0}^{m} \frac{(-1)^k}{k!(m-k)!} S^{n-k}(t,\,x_1,\,\dots,\,x_{n-k})\,g_{n-m}(x_1,\,\dots,\,x_{n-m}),$$

As an example we deduce the expression (4.3″) in the simplest cases:

$$(4.3a)\qquad (\mathcal{U}^*(t)\,g)_1(x_1) = S^1(t,\,x_1)\,g_1(x_1),$$

$$(4.3b)\qquad \begin{aligned}(\mathcal{U}^*(t)\,g)_2(x_1,\,x_2) &= S^2(t,\,x_1,\,x_2)\,g_2(x_1,\,x_2)\\ &+(S^2(t,\,x_1,\,x_2)-S^1(t,\,x_1))\,g_1(x_1)+(S^2(t,\,x_1,\,x_2)-S^1(t,\,x_2))\,g_1(x_2),\end{aligned}$$

with S^1 and S^2 defined through (4.3a) and (4.3b).

The properties of the mapping $\mathcal{U}^*(t)$ are stated in the following theorems 4.1 and 4.2.

Theorem 4.1. *If the interaction potential Φ satisfies the conditions* (3.2), *then the one-parameter family of operators $\mathcal{U}^*(t)$ is defined and bounded in the space $\mathcal{C}_2$ $\forall t \in \mathbb{R}^1$:*

$$\|\mathcal{U}^*(t)\| \leq e^2 e^{(c+1)t}.$$

Moreover it is strongly continuous and so is a quasi-bounded C_0-group of type $c+1$.

Proof. The mapping $\mathcal{U}^*(t)$, as defined by (4.1), is the product of quasi-bounded and bounded operators defined in the space $\mathcal{C}_2$; moreover:

$$\|e^{\pm\mathfrak{a}^*}\| \leq e$$

and

$$\|S(t)\| \leq e^{(c+1)t},$$

thus

$$\|\mathcal{U}^*(t)\| \leq e^2 e^{(c+1)t}.$$

Furthermore, the one-parameter family of operators $\mathcal{U}^*(t)$, $t \in \mathbb{R}^1$, has the group property. In fact:

$$\mathcal{U}^*(t_1)\,\mathcal{U}^*(t_2) = e^{-\mathfrak{a}^*} S(t_1)\,e^{\mathfrak{a}^*} e^{-\mathfrak{a}^*} S(t_2)\,e^{\mathfrak{a}^*}$$

$$= e^{-\mathfrak{a}^*} S(t_1)\,S(t_2)\,e^{\mathfrak{a}^*} = e^{-\mathfrak{a}^*} S(t_1+t_2)\,e^{\mathfrak{a}^*} = \mathcal{U}^*(t_1+t_2)$$

for arbitrary $t_1,\,t_2 \in \mathbb{R}^1$.

The strong continuity of the group $\mathcal{U}^*(t)$, as defined by Eqs. (4.1) or (4.2) follows from the strong continuity property of the group $S(t)$ and from the boundness of the operators $e^{\pm \mathcal{A}}$. ■

In order to establish further properties of the group $\mathcal{U}^*(t)$ (Theorem 4.2), we need the following preliminary lemmas:

Lemma 4.1. *For arbitrary $g \in \mathcal{C}_{2,0} \subset \mathcal{C}_2$ the following equality holds*

$$([\mathcal{L}, \mathcal{A}^*]\, g)_n(x_1, \ldots, x_n) = - \sum_{i \neq j = 1}^{n} \left\langle \frac{\partial}{\partial q_j} \Phi(q_j - q_i) \frac{\partial}{\partial p_j} \right\rangle g_{n-1}\Big(x_1, \ldots, \overset{i}{\vee}, \ldots, x_n\Big).$$

Proof. The statement follows directly from definitions (2.7) and (3.5). ■

Lemma 4.2. *If $g \in \mathcal{C}_{2,0} \subset \mathcal{C}_2$, the following identity holds*

$$[[\mathcal{L}, \mathcal{A}^*], \mathcal{A}^*]\, g = 0\,.$$

Proof. The identity follows from Lemma 4.1, from the symmetry of functions $g_n \in \mathcal{C}_{2,0}(\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n})$ with respect to their arguments and from direct calculations of the commutators. ■

Let us now give the following lemma

Lemma 4.3. *On the subspace $\mathcal{C}_{2,0} \subset \mathcal{C}_2$ the infinitesimal generator $\mathcal{G}^*$ of $\mathcal{U}^*(t)$ coincides with the operator $\mathcal{L} + [\mathcal{L}, \mathcal{A}^*]$.*

