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Generalized Brownian motion and anomalous diffusion (**)

1 - Introduction

There are many reasons to look for an equilibrium distribution different from Maxwellian distribution. Derivations of the Maxwell-Boltzmann distribution are based on several assumptions. In a kinetical approach, one assumes that: 1) the collision time be much smaller than the mean time between collisions, 2) the interaction be sufficiently local, 3) the velocities of two particles at the same point are not correlated (Boltzmann's Stosszahl-ansatz), and 4) energy is locally conserved when using only the degrees of freedom of the colliding particles. In the equilibrium statistical mechanics approach, one uses the assumption that the velocity probabilities of different particles are independent , corresponding to 3), and that the total energy of the system could be expressed as the sum of a term quadratic in the momentum of the particle and independent of the other variables, and a term independent of momentum, but if 1) and 2) are not valid the resulting effective two-body interaction is not local and depends on the momentum and energy of the particles. Finally, even when the one-particle distributions is Maxwellian, additional assumptions about correlations between particles are necessary to deduce that the relative-velocity distribution, is also Maxwellian. At last in one limit the MB distribution can be rigorously derived: systems that are dilute in the appropriate variables, whose residual interaction is small compared to the one-body energies.

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Examples among others of important physical problems where the experimental results seem to deviate from the Maxwellian behavior are: *a*) Solar core plasma and solar neutrino fluxes. *b*) Galaxy distributions and peculiar velocity function of galaxy clusters. *c*) Velocity distributions and annual-modulation signatures of weakly-interacting massive particles and existence of particle dark matter in galactic halo. d) Turbulence and non-neutral electronic plasma experiments. *e*) Quark-Gluon plasma and high energy e^+-e^- experiments. *f*) Condensed matter (charged particles in electric and magnetic fields).

Let us illustrate few of them. The solar core is a weakly-nonideal plasma where: 1) the mean Coulomb energy potential is of the order of the thermal kinetic energy; 2) the Debye screening length is $R_D \approx a$ (interparticle distance) and the Debye-Hückel conditions are only approximately verified; 3) it is not possible to separate individual and collective degrees of freedom; 4) the inverse solar plasma frequency ($t_{\rm pl} = \omega_{\rm pl}^{-1} = \sqrt{m/4\pi n e^2} \approx 10^{-17}$) is of the same order of magnitude of the collision time $t_{\rm coll} = f^{-1} = \langle n\sigma v\rangle$; 5) particles loose memory of the initial state only after many collisions: the scattering process cannot be considered Markovian; 6) the time needed to build up again the screening, after hard collisions, is not negligible.

At the thermal equilibrium reacting ions are usually described as quasi free particles with Maxwell-Boltzmann (MB) velocity distribution. But many-body effects inside the plasma could cause deviations from a pure Maxwell-Boltzmann statistics for the effective degrees of freedom. Because reacting ions belong to the high momentum tail of the distribution, at least for fusion reactions between charged ions, even tiny deviations from the MB tail can cause large modifications (enhancement or depletion) of the rates.

The value of the collision frequency f determines the possibility of two different effects that produce important deviations from the Maxwellian distribution $F_M(p)$ at high momenta:

A) Quantum uncertainty effect: When the Coulomb collisional frequency is large ($hf > kT$) the ions cannot be considered as quasi free particles: the energy and momentum distributions are different and one must decide which one is relevant for the reaction rates. The fact that the two distributions are not equivalent is related to the finite life-time of the quasi-particles and to the quantum uncertainty. Since nuclear rates should be evaluated averaging the quasi-classical cross section $\sigma(p)$ over the momentum distribution, rather than the energy distribution, even if the energy distribution is Maxwellian, the effective distribution can acquire a non-Maxwellian tail.

B) Weak nonextensivity effect: Tsallis statistics with entropic parameter q can describe systems that are not extensive due to long-range interactions or non-

Markovian memory effects; the energy distribution itself deviates from the standard free-particle statistics. When deviations are small ($q \approx 1$) the correction (enhanced or depleted tail) can be described by the factor $\exp\left[-\dfrac{1-q}{2}\left(\dfrac{\varepsilon_p}{kT}\right)^2\right]$. Deviations from the Maxwellian tail due to either quantum uncertainty effect or non extensive effect (or both) may lead to strong increase of the rates in the solar core.

From the data obtained by the Cosmic Background Explorer (COBE), it has been possible to infer the distribution of peculiar velocities of certain groups of spiral (Sc) galaxies (we recall that by *peculiar* velocity we mean the residual velocity after the global universe expansion velocity has been subtracted). Four theoretical attempts (namely Cold Dark Matter with $\Omega = 0.3$ and with $\Omega = 1.0$, Hot Dark Matter with $\Omega = 1.0$ and Primeval Barionic Isotropic with $\Omega = 0.3$) have been developed. All the attempts were done within BG statistics. The less unsatisfactory fitting was obtained for CDM model with $\Omega = 0.3$. In fact, all the attempts exhibit a long tail towards high velocities, whereas the experimental data show a pronounced cut-off at about 500 $\mathrm{Km\,s^{-1}}$. It is relevant to mention that all the models that were used had several fitting parameters, and nevertheless can not get rid of the tail. A fitting was then advanced using the Tsallis non extensive formalism (which will be outlined later), with only two free parameters, one of them being q and the other one a characteristic velocity. The q-generalized Maxwell distribution was used essentially corresponding to an ideal classical gas. The quality of the fitting is quite remarkable, far better than those corresponding to the mentioned attempts. Once again, one sees that modifications of the statistics can be sensibly more efficient than modifications of the model. A famous example along this line is provided by the completely different physics associated with a gas of free fermions or of free bosons, i.e., a Fermi-Dirac ideal gas or a Bose-Einstein gas (same model but different statistics).

