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On two theorems of Bertini for infinite-dimensional projective spaces (**)

1 - Introduction

Let V be an infinite-dimensional complex Banach space and let $\boldsymbol{P}(V)$ denote the projective space of all one-dimensional linear subspaces of V. Hence $\boldsymbol{P}(V)$ is an infinite-dimensional complex manifold. For every integer d there is a holomorphic line bundle $\boldsymbol{O}_{\boldsymbol{P}(V)}(d)$ on $\boldsymbol{P}(V)$ such that the vector space $V(d)$ of all holomorphic sections of $\boldsymbol{O}_{\boldsymbol{P}(V)}(d)$ is the set of all degree d continuous homogeneous polynomials $f: V \to \boldsymbol{C}$. Hence $V(d) = \{0\}$ if $d < 0$, $V(0) \cong \boldsymbol{C}$ (the constant functions) and $V(1)$ is the dual of V. Every $f \in V(d) \backslash \{0\}$ induces a degree d hypersurface $\{f = 0\}$ of $\boldsymbol{P}(V)$. After [**L**] and [**Ko**] it is a natural problem the existence of smooth closed subvarieties X of $\boldsymbol{P}(V)$ which are complete intersections of finitely many hypersurfaces. By the vanishing theorems proved in [**Ko**] the case in which V is a separable Hilbert space seems to be important. In [**Ko**] the smoothness of the complete intersection was essential to use complex analytic techniques (the ∂-bar operator). The existence of smooth complete intersections is a subtle problem since by [**K**] or [**B1**] Sard's theorem fails when the domain is infinite-dimensional and

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the target finite-dimensional. Even worst: by [**B2**] on each l^p space, $1 \leq p < +\infty$, there are continuous real and complex polynomials whose set of critical values has a non-empty interior. There are very interesting results concerning differential Fredholm maps ([**S**], [**QS**], [***PR***]) but they cannot be applied in our set-up because we consider maps from an infinite-dimensional domain to a finite-dimensional target. In section two we prove the following result.

Theorem 1.1. *Fix positive integers* $s, d_1, \ldots d_s$ *and a separable Hilbert space* V. *Then there exists a smooth codimension* s *complete intersection* $X \subset \boldsymbol{P}(V)$ *of* s *hypersurfaces of degree* $d_1, \ldots, d_s$.

One could hope that adding an algebricity condition, still some weak form of Bertini theorem may hold. In general this is not true: there are projective spaces $\boldsymbol{P}(V)$ with V Fréchet nuclear space such that every homogeneous hypersurface of degree at least two of $\boldsymbol{P}(V)$ is singular (see Example 3.3). In section three we consider the case $V = \boldsymbol{C}^{(N)}$ and prove the following result.

Theorem 1.2. *Fix an integer* $d \geq 2$ *and a subset* S *of* $\boldsymbol{CP}^1$ *with at most countable elements. Then there exist linearly independent homogeneous degree* d *polynomials* F *and* G *on* $\boldsymbol{C}^{(N)}$ *such that a hypersurface* $\{\lambda F + \mu G = 0\}$ *of* $\boldsymbol{P} = \boldsymbol{C}^{(N)}$ *with* $(\lambda; \mu) \in \boldsymbol{CP}^1$ *is singular if and only if* $(\lambda; \mu) \in S$.

The set-up of Theorem 1.2 (varieties over a complex vector space with countable algebraic dimension) is essentially the set-up for infinite-dimensional algebraic geometry introduced in [**S**] and [**T**]. Theorem 1.2 just describes the singular members of the pencil of degree d hypersurfaces of $\boldsymbol{P} = \boldsymbol{C}^{(N)}$ generated by the hypersurfaces $\{F = 0\}$ and $\{G = 0\}$.