Proof. Recalling the group properties of $\mathcal{U}^*(t)$ and using the representation (4.2), we obtain, $\forall g \in \mathcal{C}_{2,0}$, the following limit in the sense of the strong convergence in the space $\mathcal{C}_2$:

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} (\mathcal{U}^*(t + \Delta t)\, g - \mathcal{U}^*(t)\, g) = \lim_{\Delta t \to 0} \mathcal{U}^*(t) \frac{1}{\Delta t} (\mathcal{U}^*(\Delta t) - I)\, g$$

$$= \mathcal{U}^*(t) \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Bigg((S(\Delta t) - I)\, g + [(S(\Delta t) - I), \mathcal{A}^*]\, g$$

$$+ \sum_{n=2}^{\infty} \frac{1}{n!} [\ldots [(S(\Delta t) - I), \underbrace{\mathcal{A}^*], \ldots], \mathcal{A}^*}_{\text{n-times}}]\, g \Bigg)$$

$$= \mathcal{U}^*(t)(\mathcal{L} g + [\mathcal{L}, \mathcal{A}^*]\, g + \mathcal{R} g)\,,$$

where $\mathcal{R} g \equiv \sum_{n=2}^{\infty} \frac{1}{n!} [\ldots [\mathcal{L}, \underbrace{\mathcal{A}^*], \ldots], \mathcal{A}^*}_{\text{n-times}}]\, g$.

According to Lemma 4.2, the remainder $\mathcal{R}$ is identically equal to zero. ■

Remark. If particles are not interacting through a pair potential like (2.4), but through the general s-multiple-type interaction potential, such that the Hamiltonian of n-particle system has the form

$$H_n = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{1=i_1<\ldots<i_s}^{n} \Phi(x_{i_1}, \ldots, x_{i_s}),$$

then the generator $\mathcal{G}^*$ of the group $\mathcal{U}^*(t)$ coincides with the operator

$$\mathcal{L} + \sum_{i=1}^{s-1} [\ldots [\mathcal{L}, \underbrace{\mathcal{A}^*], \ldots], \mathcal{A}^*]}_{\text{i-times}}.$$

We are now in position to state the following main theorem:

Theorem 4.2. *Under the assumptions of Theorem* 4.1, *there exists the infinitesimal generator* $\mathcal{G}^*$ *of the group* $\mathcal{U}^*(t)$. *It is closed,* $\mathcal{G}^*\mathcal{U}^*(t) = \mathcal{U}^*(t)\mathcal{G}^*$ *and, on the subspace* $\mathcal{C}_{2,0} \subset \mathcal{C}_2$,

$$\mathcal{G}^* = \mathcal{L} + [\mathcal{L}, \mathcal{A}^*], \tag{4.4}$$

or, in componentwise form:

$$(\mathcal{G}^* g)_n(x_1, \ldots, x_n)$$

$$= (\mathcal{L}g)_n(x_1, \ldots, x_n) - \sum_{i \neq j=1}^{n} \left\langle \frac{\partial}{\partial q_j} \Phi(q_j - q_i), \frac{\partial}{\partial p_j} \right\rangle g_{n-1}\Big(x_1, \ldots, \overset{i}{\vee}, \ldots, x_n\Big),$$

where $g \in \mathcal{C}_{2,0}$ *and* $\mathcal{L}$ *is defined by* (3.5).

Proof. The proof is a straightforward consequence of Lemmas 4.1, 4.2 and 4.3. ∎

We remark that Theorem 4.2 allows to claim the existence of an infinitesimal generator, $\mathcal{G}^*$, of the group $\mathcal{U}^*(t)$, as follows from the general properties of groups of quasi-bounded operators, [3], [4]. Moreover the generator $\mathcal{G}^*$ is closed and on its domain of definition $\mathcal{C}_{2,0} \subset \mathcal{D}(\mathcal{G}^*) \subset \mathcal{C}_2$ one has that $\mathcal{G}^*\mathcal{U}^*(t) = \mathcal{U}^*(t)\,\mathcal{G}^*$.

5 - Existence and uniqueness to the Cauchy problem of the dual BBGKY hierarchy

As a consequence of Theorems 4.1 and 4.2 we can prove an existence and uniqueness theorem for the Cauchy problem of the dual BBGKY hierarcy in the space $\mathcal{C}_2$:

$$\frac{d}{dt}G(t) = \mathcal{L}G(t) + [\mathcal{L}, \mathcal{A}^*]\, G(t) \tag{5.1}$$
$$G(0) = G^0.$$

Theorem 5.1. *If the potential Φ satisfies the conditions* (3.2), *then the Cauchy problem for the dual BBGKY hierarchy* (5.1) *has a unique, global in time, solution in the space $\mathcal{C}_2$*:

$$G(t) = e^{-\mathcal{A}^*} S(t) e^{\mathcal{A}^*} G(0). \tag{5.2}$$

For initial data $G(0) = G^0 \in \mathcal{C}_{2,0} \subset \mathcal{D}(\mathcal{G}^*) \subset \mathcal{C}_2$ the solution is a strong one and for arbitrary $G^0 \in \mathcal{C}_2$ the solution is a weak one.