The presence of memory effects and color long-range forces among the many-parton system in the early stage of heavy-ion collisions can affect the particle statistical behavior at the freeze-out temperature. In this context, in the framework of the equilibrium generalized non-extensive thermostatistics, the shape of pion transverse mass spectrum and the value of the transverse momentum correlation function of the pions emitted during the central Pb+Pb collisions have been calculated and it has been shown that the experimental results are well reproduced assuming very small deviations from the standard statistics. We send the reader to our papers quoted in ref. [**1**] for details on the topics discussed above.

Tsallis distribution functions are shown to arise in the case of charged particles in electric and magnetic fields, inelastically interacting with a medium. Explicit results for Maxwell and hard sphere interactions can be provided. The mean energy and the components of the current density are functions of both the electric and the magnetic fields.

Recently, the linear Boltzmann equation for inelastic scattering has been object of interest by researchers in the field of transport theory. This equation [**2**] is derived starting from the nonlinear system of kinetic equations for a mixture of particles A (mass m) and B (mass M), where B is endowed with two internal energy levels.

A particularly simple assumption, but still preserving physical interest, consists in considering $M \gg m$ so that we can let $M \rightarrow \infty$. The resulting model is such that test particles A can gain or lose a fixed amount of kinetic energy by interacting with field particles B.

The equation for such a model raises a number of interesting problems. First of all the study of possible equilibria is not trivial at all [**3**]. The derivation of macroscopic equations from the kinetic one also shows unespected difficulties [**4**]. Moreover, it has been shown that the present problem is mathematically equivalent to the transport of electrons interacting with phonons of a crystal lattice [**5**].

In ref. [**5**] Rossani has constructed, under suitable assumptions, a Fokker-Planck approximation of this equation. Such an approximation can be applied to the case of charged test particles subjected to both an electric and a magnetic field. The equations of the model turn out to be solvable and explicit solutions can be shown for both Maxwell and hard sphere interactions. In the case of Maxwell interactions, Tsallis distribution functions (TDF) [**6**] are found. For hard sphere interactions, a Maxwellian function times a TDF is found. Tsallis distribution function *in the velocity space*, where q now is allowed to assume *many different values*, is a function of both the electric and the magnetic field.

2 - Tsallis statistics

Tsallis's thermostatistics is, actually, known to introduce a non extensive generalization of the Boltzmann-Gibbs statistics.

One of the main points in this formalism is the peculiar definition of the average value, known as normalized q-expectation value. We must mention that during the last decade different proposals have been advanced, concerning the definition of the average value. In any case, standard Boltzmann-Gibbs statistics results are always recovered in the limit $q \rightarrow 1$ (extensive limit).

We report the main relations of the Tsallis statistics without comments on technical or basic refinements that have been recently illustrated for instance in refs. [**7**], [**8**], [**9**], [**10**].

The entropy, as a function of the entropy density is defined by:

$$S_q = \int \mathcal{S}_q(p)\, dv\,, \tag{2.1}$$

being $p = p(v)$ the distribution function and the entropy density is given by:

$$\mathcal{S}_q(p) = -\,\frac{p - p^q}{1-q}\,. \tag{2.2}$$

The concavity of $\mathcal{S}_q(p)$ is:

$$\frac{\partial^2 \mathcal{S}_q(p)}{\partial p^2} < 0 \quad \text{if} \quad q \in \boldsymbol{R}^+\,. \tag{2.3}$$

When the non extensive parameter $q \to 1$ we obtain the density:

$$\mathcal{S}_1(p) = -p \log p\,, \tag{2.4}$$

and the standard Shannon entropy:

$$S_1 = -\int p \log p \, dv\,. \tag{2.5}$$

We indicate with p_A and $S_q(A)$ the distribution function and the entropy of the system A. It is clear that for statistically independent systems A and B we have $p_{A+B} = p_A p_B$, being p_{A+B} the probability associated with the system composed by the two subsystems A and B. In refs. [**11**], [**12**] is shown that the Tsallis entropy obeys to the following pseudo-additivity rule:

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)\, S_q(A)\, S_q(B)\,. \tag{2.6}$$

From Eq. (2.6) we have

$$q - 1 = \frac{S_q(A+B) - S_q(A) - S_q(B)}{S_q(A)\, S_q(B)}\,, \tag{2.7}$$

and then $q-1$ can be seen as the degree of non extensivity for the system. The q-expectation value of a physical operator is:

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}(v)\, p(v)^q\, dv}{\int p(v)^q\, dv}\,, \tag{2.8}$$

where the distribution function $p(v)$ is normalized as:

$$N = \int p(v)\, dv\,. \tag{2.9}$$

The energy constraint is postulated to be:

$$E = \int \varepsilon(v)\, p(v)^q\, dv\,. \tag{2.10}$$

By maximizing Tsallis generalized entropy:

$$\frac{\delta}{\delta p}(S_q - \beta E - \alpha N) = 0\,, \tag{2.11}$$

the following normalized distribution function is obtained:

$$p_q = \frac{1}{Z_q}[1-(1-q)\,\beta\varepsilon]^{1/1-q}, \tag{2.12}$$

where the partition function Z_q is:

$$Z_q = \int [1-(1-q)\,\beta\varepsilon]^{1/1-q}\, dv\,. \tag{2.13}$$

We must limit ourselves to the range of variability $1 \leqslant q \leqslant 3$ because of the requirements, coming from the Fokker-Planck and Boltzmann equations, of the continuity of the distribution functions and of their derivatives. In the above q-range we have:

$$Z_q = \beta^{-1/2}\sqrt{\frac{q-1}{\pi}}\,\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{3-q}{2(q-1)}\right)}\,. \tag{2.14}$$

Let us report, as an example, the average value of the quantity x^2:

$$\langle x^2\rangle = \frac{1}{2\beta}\left[\sqrt{\frac{q-1}{2\pi}}\,\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{3-q}{2(q-1)}\right)}\right]^{\frac{2(q-1)}{3-q}}\,. \tag{2.15}$$

In figure 1 is reported the Tsallis distribution given by Eqs. (2.12), (2.14) for four different values of the parameter q. The curve corresponding to $q = 1$ is the Maxwell-Boltzmann distribution.

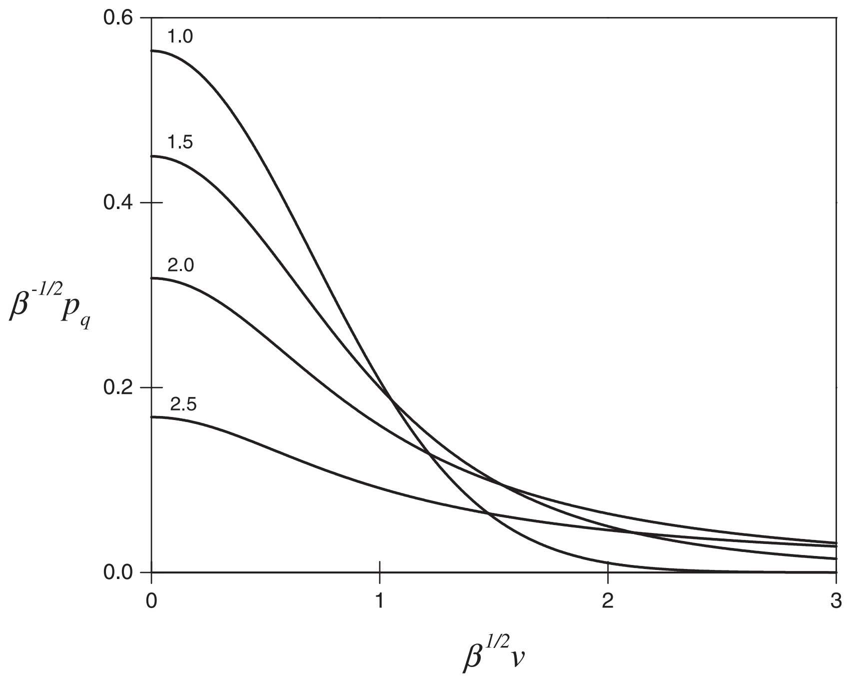

Figure 1 - Tsallis distribution given by Eqs. (2.12), (2.14) for four different values of the parameter q. The curve corresponding to $q = 1$ is the Maxwell-Boltzmann distribution.

3 - Brownian motion

Let us recall that a stochastic process can be described by the Langevin equation:

$$\frac{dv(t)}{dt} + \gamma(t)\, v = g(t)\, \Gamma(t)\,. \tag{3.1}$$

The viscous force is $-m\gamma(t)\, v$ and $mg(t)\, \Gamma(t)$ is the random force, where $\Gamma(t)$ is a Gaussian random variable

$$\langle \Gamma(t) \rangle = 0\,, \tag{3.2}$$

$$\langle \Gamma(t)\, \Gamma(t') \rangle = 2\delta(t - t')\,. \tag{3.3}$$

The macroscopic process associated to Eq. (3.1) is the Brownian motion described by the standard Fokker-Planck equation (FPE) [13], [14]:

$$\frac{\partial p(t,\, v)}{\partial t} = \frac{\partial}{\partial v}\left[\gamma(t)\, vp(t,\, v) + g(t)^2\, \frac{\partial p(t,\, v)}{\partial v}\right]. \tag{3.4}$$

After changing variable: $d\tau = \gamma(t)\, dt$ and posing $D(\tau) = g(t)^2/\gamma(t)$, Eq. (3.4) becomes

$$\frac{\partial p(\tau,\, v)}{\partial \tau} = \frac{\partial}{\partial v}\left[vp(\tau,\, v) + D(\tau)\, \frac{\partial p(\tau,\, v)}{\partial v}\right]. \tag{3.5}$$