Remark 1.3. We stress that in the statement of Theorem 1.2 we allow the case $S = \emptyset$, i.e. for every integer $d \geq 2$ we prove the existence of a pencil of degree d hypersurfaces of $\boldsymbol{P} = \boldsymbol{C}^{(N)}$ without any singular member. This is in striking contrast with the case of pencils on $\boldsymbol{CP}^n$: in that case if $d \neq 1$ every pencil has a singular member because the set of degree d singular hypersurfaces is a hypersurface in the big projective space parametrizing all degree d hypersurfaces of $\boldsymbol{CP}^n$.

2 - Proof of Theorem 1.1

Lemma 2.1. *Fix positive integers* $n, s, d_1, \ldots, d_s$ *such that* $n \geq s$. *Fix homogeneous coordinates* $z_0, \ldots, z_n$ *on* $\boldsymbol{P}^n$. *Let* $A(s, n, d_1, \ldots, d_s)$ *be the subset of* $\boldsymbol{C}^{s(n+1)}$ *formed by all* $a_{ij} \in \boldsymbol{C}$, $\leq i \leq s$, $0 \leq j \leq n$, *such that* $\{F_1 = \ldots = F_s = 0\}$ *is a* *smooth codimension s complete intersection of* $\boldsymbol{P}^n$, *where* $F_i = \sum_{0 \leqslant j \leqslant n} a_{ij} x_j^{d_i}$. *Then* $A(s, n, d_1, \ldots, d_s)$ *is a non-empty Zariski open subset of* $\boldsymbol{C}^{s(n+1)}$.

Proof. Since smoothness is an open condition in the Zariski topology and the same is true for the codimension s (i.e. the maximal possible codimension) condition, it is sufficient to show that $A(s, n, d_1, \ldots, d_s) = \emptyset$. We will use induction on s. If $s = 1$, set $B(s-1) = \boldsymbol{P}^n$. If $s \geqslant 2$ take $a_{ij} \in \boldsymbol{C}$, $1 \leqslant i \leqslant s-1$, $0 \leqslant j \leqslant n$, such that $B(s-l) := \{F_1 = \ldots = F_{s-1} = 0\}$ is a smooth codimension $s-1$ complete intersection of $\boldsymbol{P}^n$ (inductive assumption). Let $V(d_s)$ be the linear system on $B(s-1)$ spanned by the restriction to $B(s-1)$ of the degree d_s monomials $x_j^{d_s}$, $0 \leqslant j \leqslant n$. Since $V(d_s)$ has no base points and $B(s-1)$ is smooth, the general member of the linear system $V(d_s)$ is a smooth hypersurface of $B(s-1)$ by Bertini theorem ([**H**], Cor. III. 10.9, or [**K**]), i.e. for general $a_{sj} \in \boldsymbol{C}^{n+1}$, $0 \leqslant j \leqslant n$, $B(s-1) \cap F_s$ is a smooth codimension s complete intersection of $\boldsymbol{P}^n$. Thus $A(s, n, d_1, \ldots, d_s) \neq \emptyset$.

