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A new simple proof of the W. May's claim: FG determines G/G_0 (**)

The goal of the present paper is to give a smooth confirmation only in group terms of an old-standing statement of May, argued via another method in [1], which asserts the following (the notions and notations are the same as in [1]):

Theorem (W. May, 1969). *Suppose G is an abelian group with torsion part G_0, and suppose F is a field of arbitrary characteristic. Then $FG \cong FH$ as F-algebras for any group H implies $G/G_0 \cong H/H_0$.*

Proof. It is no harm in presuming that $FG = FH$. Thus, if $V(FG)$ and $V(FH)$ denote the normalized groups of units in FG and FH respectively, we extract $V(FG) = V(FH)$. On the other hand the canonical map

$$G \rightarrow G/G_0$$

induces a group homomorphism $V(FG) \rightarrow V(F(G/G_0)) = G/G_0$ with the whole kernel $[1 + I(FG;\, G_0)] \cap V(FG)$, where $I(FG;\, G_0)$ is the relative augmentation ideal of FG with respect to G_0. That is why

$$V(FG) = G([1 + I(FG;\, G_0)] \cap V(FG)).$$

By the same token $V(FH) = H([1 + I(FH;\, H_0)] \cap V(FH))$.

Thus $G([1 + I(FG;\, G_0)] \cap V(FG)) = H([1 + I(FH;\, H_0)] \cap V(FH))$. Moreover, $[1 + I(FG;\, G_0)] \cap V(FG) = [1 + I(FH;\, H_0)] \cap V(FH)$. In fact, foremost let the field F possess positive characteristic, for instance, $\mathrm{char}(F) = p \neq 0$. Then it is a

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routine matter to see that

$$[V(FG)]_p = 1 + I(FG;\, G_p) = 1 + I(FH;\, H_p) = [V(FH)]_p,$$

i.e.

$$I(FG;\, G_p) = I(FH;\, H_p),$$

hence

$$F(G/G_p) \cong FG/I(FG;\, G_p) = FH/I(FH;\, H_p) \cong F(H/H_p).$$

That is why, without loss of generality, we can assume in this case that

$$G_p = H_p = 1 \quad \text{since} \quad G/G_p/(G/G_p)_0 \cong G/G_0;$$

similaily for H.

Thus FG and FH are both semisimple, including and the situation when $\mathrm{char}\,(F) \neq 0$. Certainly, $G_0 \subseteq [V(FH)]_0$. But when FH is semisimple, i.e. H_0 has no element of order $\mathrm{char}\,(F)$, it holds valid that $[V(FH)]_0 = V(FH_0)$. Really, because the support of each element from $V(FH)$ is finite, we may presume that H is finitely generated and thus that $H = H_0 \times K$, where H_0 is finite and K is torsion-free. For a field R, the symbol R^* will designate its multiplicative group. Consequently $V(FH_0) \times F^* = F_1^* \times \ldots \times F_m^*$ for fields $F_1, \ldots, F_m$ that lie in FH_0 and which are finite algebraic extensions of F, and so

$$V(FH) \times F^* = V(F_1 K) \times F_1^* \times \ldots \times V(F_m K) \times F_m^*.$$

Furthermore,

$$V(FH) \times F^* = V(F_1 K) \times \ldots \times V(F_m K) \times V(FH_0) \times F^*$$

$$= \underbrace{K \times \ldots \times K}_{m \text{ times}} \times V(FH_0) \times F^*,$$

that ensures

$$[V(FH)]_0 = V(FH_0)\,,$$

as claimed. Therefore, we obviously yield

$$[1 + I(FG;\, G_0)] \cap V(FG) \subseteq [1 + I(FH;\, H_0)] \cap V(FH)\,.$$

By a reason of symmetry and analogous arguments, the right relation $\supseteq$ is fulfilled as well, so we have extracted the desired equality.

Finally, we detect,

$$G/G_0 \cong V(FG)/([1 + I(FG;\, G_0)] \cap V(FG))$$

$$= V(FH)/([1 + I(FH;\, H_0)] \cap V(FH)) \cong H/H_0,$$

which completes the proof in general after all.

References

[1] W. May, *Commutative group algebras*, Trans. Amer. Math. Soc. **136** (1969), 139-149.

Abstract

An easy group approach is used in this brief article to confirm once again the classical result of W. L. May (Trans. Amer. Math. Soc., 1969) that the factor-group G/G_0 of the abelian group G modulo its torsion subgroup G_0 may be retrieved from the group algebra FG over an arbitrary field F.

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