We consider now particular solutions of Eq. (3.5) by using the following ansatz:

$$p(\tau,\, v) = \frac{1}{Z(\tau)} \exp[-\beta(\tau)\, v^2], \tag{3.6}$$

where the function $Z(\tau)$, given by

$$Z(\tau) = \int_{-\infty}^{+\infty} \exp[-\beta(\tau)\, v^2]\, dv\,, \tag{3.7}$$

is the partition function and $\beta(\tau)$ defines the temperature of the system

$$T(\tau) = \frac{m}{2k\beta(\tau)}\,. \tag{3.8}$$

If we substitute the solution (3.6) into Eq. (3.5), we obtain the evolution equations

$$\frac{Z(\tau)}{Z(0)} = \left[\frac{\beta(0)}{\beta(\tau)}\right]^{1/2}, \tag{3.9}$$

$$\frac{d\beta(\tau)}{d\tau} = 2\beta(\tau) - 4D(\tau)\, \beta(\tau)^2. \tag{3.10}$$

Let us remark that, being $\lim_{\tau\to\infty} d\beta/d\tau = 0$, from Eq. (3.10) we can deduce immediately the well known Einstein relation

$$2\beta(\infty)\, D(\infty) = 1\,. \tag{3.11}$$

We note that the transformation $y = \beta^{-1}$ linearizes Eq. (3.10). After integration we obtain:

$$\beta(\tau) = \beta(\infty)\left\{1 + \left[\frac{\beta(\infty)}{\beta(0)} - 1 + b(\tau)\right]\exp(-2\tau)\right\}^{-1}, \tag{3.12}$$

$$b(\tau) = 2\int_0^\tau \left[\frac{D(\tau)}{D(\infty)} - 1\right] \exp(-2\tau)\, d\tau\,. \tag{3.13}$$

Once $\beta(\tau)$ is known, we may deduce $Z(\tau)$ from Eq. (3.9):

$$Z(\tau)\,\beta(\tau)^{1/2} = Z(0)\,\beta(0)^{1/2}. \tag{3.14}$$

Finally, we write the solution of Eq. (3.5) in the form

$$p(\tau, v) = \pi^{-1/2}\beta(\tau)^{1/2}\exp[-\beta(\tau)\, v^2]\,, \tag{3.15}$$

that represents, at any instant, a Maxwellian distribution. The temperature of the system is time dependent and the variation law of $T(\tau)$ depends on the function $D(\tau)$, as we can see from Eq. (3.12) and Eq. (3.13).

4 - General motion

We consider now a generic stochastic motion, described by the Langevin equation:

$$\frac{dv(t)}{dt} + h(t, v) = g(t, v)\,\Gamma(t)\,, \tag{4.1}$$

where $\Gamma(t)$ is a Gaussian random force

$$\langle \Gamma(t) \rangle = 0\,, \tag{4.2}$$

$$\langle \Gamma(t)\,\Gamma(t') \rangle = 2\delta(t - t')\,. \tag{4.3}$$

The viscous force has the expression

$$f_{\text{viscous}}(t, v) = -\,mh(t, v)\,, \tag{4.4}$$

and the functions $h(t, v)$ and $g(t, v)$ are, at the moment, arbitrary.

The macroscopic motion associated to Eq. (4.1) is described by the linear FPE [**15**]:

$$\frac{\partial p(t, v)}{\partial t} = \frac{\partial}{\partial v}\left\{\left[J(t, v) + \frac{\partial D(t, v)}{\partial v}\right] p(t, v) + D(t, v)\,\frac{\partial p(t, v)}{\partial v}\right\}. \tag{4.5}$$

The drift coefficient $J(t, v)$ and the diffusion coefficient $D(t, v)$ depend on the velocity v and can be expressed in terms of the functions $h(t, v)$ and $g(t, v)$:

$$J(t, v) = h(t, v) - \sigma g(t, v) \frac{\partial g(t, v)}{\partial v} , \tag{4.6}$$

$$D(t, v) = g(t, v)^2 , \tag{4.7}$$

where the parameter $\sigma = 0$ means that the Ito definition of the stochastic integral is adopted, while $\sigma = 1$ means that the Stratonovich definition is adopted (see appendix A).

Let us note that the particle current in Eq. (4.5) is the sum of two terms

$$j(t, v) = j_{\text{drift}}(t, v) + j_{\text{diffusive}}(t, v) . \tag{4.8}$$

The first is the drift current and is the sum of two contributions: one due to the coefficient $J(t, v)$, the second to $D(t, v)$

$$j_{\text{drift}}(t, v) = j_{\text{drift, J}}(t, v) + j_{\text{drift, D}}(t, v) , \tag{4.9}$$

$$j_{\text{drift, J}}(t, v) = -J(t, v)\, p(t, v) , \tag{4.10}$$

$$j_{\text{drift, D}}(t, v) = -\frac{\partial D(t, v)}{\partial v}\, p(t, v) . \tag{4.11}$$

The other contribution of the current in Eq. (4.8) is the diffusive current and is given by the Fick's law:

$$j_{\text{diffusive}}(t, v) = -D(t, v) \frac{\partial p(t, v)}{\partial v} . \tag{4.12}$$

Let us remark that in the case of Brownian motion, where $J(t, v) = \gamma(t)\, v$ and $D(t, v) = g(t)^2$, the function $h(t, v)$ can be obtained by Eq. (4.6) and assumes the value $h(t, v) = \gamma(t)\, v$ and does not depend on the definition adopted for the stochastic integral (Ito or Stratonovich).