Proof of Theorem 1.1. Fix an orthonormal basis $\{x_n\}_{n \geqslant 1}$ of V. For any $z = \sum_{n \geqslant 1} z_n x_n \in V$ set $\alpha_n(z) := z_n$. Thus $z = \sum_{n \geqslant 1} \alpha_n(z) x_n$ for every $z \in V$ and $\alpha_n \in V(1)$. Fix complex numbers $\mu_{i,j}$, $i \geqslant 1$, $1 \leqslant j \leqslant s$, which are algebraically independent over the field $\boldsymbol{Q}$ of rational numbers and such that $0 < |\mu_{i,j}| \leqslant 1$ for all i, j. This is possible because $\boldsymbol{C}$ has infinite (and even uncountable) trascendence degree over $\boldsymbol{Q}$. For every $z = \sum_{n \geqslant 1} \alpha_n(z) x_n \in V$ set $F_j(z) := \sum_{i \geqslant 1} \mu_{i,j} \alpha_n(z)^{dj}$. Hence F_j is a continuous homogeneous degree d_j polynomial on V. Set $A_j := \{F_j = 0\}$ and $X = A_1 \cap \ldots \cap A_s$. Obviously X has codimension exactly s in $\boldsymbol{P}(V)$. It is sufficient to prove that X is smooth. Fix $P \in X$ and take $z = \sum_{i \geqslant 1} \alpha_i(z)\, x_i \in V \backslash \{0\}$ representing P. Let M be the matrix with s rows and countable columns, say $M(P) = (b_{ij})$, $i \geqslant 1$, $1 \leqslant j \leqslant s$, with $b_{ij} = \partial / \partial \alpha_i(F_j)(P) = d_j \mu_{i,j} \alpha_i(z)^{d_j - 1}$. It is sufficient to prove that for every $P \in X$ the matrix $M(P)$ has rank s. First assume the existence of indices $i_1, \ldots, i_s$ such that $\alpha(z)_{ik} \neq 0$ for every k with $1 \leqslant k \leqslant s$. Call $M(P)(i_1, \ldots, i_s)$ the minor of $M(P)$ formed by the columns $i_1, \ldots, i_s$. We have $\det(M(P)(i_1, \ldots, i_s)) = d_1 \ldots d_s \alpha_{i_1}(z)^{d_1} \ldots \alpha_{i_s}(z)^{d_s} \det(B)$ where B is the $s \times s$ matrix $(\mu_{i_k, j})$, $1 \leqslant k \leqslant s$, $1 \leqslant j \leqslant s$. Since $\alpha_{i_k}(z) \neq 0$ for every k and $\det(B) \neq 0$ by the algebraic independence over $\boldsymbol{Q}$ of the complex numbers $\mu_{i,j}$, we obtain $\det(M(P)) \neq 0$. Now assume that no such indices $i_1, \ldots, i_s$ do exists. Hence at most the first s homogeneous coordinates of P are non-zero. We apply Lemma 2.1 to the case $n = s$, in which we see $\boldsymbol{P}^s$ as $\boldsymbol{P}(W)$, where $W \subset V$ is the linear span of the vectors $x_1, \ldots, x_{s+1}$. By Lemma 2.1 the submatrix of $M(P)$ formed by the first $s+1$ columns has rank s at P and hence rank $(M(P)) = s$, proving the theorem.

3 - Proof of Theorem 1.2

Proof of Theorem 1.2. First we will prove the case S infinite (and countable). Then in a completely different way we will prove the case $S = \emptyset$. Then we will adapt the proof of the case $S = \emptyset$ to the case $S \neq \emptyset$ and S finite.

Step 1. Here we assume S infinite and countable. Up to a projective transformation we may assume that $(1;0) \notin S$, $(-1;1) \notin S$ and $(0;1) \notin S$. Hence $S = \{-a_i \in \boldsymbol{N}\}$ with $a_i \in \boldsymbol{C}$, $a_i \notin \{0,1\}$. Choose homogeneous coordinates z_i, $i \geq 0$, on $\boldsymbol{P}(\boldsymbol{C}^{(N)})$. Set $F := \sum_{i \geq 0} z_i^d$ and $G := \sum_{i \geq 0} a_i z_i^d$. Hence $\{F = 0\}$ is a Fermat hypersurface and G is in diagonal form. Since every point of $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ has only finitely many non-zero entries, it is very easy to check as in the finite-dimensional case that the hypersurface $\{\lambda F + \mu G\}$ has a singular point if and only if $\lambda + \mu a_i = 0$ for some i, i.e. if and only if \$(\lambda; \mu) \in S$.