Proof. The statement of the theorem follows immediately from general results of semigroup theory, [3], [4] (see also (4.1)). ■

Remark. According to the explicit form of the dual BBGKY hierarchy (5.1) we can construct the expression of solution (5.2) by iteration. Hierarchy (5.1) is actually a recursion relation:

$$\frac{\partial}{\partial t} G_1(t, x_1) = \left\langle p_1, \frac{\partial}{\partial q_1} \right\rangle G_1(t, x_1),$$

$$\frac{\partial}{\partial t} G_2(t, x_1, x_2) = \sum_{i=1}^{2} \left(\left\langle p_i, \frac{\partial}{\partial q_i} \right\rangle - \left\langle \frac{\partial}{\partial q_i} \Phi(q_1 - q_2), \frac{\partial}{\partial p_i} \right\rangle \right) G_2(t, x_1, x_2) \tag{5.3}$$
$$- \sum_{i=1}^{2} \left\langle \frac{\partial}{\partial q_i} \Phi(q_1 - q_2), \frac{\partial}{\partial p_i} \right\rangle G_1(t, x_i),$$

and so on.

Note that each equation is coupled only with the preceding ones.

The first equation of (5.3), which represent the first component of hierarchy (5.1), is the Liouville equation for the free particle. The other equations are nonhomogeneous Liouville equations, which can be integrated with respect to time (the result is exprimed by (4.3)). In such a way, constructing the solution of the hierarchy (5.1) by iteration and performing the integration with respect to the time variable, we again succeed in obtaining the expression (5.2) for the solution to the initial value problem (5.1).

6 - An example of the dual hierarchy semigroup

We remark that the symmetry with respect to the permutations of their arguments of the Hamiltonian defined by (2.4) and of the functions belonging to the space $\mathcal{C}_2$ has played a fundamental role in the construction of the group $\mathcal{U}^*(t)$.

If we deal with non-symmetric Hamilton functions, the group generated by the dual BBGKY hierarchy has another structure. We shall consider the simplest example of such a type of dynamical system, namely the one-dimensional system of particles interacting with their next neighbours, [**2**]. The related Hamiltonian has the form

$$H_n = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} \Phi(q_i - q_{i+1}) . \tag{6.1}$$

We assume that the potential Φ satisfies the conditions (3.2), such that existence of the group $S(t)$ (3.3) and its usual properties are guaranteed. In this case the suitable function space is the space of sequences $g = (g_0, g_1(x_1), \ldots, g_n(x_1, \ldots, x_n), \ldots)$ of continuous functions $g_n(x_1, \ldots, x_n)$, $n = 1, 2, \ldots, (g_0$ is a number) defined on the phase space of the system, $\mathbb{R}^n \times \mathbb{R}^n$, with the norm

$$\|g\| = \max_{n \geq 1} \alpha^n \max_{(x_1, \ldots, x_n) \in \mathbb{R}^n \times \mathbb{R}^n} \left(1 + \sum_{i=1}^{n} (q_i^2 + p_i^2)\right)^{-1} |g_n(x_1, \ldots, x_n)|$$

where α is a number: $0 < \alpha < 1$. We denote such a Banach space by $\mathcal{C}_2^{\alpha}$.

In the space $\mathcal{C}_2^{\alpha}$ the operator $\mathcal{a}^*$ (compare with the «symmetric» case, as defined in (2.7)),

$$(\mathcal{a}^* g)_n(x_1, \ldots, x_n) = g_{n-1}(x_1, \ldots, x_{n-1}), \qquad n = 1, 2, \ldots, \tag{6.2}$$

is defined and bounded. Moreover

$$\|\mathcal{a}\|^* \leq \alpha < 1 .$$

The semigroup generated by the dual BBGKY hierarchy for a system of particles with the Hamiltonian (6.1) has the form (compare with the «symmetric» case, as defined in (4.1))

$$\mathcal{U}^*(t) = (I - \mathcal{a}^*)\, S(t)(I - \mathcal{a}^*)^{-1} \tag{6.3a}$$

or, alternatively

$$\mathcal{U}^*(t) = S(t) + \sum_{n=1}^{\infty} [S(t), \mathcal{a}^*](\mathcal{a}^*)^{n-1}, \tag{6.3b}$$

where the bracket [. , .] again denotes the commutator. The componentwise form of the mapping (6.3) is given by

$$(\mathcal{U}^*(t)\, g)_s(x_1, \ldots, x_s)$$

$$= \sum_{n=0}^{s} \sum_{k=0}^{\min(1,\, n)} (-1)^k S^{s-k}(t, x_1, \ldots, x_{s-k})\, g_{s-n}(x_1, \ldots, x_{s-n}).$$