5 - Microscopic description of generalized Brownian motion

From Eqs. (4.6) and (4.7) it is easy to see that the Ito ($\sigma = 0$) and the Stratonovich ($\sigma = 1$) definitions of $h(t, v)$ are given by:

$$h(t, v) = J(t, v) + \frac{\sigma}{2} \frac{\partial D(t, v)}{\partial v} , \tag{5.1}$$

and coincide only if $g(t, v)$ does not depend on v.

We wish to study the motions featured by the following relation of proportionality between the Ito and the Stratonovich definition of $h(t, v)$:

$$h_{\mathrm{Strat}}(t, v) = C(t)\, h_{\mathrm{Ito}}(t, v)\,. \tag{5.2}$$

Of course, when $C(t) = 1$ we obtain the motion associated with a diffusion coefficient which does not depend on v like the Brownian motion just discussed in the previous section.

In the case $C(t) \neq 1$, Eq. (5.2) is satisfied only if the condition

$$\frac{\partial D(t, v)}{\partial v} = \frac{\xi(t)}{1 - \xi(t)} J(t, v)\,, \tag{5.3}$$

is verified, being $\xi(t)$ an arbitrary function. We explain now the physical meaning of the function $\xi(t)$. From Eqs. (4.9)-(4.11) and (5.3) we have:

$$\xi(t) = \frac{j_{\mathrm{drift,\,D}}(t, v)}{j_{\mathrm{drift}}(t, v)}\,, \tag{5.4}$$

and then we can conclude that $\xi(t)$ represents the fraction of the drift current due to the coefficient $D(t, v)$ respect to the total drift current. After integration of Eq. (5.3) we can express $D(t, v)$ in terms of $J(t, v)$:

$$D(t, v) = \frac{g(t)^2}{1 - \xi(t)} + \frac{\xi(t)}{1 - \xi(t)} \int J(t, v)\, dv\,, \tag{5.5}$$

where the integration constant in Eq. (5.5) is chosen in such a way to reobtain the Brownian diffusion coefficient $D(t, v) = g(t)^2$ in the limit $\xi(t) \to 0$. By combining Eqs. (4.7) and (5.5) we have

$$g(t, v) = \left[\frac{g(t)^2}{1 - \xi(t)} + \frac{\xi(t)}{1 - \xi(t)} \int J(t, v)\, dv \right]^{1/2}, \tag{5.6}$$

while from Eqs. (4.6) and (5.1) we obtain:

$$h(t, v) = \left(1 + \frac{\sigma}{2} \frac{\xi(t)}{1 - \xi(t)} \right) J(t, v)\,. \tag{5.7}$$

In this work we make the choice that the drift coefficient $J(t, v)$ has the same expression of the Brownian motion:

$$J(t, v) = \gamma(t)\, v\,. \tag{5.8}$$

Thus, from Eq. (5.5) we have that the diffusion coefficient is given by:

$$D(t, v) = \frac{g(t)^2}{1 - \xi(t)} + \frac{1}{2} \frac{\xi(t)}{1 - \xi(t)} \gamma(t)\, v^2, \tag{5.9}$$

and consequently $g(t, v)$ and $h(t, v)$ become:

$$g(t, v) = \left[\frac{g(t)^2}{1-\xi(t)} + \frac{1}{2} \frac{\xi(t)}{1-\xi(t)} \gamma(t)\, v^2 \right]^{1/2}, \tag{5.10}$$

$$h(t, v) = \left(1 + \frac{\sigma}{2} \frac{\xi(t)}{1-\xi(t)} \right) \gamma(t)\, v\,. \tag{5.11}$$

The Langevin equation (4.1) finally assumes the form [**13**]:

$$\begin{gathered} \frac{d}{dt} v(t) + \left(1 + \frac{\sigma}{2} \frac{\xi(t)}{1-\xi(t)} \right) \gamma(t)\, v(t) \\ = \left[\frac{g(t)^2}{1-\xi(t)} + \frac{1}{2} \frac{\xi(t)}{1-\xi(t)} \gamma(t)\, v(t)^2 \right]^{1/2} \Gamma(t)\,, \end{gathered} \tag{5.12}$$

and describes a stochastic process in the presence of a multiplicative noise. We call this process generalized Brownian motion (GBM) and, in the limit $\xi(t) \to 0$, reduces to the standard Brownian motion.

6 - Macroscopic description of generalized Brownian motion

The GBM is described by a FPE which derives from Eq. (4.5) and using Eqs. (5.8) and (5.9). After introducing the variables:

$$d\tau = \frac{\gamma(t)}{1-\xi(t)}\, dt\,, \tag{6.1}$$

$$D(\tau) = \frac{g(t)^2}{\gamma(t)}\,, \tag{6.2}$$

$$2D(\tau)\, \beta(\tau)(q-1) = \xi(t)\,, \tag{6.3}$$

$$\beta(\tau) = \frac{1}{k_B\, T(\tau)}\,, \tag{6.4}$$

the drift coefficient becomes $J(v) = v$ and the diffusion coefficient is

$$D(\tau, v) = D(\tau)[1 - (1-q)\, \beta(\tau)\, v^2]\,. \tag{6.5}$$

Finally, the FPE for GBM becomes [**16**]:

$$\frac{\partial p(\tau, v)}{\partial \tau} = \frac{\partial}{\partial v}\left\{ v p(\tau, v) + D(\tau)[1-(1-q)\,\beta(\tau)\,v^2]\,\frac{\partial p(\tau, v)}{\partial v}\right\}, \tag{6.6}$$

which can be solved with the ansatz

$$p(\tau, v) = \frac{1}{Z_q(\tau)}[1-(1-q)\,\beta(\tau)\,v^2]^{1/(1-q)}, \tag{6.7}$$

where $Z_q(\tau)$ is the position function given by

$$Z_q(\tau) = \int_{-\infty}^{+\infty} [1-(1-q)\,\beta(\tau)\,v^2]^{1/(1-q)}\,dv\,. \tag{6.8}$$

The ansatz (6.7) is a generalizzation of the one given by Eq. (3.6) when $q \neq 1$ and has been used previously in refs. [**16**], [**18**] for solving the anomalous diffusion equation. Let us note that, when $q \to 1$, Eq. (6.6) reduces to Eq. (3.5) and the ansatz (6.7) to Eq. (3.6) and we recover the standard Brownian motion.

After substitution of Eq. (6.7) into Eq. (6.6) we obtain the evolution laws

$$\frac{Z_q(\tau)}{Z_q(0)} = \left[\frac{\beta(0)}{\beta(\tau)}\right]^{1/2}. \tag{6.9}$$

$$\frac{d\beta(\tau)}{d\tau} = 2\beta(\tau) - 4D(\tau)\,\beta(\tau)^2. \tag{6.10}$$

We note that Eqs. (6.9) and (6.10) are identical to the corresponding relations of the standard Brownian motion.

This justifies the name GBM to the motion described by Eq. (6.6).

The evolution equation (6.10) of the function $\beta(\tau)$ or of the temperature $T(\tau)$ does not depend on the parameter q. From Eq. (6.9) we can see that the dependence of $Z_q(\tau)$ on the parameter q is limited to the initial condition $Z_q(0)$. As for the standard Brownian motion, the Einstein relation holds:

$$2\beta(\infty)\,D(\infty) = 1\,, \tag{6.11}$$

and does not depend on the parameter q which is related to the stationary value of $\xi(\tau)$ by:

$$q - 1 = \xi(\infty)\,. \tag{6.12}$$

The time evolution of $\beta(\tau)$ is:

$$\beta(\tau) = \beta(\infty)\left\{1 + \left[\frac{\beta(\infty)}{\beta(0)} - 1 + b(\tau)\right]\exp(-2\tau)\right\}^{-1}, \tag{6.13}$$

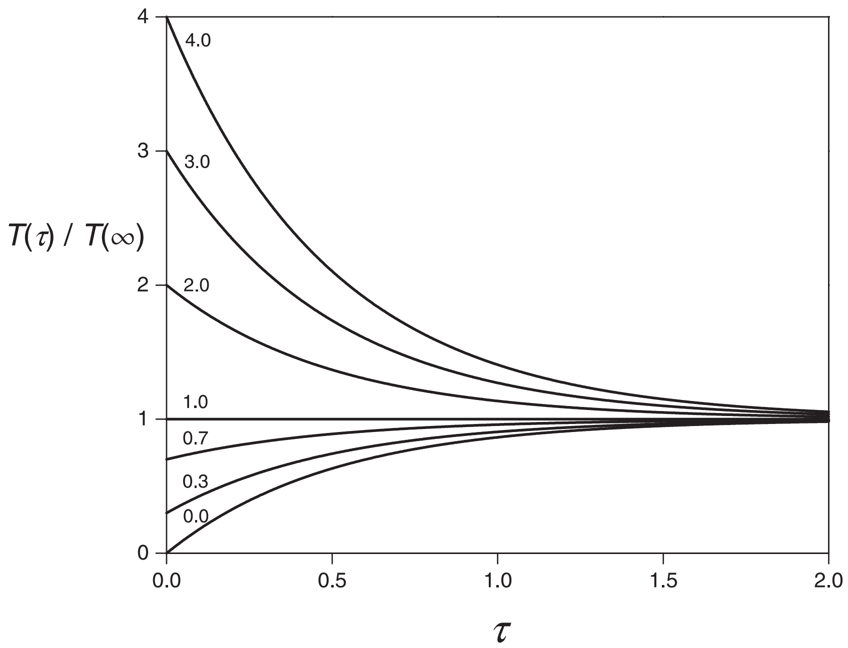

Figure 2 - Behaviour of $T(\tau)/T(\infty)$ given by Eqs. (6.4), (6.13) and (6.14) for a system where $D(\tau) = D(\infty)$. The different curves correspond to different values of $T(0)/T(\infty)$.

with

$$b(\tau) = 2 \int_0^\tau \left[\frac{D(\tau)}{D(\infty)} - 1 \right] \exp(-2\tau)\, d\tau\,, \tag{6.14}$$

and the one of $Z_q(\tau)$ is:

$$Z_q(\tau)\, \beta(\tau)^{1/2} = Z_q(0)\, \beta(0)^{1/2} = N_q^{-1}. \tag{6.15}$$

The constant N_q (when $1 \geqslant q \geqslant 3$) is given by [**17**]:

$$N_q = \sqrt{\frac{q-1}{\pi}}\; \frac{\Gamma[1/(q-1)]}{\Gamma[(3-q)/2(q-1)]}\,. \tag{6.16}$$

Now this last expression behaves as $x^{-2/(q-1)}$ as $|x| \to \infty$. In figure 2 is reported the behaviour of $T(\tau)/T(\infty)$ given by Eqs. (6.4), (6.13) and (6.14) for a system where $D(\tau) = D(\infty)$. The different curves correspond to different values of $T(0)/T(\infty)$.