Step 2. Here we assume $S = \emptyset$. Let $\boldsymbol{N}(d)$ be the set of all multi-indices α_i, $i \geq 0$, of non-negative integers with $\sum_{i \geq 0} \alpha_i = d$. Every homogeneous degree d hypersurface of $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ has an equation of the form $\sum_{\alpha \in \boldsymbol{N}(d)} a_\alpha z^\alpha$ for some complex numbers a_α. Set $F = \sum_{\alpha \in \boldsymbol{N}(d)} a_\alpha z^\alpha$ and $G = \sum_{\alpha \in \boldsymbol{N}(d)} b_\alpha z^\alpha$, where we assume that all $a_\alpha' s$ and $b_\alpha' s$ are trascendentally free over the field $\boldsymbol{Q}$ of rational numbers. This may be done because $\boldsymbol{N}(d)$ is countable, while $\boldsymbol{C}$ has even uncountable trascendence degree over $\boldsymbol{Q}$. For all $(\lambda; \mu) \in \boldsymbol{CP}^1$, set $X(\lambda, \mu) := \{\lambda F + \mu G\}$ and call L this pencil of hypersurfaces. We need to check that every $X(\lambda, \mu)$ is smooth. For any integer $n \geq 0$, set $\boldsymbol{CP}^n := \{z \in \boldsymbol{P}(\boldsymbol{C}^{(N)}) : z_i = 0$ for $i > n\}$, $X(\lambda, \mu; n) := X(\lambda, \mu) | \boldsymbol{CP}^n$ and $L(n)$ the associated pencil of $\boldsymbol{CP}^n$. Since every point of $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ has only finitely many non-zero coordinates, every singular point, P, of $X(\lambda, \mu)$ must be contained in some $X(\lambda, \mu; n)$ for some large n. It is easy to check that P must be a singular point of $X(\lambda, \mu; n)$. However, the converse is not true. Take a hypersurface Y of $\boldsymbol{CP}^{n+1}$, a hyperplane H of $\boldsymbol{CP}^{n+1}$ and $Q \in H$ such that Q is an ordinary double point of $Y \cap H$. A priori two cases may occur: either Q is an ordinary double point of Y and H is as transversal as possible to Y at Q or Y is smooth at Q and H is tangent to Y at Q. By the genericity of the coefficients a_α and b_α for every finite integer $n \geq 2$ the pencil $L(n)$ has only finitely many singular members, each singular hypersurface of $Y(n)$ has a unique singular point and this point is an ordinary double point. Now we compare the singular members of $L(n)$ and of $L(n+1)$. No singular member of $L(n)$ is the restriction of a singular member of $L(n+1)$, i.e. if $X(\lambda, \mu; n)$ is singular at Q, then $(\lambda, \mu; n+1)$ is smooth at Q and $\boldsymbol{CP}^n$ is tangent to $(\lambda, \mu; n+1)$ at Q. Hence letting n going to $+\infty$ we obtain that no $X(\lambda, \mu)$ may be singular.

Step 3. Here we assume S finite and $S \neq \emptyset$. Set $s := \mathrm{card}(S)$. We may assume, up to a projective transformation that S is given by the complex numbers $-a_j$, $1 \leq j \leq s := \mathrm{card}(S)$, with $a_j \notin \{0, 1\}$ for every j. For any positive integer n set $\boldsymbol{N}(d, n) := \{(\alpha_i) \in \boldsymbol{N}(d): \ \alpha_i = 0 \ \text{ for } \ i > n\}$. Set $F(s) := \sum_{0 \leq i \leq s} z_i^d$ and $G(s) := \sum_{0 \leq i \leq s} a_i z_i^d$ and call $L(s)$ the pencil of hypersurfaces generated by $F(s)$ and $G(s)$. The singular members of $L(s)$ are exactly the hypersurfaces $\{a_j F(s) + G(s) = 0\}$ of $\boldsymbol{CP}^s$ with $Q_j(0; \ldots; 1; 0, \ldots; 0)$ as unique singular points. Let $V(s+1)$ be the set of all extensions of $L(s)$ to a pencil of $\boldsymbol{CP}^{s+1}$ with singular members for the parameters a_j, $1 \leq j \leq s$, and respectively with Q_j as singular point. $V(s+1)$ is a finite-dimensional linear space. $V(s+1) \neq \emptyset$ (e.g. just take $a_\alpha = b_\alpha = 0$ if $\alpha \in \boldsymbol{N}(d, n+1) \backslash \boldsymbol{N}(d, n)$). Call $L(s+1)$ any general member of $V(s+1)$. The hypersurface of $L(s+1)$ corresponding to the parameter $(a_j; 1)$ have Q_j as only singular point. There will be also finitely many singular hypersurfaces in $L(s+1)$ but all of them with an ordinary double point as unique singular point. Call $L(s+2)$ a general extension of $L(s+1)$ to a pencil of hypersurfaces of $\boldsymbol{CP}^{s+2}$ with a singular member for each parameter $(a_j; 1)$ and at the point Q_j. The other singular members of the pencil $L(s+1)$ will not be singular in $\boldsymbol{CP}^{s+1}$ except on the points Q_j, $1 \leq j \leq s$, i.e. passing from $L(s+1)$ to $L(s+2)$ we have swallowed the singularities of the pencil $L(s+1)$ which were not assign in advance. And so on as in Step 2.