The one-parameter family of operators $\mathcal{U}^*(t)$ (6.3) is defined and bounded in the space $\mathcal{C}_2^\alpha$, for arbitrary $t \in \mathbb{R}^1$. In fact, since $\|\mathcal{A}\|^* \leq \alpha$ and, consequently

$$\|I - \mathcal{A}\|^* \leq 1 + \alpha\,,$$

$$\|(I - \mathcal{A}^*)^{-1}\| \leq \frac{1}{1-\alpha}\,,$$

the following estimate is obtained:

$$\|\mathcal{U}^*(t)\| \leq \frac{1+\alpha}{1-\alpha} e^{(c+1)t}.$$

It is clear that the family of operators (6.3) owns the group property and that such a group is strongly continuous in $\mathcal{C}_2^\alpha$. Thus, the mapping $\mathcal{U}^*(t)$ (6.3) forms in the space $\mathcal{C}_2^\alpha$ a quasi-bounded C_0-group.

From general properties of quasi-bounded groups of operators, [3], [4], it follows the existence of the infinitesimal generator $\mathcal{G}^*$ of the group $\mathcal{U}^*(t)$ defined by (6.3). As above, the operator $\mathcal{G}^*$ is closed on the subspace $\mathcal{C}_{2,0}^\alpha \subset \mathcal{C}_2^\alpha$ of finite sequences of continuously differentiable functions with compact support and, moreover, $\mathcal{G}^* \mathcal{U}^*(t) = \mathcal{U}^*(t)\, \mathcal{G}^*$. On the subspace $\mathcal{C}_{2,0}^\alpha$ the generator $\mathcal{G}^*$ coincides with the operator $\mathcal{L} + [\mathcal{L}, \mathcal{A}^*]$, i.e.

$$\mathcal{G}^* = \mathcal{L} + [\mathcal{L}, \mathcal{A}^*].$$

The proof of the last propositon is based on the fact that in $\mathcal{C}_{2,0}^\alpha \subset \mathcal{C}_2^\alpha$ the following identity is satisfied:

$$[\mathcal{L}, \mathcal{A}^*](\mathcal{A}^*)^n = 0\,, \qquad \text{for any } n \geq 1\,.$$

Remark. The procedure for calculating the averages of observables G, in the case of the above considered systems with non-symmetric Hamiltonian, is different from the one introduced by (2.8) and it is given by:

$$\langle G\rangle(t) = ((I - \mathcal{A})^{-1} GF(t))_0,$$

where I is the identity operator and the operator $\mathcal{A}$ is defined by (2.2). Hence, the explicit expression of the observables G is

$$G = (I - \mathcal{A}^*)\, a\,,$$

where a is a sequence of functions defined in the usual way on the phase space.

Moreover we note that the group conjugated to (6.3) is

$$\mathcal{U}(t) = (I - \mathcal{A})^{-1} S(-t)(I - \mathcal{A})\,,$$

which is generated by the BBGKY hierarchy of many particles systems with Hamiltonian (6.1).

Final Remark. As we have emphasised at the end of the Introduction, we believe that our approach in terms of evolution of observables can handle dynamics of infinitely many-particle systems. In order to properly describe the evolution of infinite particle systems it is necessary, along with the solution of Cauchy problem (5.1), to solve the additional problem of giving sense to the expression for the average $(e^{\mathcal{A}} G(t)\, F(0))_0$ (see (2.9)), if $F(0)$ is a sequence of bounded functions and $G(t) \in \mathcal{C}_2$. The componentwise formulations (4.3) of $\mathcal{U}^*(t)$ allows to establish such a fact at least for the case of short range potentials.

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Abstract

In the present paper the hierarchy of equations formally conjugated to the BBGKY hierarchy for the evolution of classical many-particle system is investigated. From such a «dual» hierarchy, characterizing the evolution of observables, it is explicitly defined a quasi-bounded C_0-semigroup (group), in both cases of symmetric and non-symmetric Hamiltonians.

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