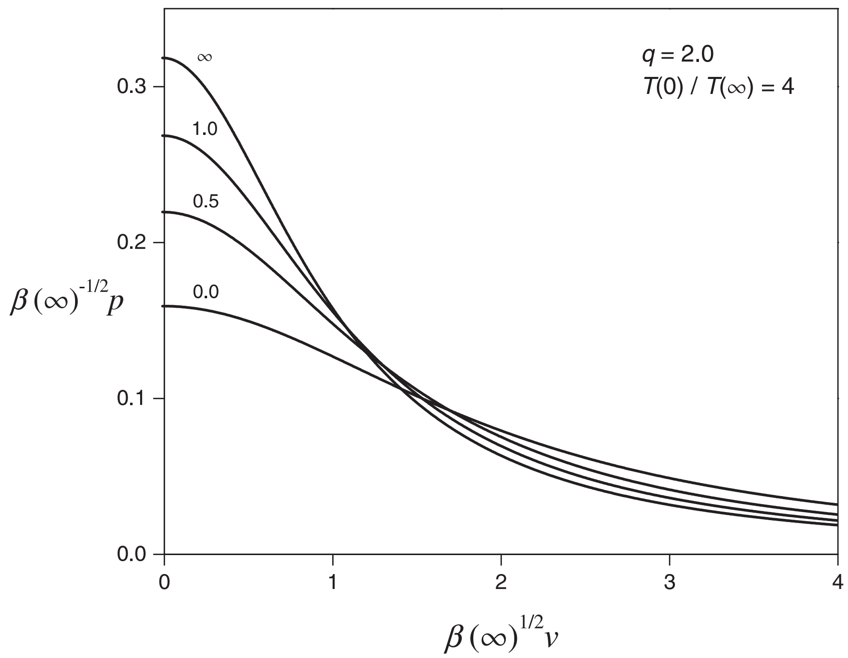

Figure 3 - Time dependent Tsallis distribution given by Eqs. (6.16), (6.17) while evolves towards equilibrium. The curves correspond to four different times ($\tau = 0$, $\tau = 0.5$, $\tau = 1$, $\tau = \infty$).

Finally the distribution function is:

$$p(\tau, v) = N_q \beta(\tau)^{1/2}[1 - (1-q)\,\beta(\tau)\,v^2]^{1/(1-q)}. \tag{6.17}$$

In figure 3 is reported the time dependent Tsallis distribution given by Eqs. (6.16), (6.17) while evolves towards equilibrium. The curves correspond to four different times ($\tau = 0$, $\tau = 0.5$, $\tau = 1$, $\tau = \infty$).

7 - Non-linear diffusion equation

Eq. (6.7) describes a particular family of solutions of Eq. (6.6) for the different values of the parameter q. The function $p(\tau, v)$ varies because $\beta(\tau)$ varies, as it happens with the standard Brownian motion. In ref. [**16**] it is shown that the solution (6.17) of Eq. (6.6) is also solution of the nonlinear Fokker-Planck equations considered in refs. [**18**], [**19**], [**20**], [**21**], [**22**], [**23**], [**24**] and also of the anomalous diffusion equation [**16**]. In fact from Eq. (6.6) we can write:

$$1-(1-q)\,\beta(\tau)\,v^2 = N_q^{q-1}\beta(\tau)^{(q-1)/2}p(\tau,\,v)^{1-q}, \tag{7.1}$$

$$vp(\tau,\,v) = \frac{N_q^{q-1}}{2(q-2)}\beta(\tau)^{(q-3)/2}\,\frac{\partial}{\partial v}\,[p(\tau,\,v)]^{2-q}. \tag{7.2}$$

After substitution of these two expressions into Eq. (6.6) we have:

$$\frac{\partial p(\tau,\,v)}{\partial\tau} = D_{\rm NL}(\tau)\,\frac{\partial^2}{\partial v^2}\,[p(\tau,\,v)]^{2-q}, \tag{7.3}$$

where the new function $D_{\rm NL}(\tau)$ is defined by means of:

$$D_{\rm NL}(\tau) = \frac{N_q^{q-1}}{q-2}\beta(\tau)^{(q-3)/2}\left[\frac{1}{2}-\beta(\tau)\,D(\tau)\right]. \tag{7.4}$$

The particle current associated to Eq. (7.3) is given by:

$$j(\tau,\,v) = -D_{\rm NL}(\tau)\,\frac{\partial}{\partial v}\,[p(\tau,\,v)]^{2-q}, \tag{7.5}$$

and represents a generalized Fick current.