Remark 3.1. In the case S infinite and countable we obtained a pencil in which all singular members have only one singular point and with a rather bad singularity (at least if $d \geq 3$). Just allowing repetitions among the complex numbers a_i, $i \geq 0$, we obtain in the same way examples in which S is the set of all singular hypersurface, but each singular hypersurface is a cone over a smooth hypersurface and the vertex of the cone may have arbitrary dimension (finite or countable). If S is finite and $S \neq \emptyset$ the construction of Step 3 of the proof of Theorem 1.2 gives hypersurfaces with a unique singular point and an ordinary one, because for every $m \geq 2s+1$ the hypersurface of the pencil $L(m)$ corresponding to the parameter $(a_j; 1)$ have an ordinary double point at Q_j. However, we may even at each step of the induction to impose a bad singularity and find examples satisfying, the thesis of Theorem 1.2 but with prescribed multiplicity at the singular points.

Remark 3.2. Let L be any pencil of degree d hypersurfaces of $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ and call $L(n)$ its restriction to $\boldsymbol{CP}^n := \{z \in \boldsymbol{P}(\boldsymbol{C}^{(N)}): z_i = 0 \text{ for } i > n\}$. Assume that for every $n \geq 2L(n)$ has no base points. Hence $L(n)$ has only finitely many singular members. Since every point of $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ has only finitely many non-zero coordinates and the restriction to $\boldsymbol{CP}^n$ of a hypersurface singular at $P \in \boldsymbol{CP}^n$ is singular at P, we see that L has at most countably many singular members.

Example 3.3. Let I be any infinite set. Since every germ of holomorphic function on $\boldsymbol{C}^I$ depends only from finitely many variables, every homogeneous polynomial on $\boldsymbol{C}^I$ depends only from finitely many variables. Hence every zero-locus of a homogeneous polynomial of $\boldsymbol{C}^I$ is a cone with infinite-dimensional vertex over a hypersurface of a finite-dimensional projective space. Hence every hypersurface of degree at least two of $\boldsymbol{C}^I$ is singular. The space $\boldsymbol{C}^N$ is a Fréchet nuclear space and hence it should be considered as a rather good locally convex space.

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Summary

Here we prove the following two results. Fix positive integers s, $d_1, \ldots, d_s$ *and a separable Hilbert space* V; *then there exists a smooth codimension* s *complete intersection* $X \subset \boldsymbol{P}(V)$ *of* s *hypersurfaces of degree* $d_1, \ldots, d_s$. *Fix an integer* $d \geqslant 2$ *and a subset* S *of* $\boldsymbol{CP}^1$ *with at most countable elements; then there exist linearly independent homogeneous degree* d *polynomials* F *and* G *on* $\boldsymbol{C}^{(N)}$ *such that a hypersurface* $\{\lambda F + \mu G = 0\}$ *of* $\boldsymbol{P}(\boldsymbol{C}^{(N)})$ *with* $(\lambda; \mu) \in \boldsymbol{CP}^1$ *is singular if and only if* $(\lambda; \mu) \in S$; *we allow the case* $S = \emptyset$, *which is in striking contrast with the corresponding problem in* $\boldsymbol{CP}^n$.

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