The evolution law of $\beta(\tau)$ in terms of $D_{\rm NL}(\tau)$ is described by the following differential equation

$$\frac{d\beta(\tau)}{d\tau} = 4(q-2)\,N_q^{1-q}D_{\rm NL}(\tau)\,\beta(\tau)^{(5-q)/2}, \tag{7.6}$$

which, after integration, gives:

$$\beta(\tau) = \beta(0)\left[1+2(q-2)(q-3)\,N_q^{1-q}\beta(0)^{(3-q)/2}\int_0^\tau D_{\rm NL}(\tau)\,d\tau\right]^{2/(q-3)}. \tag{7.7}$$

We note that using Einstein relation (6.11), from Eq. (7.4) we have $D_{\rm NL}(\infty)=0$. This implies that $j(\infty,\,v)=0$. The particle current goes to zero as $\tau\to\infty$ and therefore the stationary state $p(\infty,\,v)\neq 0$ takes place. In figure 4 is reported the time evolution of the function

$$\overline{D}_{\rm NL}(\tau) = D_{\rm NL}(\tau)\,\beta(\infty)^{(3-q)/2}. \tag{7.8}$$

The curves correspond to different values of the parameter $T(0)/T(\infty)$.

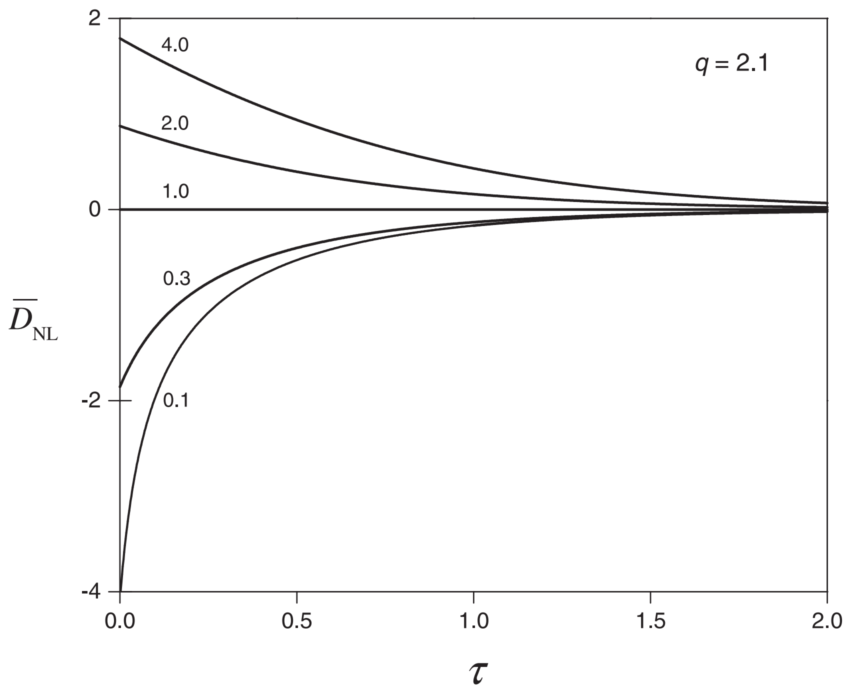

Figure 4 - Time evolution of the coefficient $\overline{D}_{\mathrm{NL}}(\tau)$ given by Eq. (7.8). The curves correspond to different values of the parameter $T(0)/T(\infty)$.

Appendix A

Let us assume that $w(\tau)$ is a Gaussian random variable. If we assume also that $\tau \geqslant 0$; $\tau_i \geqslant 0$; $\tau_j \geqslant 0$; $w(0) = 0$, it results:

(A.1) $$\langle w(\tau) \rangle = 0 \,,$$

and

(A.2) $$\langle w(\tau_i)\, w(\tau_j) \rangle = \begin{cases} 2\tau_i & \text{for } \tau_j \geqslant \tau_i, \\ 2\tau_j & \text{for } \tau_j \leqslant \tau_i. \end{cases}$$

If we introduce the quantity Δ through:

(A.3) $$\Delta = \max\{\tau_{i+1} - \tau_i;\ 0 = \tau_0 < \tau_1 < \ldots < \tau_N = \tau;\ i = 0, \ldots, N-1\} \,,$$

the Ito and the Stratonovich definitions of the stochastic integral are the following:

$$\text{(A.4)} \qquad \int_0^\tau \varphi(w(\tau'), \tau')\, dw(\tau') \overset{I}{=} \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \varphi(w(\tau_i), \tau_i)[w(\tau_{i+1}) - w(\tau_i)],$$

$$\text{(A.5)} \qquad \begin{aligned} &\int_0^\tau \varphi(w(\tau'), \tau')\, dw(\tau') \\ &\overset{S}{=} \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \varphi\left(\frac{w(\tau_i) + w(\tau_{i+1})}{2}, \frac{\tau_i + \tau_{i+1}}{2}\right)[w(\tau_{i+1}) - w(\tau_i)]. \end{aligned}$$

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Abstract

In the present contribution we consider a microscopic process described by a Langevin equation with multiplicative noise which defines a generalized Brownian motion. The associated macroscopic process is described by a linear Fokker-Planck equation with non constant coefficients. For this equation we obtain a class of solutions describing time-dependent Tsallis statistical distributions. We demonstrate that these time dependent distributions are also solutions of the standard nonlinear porous media diffusion equation.

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